C2.1a Lie algebras

Mathematical Institute, University of Oxford
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Problem Sheet 1

Throughout this sheet we assume that all Lie algebras are over a field k.

There is an algebraic was to think about the idea of "infinitesimals". The first two questions of the sheet explore this idea a little. Let k be a field and let $D_{\mathsf{k}} = \mathsf{k}[t]/(t^2)$. Write ε for the image of t in D_{k}, so that $\varepsilon^2 = 0$. We want to consider $\mathrm{Mat}_n(D_{\mathsf{k}})$ the space of $n \times n$ matrices over D_{k}.

1. Show that $\mathrm{GL}_n(D_{\mathsf{k}})$, the group of invertible matrices over D_{k} is exactly the set:

$$\{A + \varepsilon B : A \in \mathrm{GL}_n(\mathsf{k}), B \in \mathrm{Mat}_n(\mathsf{k})\}.$$

The natural homomorphism $\mathrm{e}\colon D_{\mathsf{k}} \to \mathsf{k}$ given by $\epsilon \mapsto 0$ induces a homomorphism of groups $\mathrm{e}_n : \mathrm{GL}_n(D_{\mathsf{k}}) \to \mathrm{GL}_n(\mathsf{k})$. Deduce that the kernel can be identified with $\mathrm{Mat}_n(\mathsf{k})$, *i.e.* $\mathfrak{gl}_n(\mathsf{k})$. (This does not however explain how the Lie bracket arises from the group structure on GL_n, which is slightly more involved.)

2. i) The determinant is defined for a matrix with entries in any commutative ring. For $X \in \mathrm{Mat}_n(D_{\mathsf{k}})$ find $\det(X)$ in terms of the column vectors of A, B where $X = A + \varepsilon B$, $A, B \in \mathrm{Mat}_n(\mathsf{k})$. In particular, show that if $X = I + \varepsilon B$ then $\det(X) = 1$ if and only if $\mathrm{tr}(B) = 0$. (*This is essentially the reason that the Lie algebra* $\mathfrak{sl}_n$ *is so named: the kernel of the map* $SL_n(D_{\mathsf{k}}) \to SL_n(\mathsf{k})$ *is* $\mathfrak{sl}_n(\mathsf{k})$.)

ii) The special orthogonal group is defined to be

$$\mathrm{SO}_n(\mathsf{k}) = \{A \in \mathrm{GL}_n(\mathsf{k}) : \det(A) = 1, A.A^t = I\}.$$

Show that the kernel of the map $\mathrm{SO}_n(D_{\mathsf{k}}) \to \mathrm{SO}_n(\mathsf{k})$ can be identified with

$$\mathfrak{so}_n(\mathsf{k}) = \{X \in \mathfrak{gl}_n(\mathsf{k}) : X + X^t = 0\}.$$

iii) Check that $\mathfrak{so}_n(\mathsf{k})$ is a Lie subalgebra of $\mathfrak{gl}_n(\mathsf{k})$.

3. Let A be an associative k-algebra, and $\mathrm{Der}_{\mathsf{k}}(A)$ the space of derivations of A. Check that $\mathrm{Der}_{\mathsf{k}}(A)$ is a Lie algebra under the commutator bracket.

4. Let A be an associative k-algebra as in the previous question, and, for $a \in A$, write δ_a for the map $\delta_a(b) = a.b. - b.a$, $(b \in A)$. Check that δ_a is a derivation, and the set

$$\mathrm{Inn}_{\mathsf{k}}(A) = \{\delta_a : a \in A\}$$

forms a Lie subalgebra of $\mathrm{Der}_{\mathsf{k}}(A)$.

5. Let A be as in the previous question. Check that the map $\iota\colon A \to \mathrm{Der}_{\mathsf{k}}(A)$ given by $a \mapsto \delta_a$ is a homomorphism of Lie algebras (where A is a Lie algebra via the commutator product). What is its kernel? Does it have to be surjective?

6. Suppose that $\mathfrak{g}$ is a 3-dimensional Lie algebra over a field k. Is it possible for $D(\mathfrak{g})$ to be 2-dimensional? (*Hint: consider subalgebras of* $\mathfrak{gl}_3(\mathsf{k})$.)

7. Suppose that $\mathfrak{g}$ is a 3-dimensional Lie algebra over a field k and $D(\mathfrak{g})$ is 1-dimensional, with $D(\mathfrak{g}) \subseteq \mathfrak{z}(\mathfrak{g}) \subsetneq \mathfrak{g}$ (so that $\mathfrak{g}$ is not abelian). Find the structure constants of $\mathfrak{g}$ with respect to a suitably chosen basis and deduce that up to isomorphism there is a unique such algebra.

Note here $\mathfrak{z}(\mathfrak{g}) = \{x \in \mathfrak{g} : [x, y] = 0, \forall y \in \mathfrak{g}\}$ *is the centre of* $\mathfrak{g}$ *and* $D(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}] = span\{[x, y] : x, y \in \mathfrak{g}\}]$ *is called the derived subalgebra of* $\mathfrak{g}$. *It is also the first term* $C^1(\mathfrak{g})$ *in the lower central series.*

Optional question:

8. Show that $\mathfrak{sl}_2(\mathbb{C})$ is not isomorphic to a Lie algebra obtained by equipping an associative algebra A with the commutator product.