

## C2.1a Lie algebras

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### Problem Sheet 2

1. Find the structure constants of  $\mathfrak{sl}_2$  with respect to the basis

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Show that the only ideals of  $\mathfrak{sl}_2(\mathbb{C})$  are 0 and itself.

2. If a Lie algebra  $\mathfrak{g}$  is a vector space direct sum of two Lie subalgebras  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$  such that  $[\mathfrak{g}_1, \mathfrak{g}_2] = 0$ , then we say that  $\mathfrak{g}$  is the *direct sum* of  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$  and write  $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ .

- a) Show that  $\mathfrak{gl}_2(\mathbb{C})$  is the direct sum of  $\mathfrak{sl}_2(\mathbb{C})$  and the subalgebra of scalar multiples of the identity matrix.
- b) Show that if  $\mathfrak{g}$  is the direct sum of Lie subalgebras  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$  then  $\mathfrak{g}_1$  and  $\mathfrak{g}_2$  are ideals of  $\mathfrak{g}$ . Show that taking the centre and taking the derived subalgebra respects direct sums, that is,  $\mathfrak{z}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g}_1) \oplus \mathfrak{z}(\mathfrak{g}_2)$  and  $D(\mathfrak{g}) = D(\mathfrak{g}_1) \oplus D(\mathfrak{g}_2)$ .

3. Let  $S$  be an  $n \times n$  matrix with entries in a field  $k$ . Define

$$\mathfrak{gl}_S = \{x \in \mathfrak{g} : x^t S + Sx = 0\}.$$

1. Show that  $\mathfrak{gl}_S$  is a Lie subalgebra of  $\mathfrak{gl}_n$ .
2. Let  $J_n$  be the  $n \times n$ -matrix:

$$J_n = \begin{pmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 1 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 1 & \dots & 0 & 0 \end{pmatrix}$$

Now let  $S$  be the  $2n \times 2n$  matrix:

$$S = \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix}$$

Find the conditions for a matrix to lie in  $\mathfrak{gl}_S$  and hence determine the dimension of  $\mathfrak{gl}_S$ .

4. Classify all Lie algebras  $\mathfrak{g}$  with  $\dim \mathfrak{g} = 3$  and  $\mathfrak{z}(\mathfrak{g}) \neq 0$ .
5. Let  $A \subset \mathfrak{gl}(V)$  denote a subspace consisting of commuting diagonalisable endomorphisms. Show that we may find a basis of  $V$  in which each element of  $A$  is represented by a diagonal matrix.
6. (*The classical groups*) In this course a fundamental role is played by the classical groups. In this question they will be defined, we will calculate their dimensions and we will look at some small dimensional examples. Assume throughout that  $k$  is a field of characteristic  $\neq 2$ .
- a) (*The special linear group  $\mathfrak{sl}_n$* ) Recall that  $\mathfrak{sl}_n \subset \mathfrak{gl}_n$  denotes the subspace of traceless  $n \times n$ -matrices. Check that  $\mathfrak{sl}_n$  is a subalgebra of  $\mathfrak{gl}_n$  by verifying  $\text{Tr}(ab) = \text{Tr}(ba)$  for  $n \times n$ -matrices  $a, b \in \mathfrak{gl}_n$ . Calculate the dimension of  $\mathfrak{sl}_n$ .
- b) (*The special orthogonal group  $\mathfrak{so}_n$* ) Recall the definition of  $\mathfrak{gl}_S$  and  $J_n$  from Question 3 of Sheet 1. Consider the matrix

$$S = J_n.$$

We define  $\mathfrak{so}_n$  to be  $\mathfrak{gl}_S$ . Find conditions for a matrix to belong to  $\mathfrak{so}_n$  and hence calculate its dimension.

- c) (*The symplectic group  $\mathfrak{sp}_{2n}$* ) Consider the matrix

$$S = \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix}.$$

We define  $\mathfrak{sp}_{2n}$  to be  $\mathfrak{gl}_S$ . You already calculated its dimension in question 3. Give an explicit description of  $\mathfrak{sp}_2$  in terms of another Lie algebra occurring on the list above.

d) Show that  $\mathfrak{so}_2$  is abelian and that  $\mathfrak{sl}_2 \cong \mathfrak{so}_3$

e) (*Optional ... but have a go!*) Show that  $\mathfrak{so}_4 \cong \mathfrak{sl}_2 \times \mathfrak{sl}_2$  and that  $\mathfrak{sp}_4 \cong \mathfrak{so}_5$ . (*Hint: Let  $\mathfrak{n}$  and  $\mathfrak{b}$  denote the strictly upper triangular and upper triangular matrices in  $\mathfrak{gl}_4$ . First try to understand the structure of  $\mathfrak{so}_4 \cap \mathfrak{n}$  and  $\mathfrak{sp}_4 \cap \mathfrak{n}$ . Then proceed to  $\mathfrak{so}_4 \cap \mathfrak{b}$  and  $\mathfrak{sp}_4 \cap \mathfrak{b}$ . From here it should be clearer how to construct the desired isomorphism.*)

*One may show that, as well as the list above, there is only one more “exceptional isomorphism” between the classical Lie algebras. If you are feeling courageous you might like to guess what it is (Hint: it involves  $\mathfrak{so}_6$ ). We will return to this question later on in the course.*