C2.1a Lie algebras

Mathematical Institute, University of Oxford
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Problem Sheet 3

All Lie algebras and all representations are finite dimensional unless otherwise stated.

1. Let k be an infinite field (not necessarily algebraically closed or of characteristic zero), and suppose that V is a finite dimensional k-vector space. If $U_1, U_2, \ldots, U_r$ are proper subspaces of V, show that $V \neq U_1 \cup U_2 \cup \ldots \cup U_r$.

2. a) Let k be a field of characteristic 2 and let $\mathfrak{g} = \mathfrak{sl}_2(k)$. Show that $\mathfrak{g}$ is solvable (and even nilpotent) but that the natural two-dimensional representation of $\mathfrak{g}$ is irreducible. Conclude that Lie's theorem is not true in positive characteristic.

b) Let $\mathbb{C}[x]$ denote a polynomial ring in x, and consider the Lie subalgebra $\mathfrak{g} \subset \mathfrak{gl}(\mathbb{C}[x])$ generated by the endomorphisms given by multiplication by x and $\frac{d}{dx}$. Show that $\mathfrak{g}$ is a three dimensional nilpotent Lie algebra, isomorphic to the Heisenberg algebra. Does $\mathfrak{g}$ fix a line in $\mathbb{C}[x]$? Why doesn't this contradict Lie's theorem?

3. Prove Schur's lemma: if V is a simple finite dimensional $\mathfrak{g}$-module over an algebraically closed field and $\phi \in \mathrm{End}_{\mathfrak{g}}(V)$ is an endomorphism of V commuting with the $\mathfrak{g}$-action, then ϕ is a scalar. (*Hint:* Consider the eigenspaces of ϕ.)

4. Show that $\mathfrak{sl}_n(\mathbb{C})$ is simple, that is, show that $\mathfrak{sl}_n(\mathbb{C})$ has no proper nontrivial ideals. (*Hint:* It might be easier show that $\mathfrak{gl}_n(\mathbb{C})$ has no non-trivial ideals contained in $\mathfrak{sl}_n$.)

5. Let $\mathfrak{h}$ be a nilpotent Lie algebra, over an algebraically closed field k of characteristic zero, and let (V, ρ) be a representation.

i) Show by induction on n that if $x, y \in \mathfrak{h}$, $\alpha, \beta \in \mathsf{k}$, $v \in V$ then

$$(\rho(y) - (\alpha+\beta)1)^n \rho(x)(v) = \sum_{i=0}^{n} \binom{n}{i} \rho((\mathrm{ad}(y) - \beta.1)^i(x))(\rho(y) - \alpha.1)^{n-i}(v)$$

(*Hint*: Let $x_i = (\mathrm{ad}(y) - \beta.1)^i(x)$ and calculate $(\rho(y) - (\alpha+\beta).1)\rho(x_i)$ in terms of $\rho(x_{i+1})$.)

ii) Let $y \in \mathfrak{h}$, and consider the generalized eigenspaces of $\rho(y)$ on V. Show that they are subrepresentations.

iii) Suppose that (V, ρ) is an indecomposable representation of $\mathfrak{h}$ (that is, we cannot write V as a direct sum of subrepresentations). Show that there is a basis of V with respect to which the matrices of the maps $\rho(y)$ are all upper triangular of the form

$$\rho(x) = \begin{pmatrix} \lambda(x) & * & \ldots & * \\ 0 & \lambda(x) & * & \vdots \\ \vdots & 0 & \ddots & * \\ 0 & \ldots & 0 & \lambda(x) \end{pmatrix}$$

for some $\lambda \in \mathfrak{h}^*$.

6. Let $\mathbb{H}$ denote the associative four dimensional algebra over $\mathbb{R}$ with basis e, i, j and k and multiplication given by $i^2 = j^2 = k^2 = -e$, $ij = k$, $jk = i$, $ki = j$ and requiring that e is the identity. The algebra $\mathbb{H}$ is called the algebra of *quarternions*. Determine $\mathrm{Der}\,\mathbb{H}$, the Lie algebra of derivations of $\mathbb{H}$. Can you identifiy $\mathrm{Der}\,\mathbb{H}$ with a classical Lie algebra?

Kevin McGerty, 23rd October 2012