C2.1a Lie algebras

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Problem Sheet 4

Assume throughout the problems that we work over a field k *which is algebraically closed of characteristic zero, unless the contrary is explicitly stated.*

1. Let V be a finite dimensional vector space, and let $\mathcal{F}$ be a flag $0 \subset V_1 \subset V_2 \subset \ldots \subset V_{n-1} \subset V_n = V$ of subspaces where $\dim(V_i) = i$. If $\mathfrak{n}_{\mathcal{F}} = \{x \in \mathfrak{gl}(V) : x(V_i) \subseteq V_{i-1}\}$ and $\mathfrak{b}_{\mathcal{F}} = \{x \in \mathfrak{gl}(V) : x(V_i) \subseteq V_i\}$, then we have seen in lecture that $\mathfrak{n}_{\mathcal{F}}$ is an ideal in $\mathfrak{b}_{\mathcal{F}}$ and so we have an exact sequence

$$0 \longrightarrow \mathfrak{n}_{\mathcal{F}} \longrightarrow \mathfrak{b}_{\mathcal{F}} \longrightarrow \mathfrak{t} \longrightarrow 0$$

where $\mathfrak{t}$ is defined to be the quotient $\mathfrak{b}_{\mathcal{F}}/\mathfrak{n}_{\mathcal{F}}$. Show that this sequence is split, and that there are infinitely many splitting maps $s \colon \mathfrak{t} \to \mathfrak{b}_{\mathcal{F}}$.

The next two questions study regular elements in the Lie algebra $\mathfrak{gl}(V)$. Recall an element x of a Lie algebra $\mathfrak{g}$ is said to be *regular* if it

$$\mathfrak{g}_{0,x} = \{y \in \mathfrak{g} : \exists n > 0, \mathrm{ad}(x)^n(y) = 0\}$$

has minimal possible dimension.

2. Let V be a k-vector space. If $x \in \mathrm{End}(V)$, and $V = \bigoplus_\lambda V_\lambda$ is the decomposition of V into a direct sum of generalised eigenspaces of x, we define $x_s \in \mathrm{End}(V)$ to be the linear map given by $x_s(v) = \lambda.v$ for $v \in V_\lambda$. It is called the *semisimple* part of x. Clearly it is diagonalisable.

i) Show that the element $x_n = x - x_s$ is nilpotent, and check that x_s and x_n commute.

ii) Show if $x, y \in \mathfrak{gl}(V)$ commute, and y is nilpotent, then generalised eigenspaces of x and $x + y$ coincide.

iii) In lectures we showed $\mathrm{ad}(n) \in \mathfrak{gl}(\mathfrak{gl}(V))$ is nilpotent if $n \in \mathfrak{gl}(V)$ is. Show that if s is diagonalizable then $\mathrm{ad}(s)$ is diagonalizable. (*Hint: take a sensible basis of* V *to reduce the question to one about* $\mathfrak{gl}_n(\mathsf{k})$).

3. Let V be a k-vector space as above.

i) Show that x is regular if and only if x_s is regular.

ii) When is a semisimple (*i.e.* diagonalisable) element of $\mathfrak{gl}(V)$ regular?

iii) Exhibit a Cartan subalgebra of $\mathfrak{gl}(V)$, and describe the set of all regular elements of $\mathfrak{gl}(V)$.

4. Let $\mathfrak{g}$ be a nilpotent Lie algebra. Show that the Killing form on $\mathfrak{g}$ is identically zero.

5. Let κ denote the Killing form on $\mathfrak{gl}_n(\mathbb{C})$ and let $\mathfrak{h}, \mathfrak{n}_+, \mathfrak{n}_-$ denote the subspaces of diagonal, strictly upper triangular and strictly lower triangular matrices respectively.

i) Show that $\mathfrak{h}$ is orthogonal to $\mathfrak{n}_+ \oplus \mathfrak{n}_-$ and that the restriction of κ to $\mathfrak{n}_+ \oplus \mathfrak{n}_-$ is nondegenerate. (*Hint*: It is probably useful to calculate the values of the Killing form on matrix coefficients).

ii) Calculate $\mathfrak{n}_+^\perp$.

iii) Describe the radical of the restriction of κ to $\mathfrak{h}$ and conclude that the restriction of κ to $\mathfrak{sl}_n(\mathbb{C})$ is nondegenerate.

6. Let k be a field and let $\mathfrak{s}_\mathsf{k}$ be the 3-dimensional k-Lie algebra with basis $\{e_0, e_1, e_2\}$ and structure constants $[e_i, e_{i+1}] = e_{i+2}$ (where we read the indices modulo 3, so that we have for example $[e_2, e_0] = e_1$). Show that $\mathfrak{s}_\mathsf{k}$ is a simple Lie algebra. Show that $\mathfrak{s}_\mathbb{R}$ is not isomorphic to $\mathfrak{sl}_2(\mathbb{R})$ but that $\mathfrak{s}_\mathbb{C} \cong \mathfrak{sl}_2(\mathbb{C})$. (*Hint: Consider characteristic polynomials.*)