

C2.1a Lie algebras

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Problem Sheet 5

Throughout this sheet we assume that k is an algebraically closed field of characteristic zero, and all Lie algebras and representations discussed are finite dimensional unless the contrary is explicitly stated.

1. Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_k$ be a direct sum of simple Lie algebras over a field of characteristic zero. Show that \mathfrak{g} is semisimple (i.e. establish the converse to the result we had in class that any semisimple Lie algebra is a direct sum of simple Lie algebras).

The next few questions of this exercise sheet classify all the irreducible finite dimensional representations of $\mathfrak{sl}_2(\mathbb{C})$.

Recall that if we let

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

then e, f and h give a basis of \mathfrak{sl}_2 with relations

$$[h, e] = 2e, [h, f] = -2f \quad \text{and} \quad [e, f] = h.$$

Hence, a representation of $\mathfrak{sl}_2(\mathbb{C})$ consists of a vector space V over \mathbb{C} together with three endomorphisms E, F and H satisfying

$$HE - EH = 2E, HF - FH = -2F \quad \text{and} \quad EF - FE = H.$$

(We recover the representation $\phi : \mathfrak{sl}_2(\mathbb{C}) \rightarrow \mathfrak{gl}(V)$ by setting $\phi(e) = E, \phi(f) = F$ and $\phi(h) = H$.)

We will also need a partial ordering on k : since k has characteristic zero it contains a copy of \mathbb{Q} , and we will say that $a < b$ if $b - a \in \mathbb{Q}_{>0}$. If $I \subset k$ is a finite subset of k we say $\lambda \in I$ is *maximal* if $\lambda < \mu$ implies $\mu \notin I$.

In the rest of this problem set we always assume that V is *finite dimensional*.

2. a) Show that the endomorphisms E and H satisfy the relation

$$(H - (\lambda + 2))^k E = E(H - \lambda)^k.$$

(Here $\lambda \in \mathbb{C}$ and we write λ instead of $\lambda \cdot \text{id}_V$.) Deduce that if $v \in V$ belongs to the generalised λ -eigenspace of H , then Ev belongs to the generalised $(\lambda + 2)$ -eigenspace.

b) Deduce a similar statement for the action of F on the generalised eigenspaces of H .

c) Let λ be an eigenvalue for H which is a maximal element of the set of eigenvalues of H in the sense described above. Use a) to show that $EV_\lambda = 0$.

d) Use b) to deduce that for large enough n we have $F^n(v) = 0$.

3. a) Show the relation (for $n \geq 1$)

$$HF^n = F^n H - 2nF^n.$$

b) Show ($n \geq 1$ as before)

$$EF^n = F^n E + nF^{n-1}H - n(n-1)F^{n-1}.$$

c) Deduce that, if $v \in V$ is a vector such that $Ev = 0$ then

$$E^n F^n v = nE^{n-1}F^{n-1}(H - (n-1))v = n! \prod_{i=1}^n (H - (i-1))v.$$

4. Let λ be a maximal eigenvalue of H (in the above sense) and let V_λ denote the generalised λ -eigenspace. Use 2(d) and 3(c) to deduce that H acts diagonalisably on V_λ and that λ is a non-negative integer.

5. a) Let λ be a maximal eigenvalue of H as in the previous question, and choose a non-zero vector $v \in V_\lambda$. We know by Questions 2 and 3 that $Ev = 0$ and that λ is an non-negative integer. Show the relations:

$$HF^k v = (\lambda - 2k)F^k v,$$

$$EF^k v = k(\lambda - (k - 1))F^{k-1} v.$$

Deduce that $F^{\lambda+1}v = 0$ and that the $F^i v$ for $0 \leq i \leq \lambda$ are linearly independent and span a simple submodule of V .

- b) Check that the above relations define an $\mathfrak{sl}_2(\mathbb{C})$ -module for any non-negative integer λ . Deduce that there is (up to isomorphism) a unique simple module $V(\lambda)$ of dimension $\lambda + 1$ for all non-negative integers λ .

6. Let \mathfrak{g} be a semisimple Lie algebra, and $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$ be its Cartan decomposition. To each $\alpha \in \Phi$ one may attach an element $h_\alpha = \frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)} \in \mathfrak{h}$ (where $\lambda \mapsto t_\lambda$ is the isomorphism $\mathfrak{h}^* \rightarrow \mathfrak{h}$ induced by the Killing form restricted to \mathfrak{h}). Show that if V is a finite-dimensional representation of \mathfrak{g} then the eigenvalues of h_α on V are all integers. (Here the elements $h_\alpha \in \mathfrak{h}$ are as given in Proposition 2.5 of the notes for Lecture 10 – you may just assume that Proposition when doing this question.)

7. (*Optional harder exercise*) Let V be an arbitrary finite dimensional representation of $\mathfrak{sl}_2(\mathbb{C})$.

- a) Let $\lambda \in \mathbb{Z}$ be maximal amongst the eigenvalues of H , and let $V_\lambda \subset V$ denote the λ -eigenspace. Suppose that V has the property that $Ev = 0$ implies that $v \in V_\lambda$. Show that V is completely reducible.
- b) Consider the endomorphism $c = EF + FE + \frac{1}{2}H^2$. Show that c commutes with E , F and H . (c is called the *Casimir* element.) Deduce from Schur's lemma that c acts as a scalar on any irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$. Compute the scalar with which c acts on $V(m)$.
- c) Show that V is completely reducible.

(*This result also follows from the more general complete reducibility result due to Weyl, but it is nice to see a more explicit proof for an algebra as small as \mathfrak{sl}_2 .*)