C2.1a Lie algebras

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Problem Sheet 7

Throughout this sheet we assume that all Lie algebras and all representations discussed are finite dimensional unless the contrary is explicitly stated, and we work over a field k which is algebraically closed of characteristic zero.

1. Let $\mathfrak{g}$ be a Lie algebra. We say that a representation (V, ρ) of $\mathfrak{g}$ is *semisimple* if any subrepresentation $W < V$ has a complement, that is, if W is any subrepresentation of V, then there is a subrepresentation $U < V$ such that $V = U \oplus W$.

i) Show that if V is semisimple then any subrepresentation or quotient of V is also semisimple.

ii) Show that if V is semisimple it is completely reducible.

2. Let $\mathfrak{g}$ be a Lie algebra. We say that an injective map of $\mathfrak{g}$-representations $i\colon U \to V$ *splits* if i has a left inverse $t\colon V \to U$ so that $t \circ i = \mathrm{id}_U$. Show that if every injective map of $\mathfrak{g}$-representations splits, then $\mathfrak{g}$-representations are completely reducible. Prove using properties of Casimir operators that any injective map of $\mathfrak{g}$-representations of a semisimple Lie algebra splits by considering the $\mathfrak{g}$-representations $S = \{\phi \in \mathrm{Hom}_{\mathsf{k}}(V, U) : \phi \text{ act as a scalar on } U\}$ and $T = \{\phi \in \mathrm{Hom}_{\mathsf{k}}(V, U) : \phi(U) = 0\}$. (You should check they are indeed $\mathfrak{g}$-representations.)

3. Suppose that $\mathfrak{g}$ is a Lie algebra and (V, ρ) is a faithful finite-dimensional representation (so that we may think of $\mathfrak{g}$ as a subalgebra of $\mathfrak{gl}(V)$). Show that if V is irreducible then $\mathfrak{g}$ is semisimple.

4. Let $\mathfrak{g} = \mathfrak{sp}_{2n}$ be the symplectic Lie algebra. Show that $\mathfrak{h}$, the space of matrices in $\mathfrak{g}$ which are diagonal, is a Cartan subalgebra, and thus find the roots of $\mathfrak{sp}_{2n}$. Do the same for the Lie algebras $\mathfrak{so}_{2n}$. (*Use the definitions of Problem set* 2, *question* 6).

5. Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra. If $\Phi \subset \mathfrak{h}^*$ is the corresponding root system find an expression for the dimension of $\mathfrak{g}$ in terms of Φ. (TIn particular, the dimension of $\mathfrak{g}$ is determined by the root system).

6. Let V be a $\mathbb{Q}$-vector space. A *lattice* in V is a discrete subgroup[1] $Q \subset V$ which spans V over $\mathbb{Q}$. Equivalently, a lattice is a subgroup Q of V of the form

$$\{\sum_i \lambda_i \beta_i : \lambda_i \in \mathbb{Z}\}$$

where $\{\beta_i\}_{i=1}^n$ is a basis of V. (You do not have to prove this).

Assume that V is equipped with an positive definite inner product $(-,-)$. A lattice $Q \subset V$ is called *integral* if $(\alpha, \beta) \in \mathbb{Z}$ for all $\alpha, \beta \in Q$. A lattice Q is called *even* if $(\alpha, \alpha) \in 2\mathbb{Z}$ for all $\alpha \in Q$.

i) Show that an even lattice is integral.

ii) Let $Q \subset V$ be an even lattice. Assume that the set $R_Q = \{\alpha \in Q : (\alpha, \alpha) = 2\}$ spans V. Show that R_Q is a root system in V.

iii) Let $V = \bigoplus_{i=1}^r \mathbb{Q} e_i$ equipped with the standard inner product $(e_i, e_j) = \delta_{ij}$. Let

$$\Gamma_r = \{\sum a_i e_i : \sum a_i \in 2\mathbb{Z} \text{ and either all } a_i \in \mathbb{Z} \text{ or all } a_i \in \mathbb{Z} + \frac{1}{2}.\}$$

Show that Γ_r is an even lattice if r is divisible by 8.

iv) Consider $\Gamma = \Gamma_8 \subset \mathbb{Q}^8$. Show that V is spanned by the vectors $v \in \Gamma$ such that $(v, v) = 2$, and describe the roots in the resulting root system R_Γ.

[1] That is, each element has an open ball around it which contains no other element of Q.

v) Consider the functional $t \in V^*$ given by

$$t = \sum_{i=1}^{7} (i-1)e_i^* + 23e_8^*,$$

where $\{e_i^*\}_{i=1}^8$ is the dual basis to $\{e_i\}_{i=1}^8$. Show that $0 \neq t(\alpha) \in \mathbb{Z}$ for all $\alpha \in R_\Gamma$. Calculate the set of roots $\alpha \in R_\Gamma$ with $t(\alpha) = 1$ and check it is a basis of V. Compute the matrix of the inner product with respect to this basis. (This step is similar to the proof that a root system has a base.)

vi) Let H_7 denote the hyperplane V orthogonal to $e_7 + e_8$. Show that $R_\Gamma \cap H_7$ is a root system of and find a basis for H_7 contained in it (*hint: start with the basis in the previous part.*)

vii) Let H_6 be the subspace orthogonal to $e_6 + e_7 + 2e_8$ and $e_7 + e_8$. Show that $R_\Gamma \cap H_6$ is a root system and calculate a basis for H_6 contained in it.