BACKGROUND MATERIAL FOR LIE ALGEBRAS

1. Background on Symmetric bilinear forms

In this section we review the basics of symmetric bilinear forms over a field k. It is all material that is almost in Part A Algebra, but perhaps not quite phrased there the way we use it. We shall work to begin with over an arbitrary field k.

Definition 1.1. Let V be a k-vector space. A function $B\colon V \times V \to \mathsf{k}$ is said to be bilinear if it is linear in each factor, that is if

$$B(\lambda_1 v_2 + \lambda_2 v_2, w) = \lambda_1 B(v_1, w) + \lambda_2 B(v_2, w),$$

and

$$B(v, \lambda_1 w_1 + \lambda_2 w_2) = \lambda_1 B(v, w_1) + \lambda_2 B(v, w_2),$$

for all $v, v_1, v_2, w, w_1, w_2 \in V$, $\lambda_1, \lambda_2 \in \mathsf{k}$. We say that it is *symmetric* if $B(v,w) = B(w,v)$. The *radical* of a symmetric bilinear form is the subspace:

$$\text{rad}(B) = \{v \in V : B(v,w) = 0, \forall w \in V\}.$$

We say that B is *nondegenerate* if $\text{rad}(B) = \{0\}$. Let $\text{Bil}(V)$ denote the vector space of bilinear forms on V, and[1] $\text{SBil}(V)$ for the space of symmetric bilinear forms on V.

Lemma 1.2. *There is a natural isomorphism $\Theta\colon Bil(V) \to Hom(V, V^*)$.*

Proof. Suppose that $B \in \text{Bil}(V)$. Then define $\Theta(B) = \theta \in \text{Hom}(V, V^*)$ as follows: given $v \in V$, let $\theta(v)\colon V \to \mathsf{k}$ be the function given by $\theta(v)(w) = B(v,w)$. The linearity of B in the second variable then ensures that $\theta(v) \in V^*$, while the linearity of B in the first variable ensures that the map $v \mapsto \theta(v)$ is linear, so that θ is induce a linear map from V to V^*.

Conversely, given $\theta\colon V \to V^*$, define B_θ by $B_\theta(v,w) = \theta(v)(w)$. It is easy to see that $B_\theta \in \text{Bil}(V)$, and that $\theta \mapsto B_\theta$ gives a linear map $\text{Hom}(V, V^*) \to \text{Bil}(V)$ which is clearly inverse to the map Θ, so they are both isomorphisms. □

Remark 1.3. The lemma shows that giving a bilinear form on V is equivalent to giving a linear map from V to V^*. Note that we made a choice in the above construction, since given $B \in \text{Bil}(V)$ we could have defined θ by $\theta(v)(w) = B(w,v)$. For symmetric bilinear forms the two possible choices agree, but for arbitrary bilinear forms they establish different isomorphisms.

From now on we will only work with symmetric bilinear forms. The kernel of $\theta = \Theta(B)$ is clearly the subspace of $v \in V$ such that $\theta(v)(w) = 0$ for all $w \in V$, which is exactly the definition of $\text{rad}(B)$ as required.

Now fix $B \in \text{SBil}(V)$. Then if U is a subspace of V, we define

$$U^\perp = \{v \in V : B(v,w) = 0, \forall w \in U\}.$$

If $\theta\colon V \to V^*$ is the associated map from V to V^*, then clearly $U^\perp = \theta^{-1}(U^0)$. When B is nondegenerate, so that θ is an isomorphism, this shows that $\dim(U^\perp) = \dim(V) - \dim(U)$. The next Lemma shows that this can be refined slightly.

Lemma 1.4. *Let V be a finite-dimensional k-vector space equipped with a symmetric bilinear form B. Then for any subspace U of V we have the following:*

(1) $\dim(U) + \dim(U^\perp) \geq \dim(V)$.
(2) *The restriction of B to U is nondegenerate if and only if $V = U \oplus U^\perp$.*

Date: November, 2012.

[1] This is not standard notation – it would be more normal to write something like $\text{Sym}^2(V^*)$ but then I'd have to explain why...

Proof. For the first part, define a map $\phi\colon U \to V^*$ by $\phi(u)(v) = B(u,v)$, $(u \in U, v \in V)$. Then $U^\perp$ is by definition exactly the subspace of $(\text{im}(\phi))^0$ (*i.e.* the annihilator of $\text{im}(\phi)$ under the natural identification of V with its double dual $(V^*)^*$). It follows that $\dim(U^\perp) + \dim(\text{im}(\phi)) = \dim(V)$, and hence certainly

$$\dim(U^\perp) + \dim(U) \geq \dim(V).$$

For the second part, it is immediate from the definitions that B is nondegenerate when restricted to U if and only if $U \cap U^\perp = \{0\}$, *i.e.* if and only if the sum of U and $U^\perp$ is direct. But then the equivalence follows immediately from the dimension inequality in the first part. □

1.1. **Classification of symmetric bilinear forms.** This subsection is not needed for the course[2] but might be clarifying. There is a natual linear action of GL(V) on the space Bil(V): if $g \in \text{GL}(V)$ and $B \in \text{Bil}(V)$ then we set $g(B)$ to be the bilinear form given by

$$g(B)(v,w) = B(g^{-1}(v), g^{-1}(w)), \quad (v,w \in V),$$

where the inverses ensure that the above equation defines a left action. It is clear the action preserves the subspace of symmetric bilinear forms.

Since we can find a invertible map taking any basis of a vector space to any other basis, the next lemma says that over an algebraically closed field there is only one nondegenerate symmetric bilinear form up to the action of GL(V), that is, when k is algebraically closed the nondegenerate symmetric bilinear forms are a single orbit for the action of GL(V).

Lemma 1.5. *Let V be a* k*-vector space equipped with a nondegenerate symmetric bilinear form B. Then if char*(k) $\neq 2$*, there is an orthonormal basis of V, i.e a basis $\{v_1, \ldots, v_n\}$ of V such that $B(v_i, v_j) = \delta_{ij}$.*

Proof. We use induction on $\dim(V)$. The identity[3]

$$B(v,w) = \frac{1}{2}\big(B(v+w, v+w) - B(v,v) - B(w,w)\big),$$

shows that if $B \neq 0$ we may find a vector $v \in V$ such that $B(v,v) \neq 0$. Rescaling by a choice of square root of $B(v,v)$ (which is possible since k is algebraically closed) we may assume that $B(v,v) = 1$. But if $L = \mathsf{k}.v$ then since $B_{|L}$ is nondegenerate, the previous lemma shows that $V = L \oplus L^\perp$, and if B is nondegenerate on V it must also be so on $L^\perp$. But $\dim(L^\perp) = \dim(V) - 1$, and so $L^\perp$ has an orthonormal basis $\{v_1, \ldots, v_{n-1}$. Setting $v = v_n$, it then follows $\{v_1, \ldots, v_n\}$ is an orthonormal basis of V as required. □

Remark 1.6. Over the real numbers, for example, there is more than one orbit of nondegenerate symmetric bilinear form, but the above proof can be modified to give a classification and it turns out that there are $\dim(V) + 1$ orbits.

2. Representation theory background

We recall here some basics of representation theory used in the course, all of which is covered (in much more detail than we need) in the Part B course on Representation theory. Let $\mathfrak{g}$ be a Lie algebra. We will always assume our representations are finite dimensional unless we explicitly say otherwise.

Definition 2.1. A representation is *irreducible* if it has no proper nonzero subrepresentations. A representation (V, ρ) is said to be *indecomposable* if it cannot be written as a direct sum of two proper subrepresentations. A representation is said to be *completely reducible* if is a direct sum of irreducible representations.

Clearly an irreducible representation is indecomposable, but the converse is not in general true. For example k^2 is naturally a representation for the nilpotent Lie algebra of strictly upper triangular matrices $\mathfrak{n}_2 \subset \mathfrak{gl}_2(\mathsf{k})$ and it is not hard to see that it has a unique 1-dimensional sub representation, hence it is indecomposable, but not irreducible.

Definition 2.2. A representation V is said to be *semisimple* if every subrepresentation U has a complement, that is, there exists a subrepresentation W of V such that $V = U \oplus W$.

Lemma 2.3. *Let $\mathfrak{g}$ be a Lie algebra.*

---

[2]So in particular you don't need to know it for any exam...

[3]Note that this identity holds unless char(k) $= 2$. It might be useful to remember this identity when understanding the Proposition which is the key to the proof of the Cartan Criterion: it claims that if $\mathfrak{g} = D\mathfrak{g}$ then there is an element $x \in \mathfrak{g}$ with $\kappa(x,x) \neq 0$. Noting the above identity, we see this is equivalent to asserting that κ is nonzero.

i) *A representation is semisimple if and only if every surjective map of representations $q \colon V \to W$ has a right inverse, that is there is a map of representations $s \colon W \to V$ such that $q \circ s = id_W$.*
ii) *Suppose every representation (V, ρ) of $\mathfrak{g}$ is semisimple. Then all representations of $\mathfrak{g}$ are completely reducible.*

Proof. For the first part, suppose that V is semisimple. Then given a surjection $q \colon V \to W$, we may find a complement U to the subrepresentation ker(q). Clearly then $q_{|U}$ is an isomorphism from U to W, and its inverse gives the required homomorphism s. For the converse, if U is a subrepresentation, then the quotient map $q \colon V \to V/U$ has a right inverse s. We claim that $V = U \oplus s(V/U)$. Suppose that $u \in V$ lies in the intersection. Then $u = s(w)$ for some $w \in V/U$, and hence (since U is the kernel of q) we have $0 = q(u) = q(s(w) = w$, so that $u = 0$. Thus $U \cap s(V/U) = \{0\}$, and thus by dimension counting we see that $V = U \oplus s(V/U)$ and $s(V/U)$ is a complement as required.

We prove the second part by induction on $\dim(V)$ the dimension of the representation. If V is irreducible then clearly we are done. Otherwise we may find a nontrivial proper subrepresentation $U < V$, which because V is semisimple has a complement W say. But $V = U \oplus W$ and $\dim(U), \dim(W)$ are both less than $\dim(V)$, so by induction they are completely reducible, and hence V is also. □

Remark 2.4. Our proof of Weyl's theorem shows that if $\mathfrak{g}$ is semisimple then every surjection of $\mathfrak{g}$-representations has a right inverse, and hence all representations of $\mathfrak{g}$ are semisimple and this completely reducible. One can show that any completely reducible representation is semisimple, so that in fact the two statements are equivalent, but our argument to prove Weyl's theorem does not require that implication.

Even if a representation is not completely reducible, we may still think of it as being built out of irreducibles. Indeed using induction on dimension, it is easy to see that any representation may be *filtered* by subrepresentations $0 = V_0 \subsetneq V_1 \subsetneq \ldots \subsetneq V_k = V$ where each successive quotient V_i/V_{i-1} is irreducible. We call such a filtration a *composition series* for V, and the irreducible representations V_i/V_{i-1} $(1 \leq i \leq k)$ are called *composition factors*.

Although we don't use it in the course, the following result is shows that the composition factors are actually independent of the filtration: If L is a simple representation then write $[L, V]$ for the number of times L occurs in the composition series (*i.e.* $[L, V] = \#\{i : 1 \leq i \leq k, V_i/V_{i-1} \cong L\}$.

Lemma 2.5. *(Jordan-Holder). The numbers $[L, V]$ are independent of the composition series.*

Proof. Use induction on the dimension of V. Clearly if this is 1, then V is irreducible and the result is clear. Now suppose that $(V_i)_{i=1}^k)$ and $(W_i)_{i=1}^l$ are two composition series for V. There is a smallest j such that $W_j \cap V_1$ is nonzero, and then since V_1 is irreducible we must have $V_1 \cap W_j = V_1$, that is, $V_1 \subseteq W_j$. But then the induced map $V_1 \to W_j/W_{j-1}$ must be an isomorphism by Schur's Lemma, so that $W_j = W_{j-1} \oplus V_1$. and so setting

$$W_i' = \left\{ \begin{array}{cc} (W_i \oplus V_1)/V_1 & \text{if } i < j, \\ W_{i+1}/V_1 & \text{if } j \leq l-1. \end{array} \right.$$

we obtain a composition series of V/V_1, whose composition factors are those of the composition series $(W_i)_{i=1}^l$ for V, with one fewer copy of the isomorphism class of $V_1 \cong W_j/W_{j-1}$. By induction it has the same composition factors as the filtration $\{V_i/V_1 : 1 < i \leq k\}$, and we are done.

For example, Lie's theorem for a solvable Lie algebra $\mathfrak{g}$ (over k an algebraically closed field of characteristic zero) shows that every irreducible representation of $\mathfrak{g}$ is one-dimensional, thus a composition series will have 1-dimensional composition factors, and these factors are independent of the choice of composition series. Note that in this case the representation may well not be a direct sum of 1-dimensional subrepresentations, so you really need to use a filtration by subrepresentations. On the other hand, Weyl's theorem for semisimple Lie algebras says that any representation of such a Lie algebra is in fact a direct sum of irreducible subrepresentations,

□