SYMMETRIC GROUPS AND THE STEINBERG VARIETY.

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1. Geometry of flag varieties

Much of what we say will make sense for any reductive algebraic group G and its Weyl group W, however I have proofs only for GL_n, and everything there can be made quite explicit. All our varieties are over $\mathbb{C}$. Our goal is to produce representations of the Weyl group W from geometry attached to G.

The protagonists of the story are the following.

Definition 1.1. Let $\mathcal{F}$ be the *flag variety* of GL_n, that is

$$\mathcal{F} = \{(0 \subset V_1 \subset V_2 \subset \ldots \subset V_n = \mathbb{C}^n) : \dim(V_i) = i\}.$$

Clearly $\mathcal{F}$ is a smooth projective variety. The group GL_n acts diagonally on $\mathcal{F} \times \mathcal{F}$ with finitely many orbits, each naturally indexed by an element of the symmetric group $W = S_n$. For $w \in S_n$, let $\mathcal{O}_w$ denote the corresponding orbit.

The *nilpotent cone* of GL_n is the variety:

$$\mathcal{N} = \{x \in \mathfrak{gl}_n : x^n = 0\},$$

a conic subvariety of $\mathfrak{gl}_n$. One can naturally identify the cotangent bundle of $\mathcal{F}$ with the variety

$$\tilde{\mathcal{N}} = \{(e, F) \in \mathfrak{gl}_n \times \mathcal{N} : e(F_i) \subset F_{i-1}\}.$$

Moreover, the obvious map $\mu \colon \tilde{\mathcal{N}} \to \mathcal{N}$ is a resolution of singularities (even with normal crossings exceptional divisor). The *Steinberg variety* $\mathcal{Z}$ is the fiber product:

$$\tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}} = \{(e, F^1, F^2) \in \mathcal{N} \times \mathcal{F} \times \mathcal{F} : e(F^i_j) \subset F^i_{j-1}, \text{ for } 1 \leq j \leq n, i = 1, 2\}.$$

Proposition 1.2. *The variety $\mathcal{Z}$ is pure dimensional with each component of dimension $n(n-1)$. Moreover the irreducible components of $\mathcal{Z}$ are the closures of*

$$\mathcal{Z}_w = \{(e, F_1, F_2) \in \mathcal{Z} : (F_1, F_2) \in \mathcal{O}_w\}.$$

Proof. Identify $\mathcal{Z}$ with the conormal bundles of the orbits $\mathcal{O}$ in $\mathcal{F} \times \mathcal{F}$. □

Lusztig showed that one can construct the group algebra $\mathbb{Z}[S_n]$ as a convolution algebra of constructible functions on $\mathcal{Z}$.

Definition 1.3. Let X be a complex algebraic variety. For any closed subvariety Z, let 1_Z be the characteristic function of Z, that is

$$1_Z(x) = \begin{cases} 1 & \text{if } x \in Z \\ 0 & \text{otherwise.} \end{cases}$$

The abelian group generated by the functions 1_Z as Z runs over the closed subvarieties of X is denoted $Con(X)$, and if $f \in Con(X)$ we say that f is constructible.

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Now *Con*$(\mathcal{Z})$ is clearly a ring under multiplication of functions, but this is not the algebra structure we wish to use, instead we want to use the fact that $\mathcal{Z}$ is a groupoid. To do this we need "functorial" operations on *Con*(X). Let $f\colon X \to Y$ be a morphism of varieties. Then if $\alpha \in$ *Con*(Y), the function $f^*(\alpha)$ given by

$$f^*(\alpha)(x) = \alpha(f(x)), \quad x \in X,$$

lies in *Con*(X), thus we have a "pull-back" operation.

Slightly less obviously we can also "push-forward" constructible functions. This essentially requires a notion of integration (really measures push forward and functions pull back). The integration we use is given by the Euler characteristic: we define for $f \in$ *Con*(X)

$$\int_X f = \sum_{n \in \mathbb{Z}} n.\chi(\alpha^{-1}(n)),$$

where χ denotes the Euler characteristic. This gives an additive functional on *Con*(X). We then define, for $f\colon X \to Y$ the pushforward $f_!\colon$ *Con*$(X) \to$ *Con*(Y) by setting

$$f_!(\alpha)(y) = \int_{f^{-1}(y)} \alpha.$$

It follows from basic stratification theory that the function $f_!(\alpha)$ is indeed constructible.

Using these operations it is easy to define a convolution product on *Con*$(\mathcal{Z})$: for $f, g \in$ *Con*(Z) we set

$$(f \star g)(e, F_1, F_2) = \int_{F \in \mathcal{F}_e} f(e, F_1, F)g(e, F, F_2),$$

where $\mathcal{F}_e = \{F \in \mathcal{F} : e(F_i) \subset F_{i-1}\}$. (One can also define this via a pull-back/push-forward diagram involving the variety

$$\mathcal{Z}_3 = \{(e, F^1, F^2, F^3) \in \mathcal{N} \times \mathcal{F}^3 : e \textit{ preserves each flag } F^i\},$$

using the three maps $q_{ij}\colon \mathcal{Z}_3 \to \mathcal{Z}$, where i, j are distinct elements of $\{1, 2, 3\}$.

It is easy to see that *Con*$(\mathcal{Z})$ becomes an associative algebra under $\star$, with unit $1_{\mathcal{Z}_e}$. It is of course, very big, but nevertheless Lusztig showed that one could construct $\mathbb{Z}[S_n]$ as a subalgebra. For $s \in W$ a transposition of consecutive integers, that is $s_i = (i, i+1)$ say,

$$\overline{\mathcal{O}}_{s_i} = \{(F, F') : F_j = F'_j \text{ if } j \neq i\},$$

so it is a $\mathbb{P}^1$ bundle over $\mathcal{F}$, and $\overline{\mathcal{Z}_{s_i}}$ is a smooth component of $\mathcal{Z}$. Let f_{s_i} be its characteristic function.

Theorem 1.4. *(Lusztig) The functions $\{f_{s_i} : 1 \leq i \leq n-1\}$ generate a subalgebra $\mathcal{W}$ of $($Con$(\mathcal{Z}), \star)$ isomorphic to $\mathbb{Z}[S_n]$, with the isomorphism given by $f_{s_i} \mapsto 1 - s_i$. Moreover the algebra $\mathcal{W}$ has a distinguished basis $\{f_w : w \in S_n\}$ which is characterized by the property that f_w is generically 1 on $\mathcal{Z}_w$ and generically 0 on the other components $\mathcal{Z}_v$.*

Now notice that if $e \in \mathcal{N}$, then *Con*$(\mathcal{F}_e)$ is naturally a module for *Con*$(\mathcal{Z})$: if $f \in$ *Con*$(\mathcal{Z})$ and $g \in$ *Con*$(\mathcal{F}_e)$, then

$$(f \star g)(F) = \int_{F' \in \mathcal{F}_e} f(e, F, F')g(F').$$

Thus for each $e \in \mathcal{N}$ we have $\mathcal{W}$-modules *Con*$(\mathcal{F}_e)$ (these are of course these are again infinite dimensional). Since all the functions in $\mathcal{W}$ are GL_n-invariant, (as each f_{s_i} is, and $\star$ is compatible with the GL_n action), we may restrict our attention to functions $Con^{\mathrm{GL}_n}(\mathcal{F}_e)$, and then the modules *Con*$(\mathcal{F}_e)$ and *Con*$(\mathcal{F}_{e'})$ are isomorphic if e and e' are conjugate. Now there are $p(n)$ orbits of GL_n on $\mathcal{N}$, and also exactly $p(n)$ irreducible representations of S_n, so it is tempting to seek to find an irreducible representation of S_n in $Con^G(\mathcal{F}_e)$ and thus construct all irreducible representations of S_n.

Theorem 1.5. *Let $e \in \mathcal{N}$. Then there exists a subgroup of $\mathcal{M}_e$ of $Con^G(\mathcal{F}_e)$ which is a module for $\mathcal{W}$. Moreover, as a representation of S_n it is irreducible, and $\mathcal{M}_e$ has a basis $\{m_c : c \in \mathcal{P}_e\}$ where $\mathcal{P}_e$ denotes the irreducible components of $\mathcal{F}_e$ which is characterized by the condition that m_c is generically one on the component c and generically zero on every other component.*

The difficulty in proving such a theorem is that it is not at all clear how one might construct such functions. To find them for GL_n we use a "dirty type A trick"[1]. Consider instead of $\mathcal{F}$ the larger variety

$$\mathcal{P} = \{(F_1 \subseteq F_2 \subseteq \ldots \subseteq F_n = \mathbb{C}^n : F_i \text{ subspaces of } V\}$$

Thus $\mathcal{P}$ is a disjoint union of components indexed by the compositions Λ_n of n: for each composition $\lambda \in \Lambda_n$ say $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of n, the corresponding component $\mathcal{P}_\lambda$ of $\mathcal{P}$, consists of the flags (F_i) for which $\dim(F_i) - \dim(F_{i-1}) = \lambda_i$. Thus $\mathcal{F}$ is the component $\mathcal{P}_{(1,\ldots,1)}$.

In exactly the same fashion as before, one can check that $T^*\mathcal{P}$, the cotangent bundle of $\mathcal{P}$ is

$$T^*\mathcal{P} = \{(e, F) \in \mathcal{N} \times \mathcal{P} : e(F_i) \subseteq F_{i-1}\}.$$

One then forms $\mathcal{Z}_P = T^*\mathcal{P} \times_{\mathcal{N}} T^*\mathcal{P}$, *i.e.*

$$\mathcal{Z}_P = \{(e, F, F') \in \mathcal{N} \times \mathcal{P} \times \mathcal{P} : e(F_i) \subseteq F_{i-1}, e(F'_i) \subseteq F'_{i-1}\}.$$

It is again the case that $\mathcal{Z}_P$ can be identified with the union of the conormal bundles of the GL_n-orbits on $\mathcal{P} \times \mathcal{P}$, and the connected components of $\mathcal{Z}_P$ are indexed by pairs of compositions (λ, μ) corresponding to the varieties $\mathcal{P}_\lambda \times \mathcal{P}_\mu$.

We define $e_i \in \mathrm{Con}(\mathcal{Z}_P)$ by $e_i = \sum_{\lambda \in \Lambda_n} 1_{E_i^\lambda}$, where

$$\begin{aligned} E_i^\lambda = \{(e, F, F') \in \mathcal{Z}_P : &(e, F') \in T^*\mathcal{P}_\lambda, F_j = F'_j, j \neq i, F'_i \subset F_i, \\ &\text{and } \dim(F_i/F'_i) = 1\} \end{aligned}$$

Similarly define $f_i \in \mathrm{Con}(\mathcal{Z}_P)$ by $f_i = \sum_{\lambda \in \Lambda_n} 1_{F_i^\lambda}$ where

$$\begin{aligned} F_i^\lambda = \{(e, F, F') \in \mathcal{Z}_P : &(e, F') \in T^*\mathcal{P}_\lambda, F_j = F'_j, j \neq i, F'_i \subset F_i, \\ &\text{and } \dim(F'_i/F_i) = 1\} \end{aligned}$$

Finally, set $h_i = \sum_{\lambda \in \Lambda_n} (\lambda_i - \lambda_{i+1}) 1_{H_i^\lambda}$ where

$$H_i^\lambda = \{(e, F, F) \in \mathcal{Z}_P : (e, F) \in T^*\mathcal{P}_\lambda\}$$

Let $\mathcal{U}$ be the algebra these functions generate under convolution (defined as for the case of $\mathcal{Z}$). We have the following theorem:

[1] a phrase stolen from A. Kleshchev.

Theorem 1.6. *(J. Chislenko) Let $\mathbb{U}$ be the enveloping algebra of $\mathfrak{sl}_n$, with Chevalley generators $\{E_i, F_i, H_i : 1 \leq i \leq n-1\}$. Then the assignment*

$$E_i \mapsto e_i, \quad F_i \mapsto f_i, \quad H_i \mapsto h_i,$$

extends to an algebra homomorphism $\mathbb{U} \to \mathcal{U}$. Moreover the kernel I_n is exactly the kernel of the natural map $\mathbb{U} \to End((\mathbb{C}^n)^{\otimes n})$.

Using this one can realize the highest weight representations which occur in $(\mathbb{C}^n)^{\otimes n}$ in constructible functions on the varieties $\mathcal{P}(e) = \{F \in \mathcal{P} : e(F_i) \subseteq F_{i-1}\}$, and moreover one gets a basis of the space of such functions which is in bijection with the irreducible components of the varieties $\mathcal{P}(e)$, with the bijection being given by assigning to each function the unique component on which its generic value is 1.

Now the following was observed by Kostant:

Lemma 1.7. *The zero weight space of a representation of $\mathfrak{sl}_n$ is a representation of the Weyl group S_n. Moreover, if λ is a partition on n, then the irreducible representation of highest weight corresponding to λ has as zero weight space an irreducible representation of S_n, and every irreducible representation of S_n occurs in this way.*

Now the weight spaces of the $\mathbb{U}$ representations correspond to the connected components of $\mathcal{P}(e)$, and the zero weight space is the component in $\mathcal{F}$, that is, the functions on $\mathcal{F}_e$ are exactly the zero weight space of $\mathcal{U}$.

The main theorem now follows by checking the Weyl group action given by $\mathcal{W}$ is compatible with the action of $\mathcal{U}$.

Example 1.8. If $n = 3$, then S_3 has three irreducible representations: the trivial, the sign, and the "reflection" representation. The variety $\mathcal{N}$ has 3 orbits – the zero orbit $\mathcal{O}_{1^3}$, the orbit of rank one matrices $\mathcal{O}_{21}$ and the orbit of rank two matrices $\mathcal{O}_3$. The corresponding varieties $\mathcal{F}_e$ are, respectively, the whole flag variety $\mathcal{F}$ for $e \in \mathcal{O}_{1^3}$, a single point for $e \in \mathcal{O}_3$, and two copies of $\mathbb{P}^1$ joined at a point for $e \in \mathcal{O}_{21}$. The modules $\mathcal{M}_e$ in each case are just the characteristic functions of the components of the $\mathcal{F}_e$.

We now give a more intrinsic definition of bimodules for $\mathcal{W}$ via a filtration of $\mathcal{F}$. Let $\pi \colon \mathcal{Z} \to \mathcal{N}$ be the obvious map. Note that if Z is a constructible subset of $\mathcal{N}$, then *Con*$(\pi^{-1}(Z))$ is obviously a module (even bi-module) for *Con*$(\mathcal{Z})$. Moreover, it is known that if we take $Z = \mathcal{O}_e$ a nilpotent orbit of $\mathcal{N}$, then $\pi^{-1}(\mathcal{O}_e)$ is pure dimensional of dimension $n(n-1)$ – that is, its closure is a union of components of $\mathcal{Z}$. This gives a partition of the elements of S_n, which label the components of $\mathcal{Z}$, into pieces known as *geometric cells*.

Conjecture 1.9. Let f_w be an element of the distinguished basis of $\mathcal{W}$. Then if $w \in \mathcal{C}_e$ the geometric cell corresponding to $\mathcal{O}_e \subset \mathcal{N}$, then f_w vanishes on the subset $\pi^{-1}(\overline{\mathcal{O}}_e)$.

Assuming this conjecture, it follows that if we take the functions $\{f_w : w \in \mathcal{C}_e\}$ and restrict them to the set $\pi^{-1}(e)$, then we obtain a bimodule for $\mathcal{W}$ which is isomorphic to $\mathrm{End}(V_e)$ where V_e is the irreducible representation attached to e by the above theorem.

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