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Performance of utility-based strategies for hedging basis risk

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Abstract

The performance of optimal strategies for hedging a claim on a non-traded asset is analysed. The claim is valued and hedged in a utility maximization framework, using exponential utility. A traded asset, correlated with that underlying the claim, is used for hedging, with the correlation ρ typically close to 1. Using a distortion method (Zariphopoulou 2001 *Finance Stochastics* **5** 61–82) we derive a nonlinear expectation representation for the claim's ask price and a formula for the optimal hedging strategy. We generate a perturbation expansion for the price and hedging strategy in powers of $\epsilon^2 = 1 - \rho^2$. The terms in the price expansion are proportional to the central moments of the claim payoff under the minimal martingale measure. The resulting fast computation capability is used to carry out a simulation-based test of the optimal hedging program, computing the terminal hedging error over many asset price paths. These errors are compared with those from a naive strategy which uses the traded asset as a proxy for the non-traded one. The distribution of the hedging error acts as a suitable metric to analyse hedging performance. We find that the optimal policy improves hedging performance, in that the hedging error distribution is more sharply peaked around a non-negative profit. The frequency of profits over losses is increased, and this is measured by the median of the distribution, which is always increased by the optimal strategies. An empirical example illustrates the application of the method to the hedging of a stock basket using index futures.

1. Introduction

This paper investigates the extent to which the use of an optimal hedging method, based on utility maximization, can improve the management of *basis risk*. By this term we mean the risk associated with the trading of a derivative security on an underlying asset that is not traded. Examples include weather derivatives, or options on baskets of stocks, where the basket is illiquid. In such a scenario, a correlated traded asset might be used for hedging purposes. (In the stock basket example, the claim on the basket might be hedged using liquid futures on a stock index, where the composition of the basket and the index are similar but not identical.)

In such a situation perfect hedging will not generally be possible, and to approach the problem systematically some optimal hedging method is sought. This can be done by embedding the problem in a utility maximization framework, in a manner that is now well established in derivative pricing. Indeed, the optimal valuation and hedging of claims on non-traded assets has been studied by other authors [3, 4, 8, 11, 18]. These papers have been concerned with solving the associated utility maximization problems, involving a portfolio of the

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traded asset and a random endowment of the claim payoff, from a variety of perspectives.

This paper takes the solution of the utility maximization problem as given, though we do present it briefly for completeness. Our main contribution is, first, to derive a perturbation series which gives accurate analytic approximations for the price and hedging strategy of the claim. Second, we use the ensuing fast computation of prices and hedging strategies to conduct a simulation-based test of the efficacy of the optimal hedge relative to a naive strategy which simply uses the traded asset as a proxy for the non-traded one. We take the view that it is important to establish whether optimal risk management procedures offer a significant improvement over more *ad hoc* procedures.

We use an exponential utility function to express the investor's risk preferences, though future work will explore strategies across different preferences and risk measures, such as 'expected shortfall' [5]. This risk measure has recently been analysed in the context of hedging in a stochastic volatility model [12], though a full-blooded test over many asset path histories was not carried out. This is also a fertile topic for future research.

Our testing procedure is to simulate many paths for the traded and non-traded asset prices, and to implement a self-financing hedging strategy implied by both optimal and naive methods. We compute the terminal tracking error for each path, plot the histogram for the tracking error distribution and compute some relevant statistics of the distribution. Recall that in the Black–Scholes (BS) [2] world the hedging error is zero with probability one, implying a Dirac δ-function distribution for the terminal hedging error.

We do indeed find that the optimal method improves hedging performance over the naive method, and the improvement is greater for lower absolute values of the correlation, and for higher values of risk aversion. The hedging error distribution has a lower standard deviation under the optimal strategy, and a higher median, indicating a higher relative occurrence of positive hedging errors.

The structure of the paper is as follows. In section 2 we set up the model, give utility-based pricing and hedging formulae and define the minimal martingale measure that arises in the remainder of the paper. In section 3 we derive representations for the asking price and optimal hedging strategy for the claim, and perturbation expansions are derived in section 4, with explicit results for a put option on the non-traded asset. Section 5 analyses hedging performance via simulation, and section 6 gives an empirical example of hedging a stock basket using index futures, to illustrate the methodology working on real data. Section 7 concludes.

2. The basis risk model

Two asset prices $(S, Y) := (S_t, Y_t)_{0\leqslant t\leqslant T}$ follow log-normal diffusions:

$$dS_t = \mu S_t\, dt + \sigma S_t\, dw_t, \tag{1}$$

$$dY_t = \mu_0 Y_t\, dt + \sigma_0 Y_t\, dw_t^0, \tag{2}$$

for $0 \leqslant t \leqslant T$, where the Brownian motions $(w, w^0) = (w_t, w_t^0)_{0\leqslant t\leqslant T}$ have correlation ρ, so that $dw_t^0\, dw_t = \rho\, dt$, with $-1 \leqslant \rho \leqslant 1$. The parameters $\mu, \sigma, \mu_0, \sigma_0, \rho$ are constants, and equations (1) and (2) are written in the physical measure $\mathbb{P}$. The riskless interest rate r is constant. The asset with price S is a *traded* asset but the asset with price Y is *non-traded*. A European option on asset Y has non-negative payoff $h(Y_T)$ at maturity time T, where h is a function.

Denote by $(w, w') := (w_t, w'_t)_{0\leqslant t\leqslant T}$ a two-dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leqslant t\leqslant T}, \mathbb{P})$, and let the filtration $(\mathcal{F}_t)_{0\leqslant t\leqslant T}$ be the one generated by $(w_t, w'_t)_{0\leqslant t\leqslant T}$. Then w' is independent of w and we can write w_t^0 in (2) as

$$w_t^0 = \rho w_t + \epsilon w'_t, \tag{3}$$

where $\epsilon = \sqrt{1-\rho^2}$.

An agent with risk preferences expressed via an exponential utility function

$$U(x) = -\exp(-\gamma x), \tag{4}$$

with constant risk aversion parameter $\gamma \in (0, 1)$, has the objective of maximizing expected utility of terminal wealth at time T. The investor can trade a dynamic self-financing portfolio containing Δ_t shares of the traded asset S_t at time $t \in [0, T]$, with the remainder invested in a cash account at interest rate r. In addition, the investor's account is credited at time T with n units of the derivative payoff $h(Y_T)$.

The wealth in the investor's cash and share portfolio, $(X_t)_{0\leqslant t\leqslant T}$, then follows the process

$$dX_t = rX_t\, dt + \pi_t((\mu - r)\, dt + \sigma\, dw_t), \tag{5}$$

where we have defined $\pi_t := \Delta_t S_t, 0 \leqslant t \leqslant T$, as the wealth invested in the stock. We note that there is no explicit dependence on S in (5), so that we may use (5) in place of (1) in the equations describing the dynamics of the state variables (X, Y) instead of (S, Y).

The investor's optimization problem is as follows: starting at time $t \in [0, T]$ with endowment $X_t = x$, and with initial non-traded asset price $Y_t = y$, the investor seeks a trading strategy $\pi := (\pi_t)_{0\leqslant t\leqslant T}$ in the class of admissible strategies $\mathcal{P}$ to achieve the supremum

$$F^n(t, x, y) := \sup_{\pi\in\mathcal{P}} \mathbb{E}_{t,x,y} U(X_T + nh(Y_T)), \tag{6}$$

where $\mathbb{E}_{t,x,y}$ denotes $\mathbb{P}$-expectation conditional on $X_t = x, Y_t = y$. The superscript n on the left-hand side of (6) will denote the number of derivative payoffs credited at time T, and the cases $n = 0$ and -1 will concern us in the remainder of the paper.

As is well known [4, 8], to ensure that (6) results in a meaningful optimization problem with exponential utility, we must assume that the random endowment $nh(Y_T)$ is bounded below. This covers long positions in calls and puts, short positions in puts, but excludes short call positions. The case of hedging short calls on the non-traded asset will be revisited in future papers.

A trading strategy is an adapted process $(\pi_t)_{0\leqslant t\leqslant T}$ satisfying $\int_0^T \pi_t^2\,dt < \infty$ almost surely. The class $\mathcal{P}$ of admissible trading strategies in (6) includes all those whose gains processes are bounded below, in order to eliminate doubling strategies [7]. However, this class is not big enough to ensure locating the optimal strategy by searching only within it [4, 19]. When the utility function $U(x)$ is defined for all $x \in \mathbb{R}$, the admissible class is enlarged to include some strategies with wealths which are not necessarily bounded from below. See [4, 19] for further details.

We shall denote the optimal trading strategy that achieves the supremum in (6) by $\pi^n = (\pi_t^n)_{0\leqslant t\leqslant T}$.

2.1. The case of perfect correlation

If $\rho = 1$, then as shown in [3], absence of arbitrage implies that, given σ, σ_0, the drifts μ, μ_0 are related by

$$\frac{\mu_0 - r}{\sigma_0} = \frac{\mu - r}{\sigma}. \tag{7}$$

In this case, perfect hedging of the claim on Y is possible by trading S, the hedging strategy at time $t \in [0, T]$ being to hold a number of shares given by

$$\frac{\sigma_0 Y_t}{\sigma S_t}\frac{\partial}{\partial s}\mathrm{BS}(Y_t, 0, \sigma_0), \tag{8}$$

where $\mathrm{BS}(s, q, \sigma)$ denotes the BS formula with underlying asset price s, dividend yield q and volatility σ.

2.2. Utility-based pricing and hedging

Consider two special cases of the optimization problem (6). For $n = 0$ there is no dependence on the claim. The dynamics of the non-traded asset Y do not influence the problem at all and we recover a variant of the classical Merton problem [13, 14]. We set $F^0(t, x, y) =: F(t, x)$ to signify that there is no dependence on n or y in this case. Denote by $\pi^0 = (\pi_t^0)_{0\leqslant t\leqslant T}$ the optimal trading strategy that achieves the supremum in (6) when $n = 0$.

The case $n = -1$ corresponds to a debit of one unit of the option payoff $h(Y_T)$, so when accompanied with a credit to the initial endowment of $p(t, x, y)$, represents the case where the investor sells one claim for price $p(t, x, y)$. The (by now classical) definition of the time-t utility indifference selling price (or simply the *ask* price) of the claim, $p^a(t, x, y)$, is as the solution of

$$F(t, x) = F^{-1}(t, x + p^a(t, x, y), y). \tag{9}$$

There is a natural definition of a hedging strategy associated with the sale of the claim for the utility indifference ask price, introduced in [15, 16]. Let the optimal trading strategy for the optimization problem with value function $F^{-1}(t, x + p^a(t, x, y), y)$ be $\pi^{-1} := (\pi_t^{-1})_{0\leqslant t\leqslant T}$. The difference in stock holdings between the optimal strategies for $n = -1$ and 0 represents the additional position taken in the hedging instrument as a result of the sale of the claim. This is therefore a natural analogue of the hedging strategy of a claim in a complete market, and motivates the definition below.

Definition 1. *The hedging strategy $\pi^h = (\pi_t^h)_{0\leqslant t\leqslant T}$ associated with the sale of the claim at the ask price $p^a(t, x, y)$ is given by*

$$\pi_t^h := \pi_t^{-1} - \pi_t^0, \qquad 0 \leqslant t \leqslant T. \tag{10}$$

The strategy π^h reduces to the BS hedging strategy in a complete market situation. See [15, 16] for further details on utility indifference pricing and hedging, in which the above definitions are applied to a model with transaction costs on stock trades.

2.3. Minimal martingale measure

Denote by $\mathcal{M}$ the set of equivalent local martingale measures, under which $(\mathrm{e}^{-rt}S_t)_{0\leqslant t\leqslant T}$ is a local martingale. The asset price dynamics under measures $\mathbb{Q} \in \mathcal{M}$ are

$$\mathrm{d}S_t = rS_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}\tilde{w}_t, \tag{11}$$

$$\mathrm{d}Y_t = (\mu_0 - \sigma_0(\rho\lambda + \epsilon g_t))Y_t\,\mathrm{d}t + \sigma_0 Y_t\,\mathrm{d}\tilde{w}_t^0, \tag{12}$$

where

$$\lambda := \frac{\mu - r}{\sigma}, \tag{13}$$

$(g_t)_{0\leqslant t\leqslant T}$ is an $\mathcal{F}_t$-adapted process satisfying $\int_0^T g_t^2\,\mathrm{d}t < \infty$, $\mathbb{P}$-almost surely, and $\tilde{w}_t^0$ is a Brownian motion defined by

$$\tilde{w}_t^0 = \rho\tilde{w}_t + \epsilon\tilde{w}_t', \tag{14}$$

with $(\tilde{w}, \tilde{w}') := (\tilde{w}_t, \tilde{w}_t')_{0\leqslant t\leqslant T}$ a two-dimensional $\mathbb{Q}$-Brownian motion defined by

$$\tilde{w}_t := w_t + \lambda t, \tag{15}$$

$$\tilde{w}_t' := w_t' + \int_0^t g_u\,\mathrm{d}u. \tag{16}$$

Then $\mathrm{d}\tilde{w}_t^0\,\mathrm{d}\tilde{w}_t = \rho\,\mathrm{d}t$, and the set $\mathcal{M}$ is in one-to-one correspondence with the set of processes g_t.

Definition 2 (Minimal martingale measure). *The minimal martingale measure $\mathbb{Q}^0 \in \mathcal{M}$ corresponds to $g_t = 0, 0 \leqslant t \leqslant T$.*

There are many characterizations of the minimal martingale measure, and the reader is referred to the review by Schweizer [20] for further details.

3. The asking price of a claim

In this section we briefly review the solution to the optimization problem (6), based on the Hamilton–Jacobi–Bellman (HJB) equation of dynamic programming. For more details see [8, 9, 18], or [4] for a dual approach to the problem. Connections between these solution methods are discussed in [17].

3.1. The Hamilton–Jacobi–Bellman equation

The value function $F^n(t,x,y)$ satisfies the PDE

$$
\begin{aligned}
&F^n_t(t,x,y) + rxF^n_x(t,x,y) + \mu_0 y F^n_y(t,x,y) \\
&\quad + \frac{1}{2}\sigma_0^2 y^2 F^n_{yy}(t,x,y) - \frac{1}{2F^n_{xx}(t,x,y)}[\lambda F^n_x(t,x,y) \\
&\quad + \rho\sigma_0 y F^n_{xy}(t,x,y)]^2 = 0, \qquad (17)
\end{aligned}
$$

with terminal boundary condition $F^n(T,x,y) = -\mathrm{e}^{-\gamma(x+nh(y))}$.

The optimal trading strategy π^*_t is given by

$$\pi^*_t = -\frac{[(\mu-r)F^n_x(t,x,y) + \rho\sigma\sigma_0 y F^n_{xy}(t,x,y)]}{\sigma^2 F^n_{xx}(t,x,y)}. \qquad (18)$$

Under exponential utility, it turns out that one can find a solution to (17) of the form

$$F^n(t,x,y) = -\mathrm{e}^{-\gamma\beta(t,T)x}(f^n(t,y))^\delta, \qquad (19)$$

where $\beta(t,T) := \mathrm{e}^{r(T-t)}$, $0 \leqslant t \leqslant T$, and where the parameter δ can be chosen so that the function $f^n(t,y)$ satisfies a linear PDE. This technique is called *distortion* by Zariphopoulou [22] and is also employed in [8, 9]. There are links to the dual approach to solving the optimization problem, involving the Legendre transform of the value function. These links are discussed further in [17].

The distortion method gives the solution for the value function in (6) as

$$F^n(t,x,y) = -\mathrm{e}^{-\gamma\beta(t,T)x}[\mathbb{E}^0_{t,y}(\mathrm{e}^{-\alpha(T-t)-\gamma\epsilon^2 nh(Y_T)})]^{1/\epsilon^2}, \qquad (20)$$

where

$$\alpha = \frac{1}{2}\lambda^2\epsilon^2 = \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2(1-\rho^2), \qquad (21)$$

and $\mathbb{E}^0_{t,y}$ denotes expectation under the minimal martingale measure $\mathbb{Q}_0$, conditional on $Y_T = y$. Under $\mathbb{Q}_0$ the dynamics of Y are

$$\mathrm{d}Y_t = (\mu_0 - \sigma_0\rho\lambda)Y_t\,\mathrm{d}t + \sigma_0 Y_t\,\mathrm{d}\tilde{w}^0_t. \qquad (22)$$

Using (20) along with (9) we obtain the following representation for the ask price of the claim.

Theorem 1. *The utility indifference asking price at time $t \leqslant T$ of a European claim with payoff $h(Y_T)$ is given by*

$$p^a(t,y) = \frac{\mathrm{e}^{-r(T-t)}}{\gamma(1-\rho^2)}\log[\mathbb{E}^0_{t,y}(\mathrm{e}^{\gamma(1-\rho^2)h(Y_T)})], \qquad (23)$$

where $\mathbb{E}^0_{t,y}$ denotes expectation conditional on $Y_t = y$ under the minimal martingale measure $\mathbb{Q}^0 \in \mathcal{M}$.

We observe that $p^a(t,y)$ is independent of the agent's initial cash endowment x, as is always the case under exponential preferences. Henderson [8] and Musiela and Zariphopoulou [18] give similar representations to (23) for the ask price.

3.2. Optimal hedging strategy

The optimal trading strategy in the presence of the random endowment $nh(Y_t)$ at the terminal time is given by (18). For $n = 0$, and using (20), this gives the optimal trading strategy in the absence of the claim as

$$\pi^0_t = \mathrm{e}^{-r(T-t)}\left(\frac{\mu-r}{\sigma^2\gamma}\right), \qquad (24)$$

which is the well-known solution to the Merton optimal investment problem with exponential utility.

For the case of the writer of a claim, we take $n = -1$ in (18). Now, for general n, differentiating (20) yields

$$F^n_x(t,x,y) = -\gamma\beta(t,T)F^n(t,x,y), \qquad (25)$$

$$F^n_{xx}(t,x,y) = \gamma^2\beta^2(t,T)F^n(t,x,y), \qquad (26)$$

$$F^n_{xy}(t,x,y) = -\gamma\beta(t,T)F^n_y(t,x,y). \qquad (27)$$

The derivatives of the value function with respect to the initial capital x are proportional to the value function itself. To get a similar result for the mixed derivative $F^n_{xy}(t,x,y)$ in the case $n = -1$, proceed as follows. Differentiate (9) with respect to y, and recall that the ask price is independent of the initial capital (i.e. $p^a(t,x,y) = p^a(t,y)$), to give

$$F^{-1}_y(t,\tilde{x},y) = -F^{-1}_x(t,\tilde{x},y)p^a_y(t,y), \qquad (28)$$

where we have put $\tilde{x} = x + p^a(t,y)$. Using this in (27), along with (25), (26) and (18), all evaluated at initial capital $\tilde{x}$, gives the optimal trading strategy of the writer as

$$\pi^{-1}_t = \mathrm{e}^{-r(T-t)}\left(\frac{\mu-r}{\sigma^2\gamma}\right) + \frac{\rho\sigma_0 y}{\sigma}p^a_y(t,y). \qquad (29)$$

The strategy in (29) is very intuitive. The first term represents the optimal investment strategy in the absence of a claim. The second term is the adjustment to this strategy caused by the introduction of the claim, that is, the *hedging strategy* for the claim, in precise accordance with definition 1. Applying this definition immediately gives the following result.

Theorem 2. *The hedging strategy for the sale of the claim at the asking price $p^a(t,y)$ at time $t \in [0,T]$ is to hold Δ^a_u shares of the traded asset S at time $u \geqslant t$, given by*

$$\Delta^a_u = \frac{\rho\sigma_0 Y_u}{\sigma S_u}\frac{\partial p^a}{\partial y}(u,Y_u), \qquad t \leqslant u < T. \qquad (30)$$

It is easy to see that this reduces to the strategy in (8) when $\rho = 1$.

4. Perturbation expansions

From the representation (23) for the ask price of the claim, we proceed to derive a power series expansion for the price, and also for its derivative with respect to y, which has application in hedging, as given by theorem 2.

Let a random variable X have variance Σ^2 and write $\mu_k = \mathbb{E}(X^k), k \in \mathbb{N}$. Define the skewness $\mathrm{skw}(X)$ and kurtosis $\mathrm{kur}(X)$ of X by

$$\mathrm{skw}(X) := \frac{\mathbb{E}[(X-\mu_1)^3]}{\Sigma^3}, \qquad (31)$$

$$\mathrm{kur}(X) := \frac{\mathbb{E}[(X-\mu_1)^4]}{\Sigma^4} - 3. \tag{32}$$

Observe that with the above definitions we have the identities

$$\Sigma^3 \mathrm{skw}(X) = \mu_3 - 3\mu_1\mu_2 + 2\mu_1^3 \tag{33}$$

$$\Sigma^4 \mathrm{kur}(X) = \mu_4 - 3\mu_2^2 + 12\mu_1^2\mu_2 - 4\mu_1\mu_3 - 6\mu_1^4. \tag{34}$$

We then have the following expansion for the asking price $p^a(t, y)$ of the claim on the non-traded asset with payoff $h(Y_T)$.

Theorem 3. *The function $p^a(t, y)$ representing the asking price of the claim with payoff $h(Y_T)$ at time $T \geqslant t$ has the perturbative representation*

$$\begin{aligned} p^a(t, y) = \frac{1}{\beta(t,T)}\bigg[&\mathbb{E}^0_{t,y}h(Y_T) + \frac{1}{2}\gamma\epsilon^2 \,\mathrm{var}^0_{t,y}\, h(Y_T) \\ &+ \frac{1}{3!}(\gamma\epsilon^2)^2\Sigma_0^3 \,\mathrm{skw}^0_{t,y}\, h(Y_T) \\ &+ \frac{1}{4!}(\gamma\epsilon^2)^3\Sigma_0^4 \,\mathrm{kur}^0_{t,y}\, h(Y_T) + \mathrm{O}(\epsilon^8)\bigg], \end{aligned} \tag{35}$$

where $O(\epsilon^8)$ denotes terms proportional to ϵ^8 and to higher powers of ϵ. The expansion is valid for model parameters satisfying $\mathbb{E}^0_{t,y} \exp(\gamma\epsilon^2 h(Y_T)) \leqslant 2$.

In (35), $\mathrm{var}^0_{t,y}$ denotes the variance operator conditional on $Y_t = y$, under the minimal martingale measure $\mathbb{Q}^0$, with a similar convention for $\mathrm{skw}^0_{t,y}$ and $\mathrm{kur}^0_{t,y}$. We have used the notation $\mathrm{var}^0_{t,y} h(Y_T) =: \Sigma_0^2$ in the third and fourth terms of the expansion.

Remark 1. For $\rho = 1$ the asking price becomes the BS price with volatility σ_0, since all but the leading term in the price expansion disappear and, by (7), the drift of Y under the minimal measure becomes the risk-free rate r.

Proof of theorem 3. Expanding the exponential in (23) using Taylor's theorem gives

$$\begin{aligned} p^a(t, y) = \frac{1}{\beta(t,T)\gamma\epsilon^2}\log\bigg(&1 + \gamma\epsilon^2\mathbb{E}^0_{t,y}h(Y_T) \\ &+ \frac{1}{2}\gamma^2\epsilon^4\mathbb{E}^0_{t,y}h^2(Y_T) + \frac{1}{3!}\gamma^3\epsilon^6\mathbb{E}^0_{t,y}h^3(Y_T) \\ &+ \frac{1}{4!}\gamma^4\epsilon^8\mathbb{E}^0_{t,y}h^4(Y_T) + \mathrm{O}(\epsilon^{10})\bigg). \end{aligned} \tag{36}$$

The power series expansion of $f(x) = \log(1+x)$ is valid for $-1 < x \leqslant 1$. The terms inside the logarithm in (36) are non-negative, and when summed over all powers of ϵ^2 they give the exponential in (23). This implies that the logarithm in (36) can be expanded as a Taylor series provided $\mathbb{E}^0_{t,y} \exp(\gamma\epsilon^2 h(Y_T)) \leqslant 2$. This proves the last assertion in the theorem.

Expanding (36), initially keeping all terms up to order ϵ^{10}, then simplifying, gives

$$\begin{aligned} p^a(t, y) = \frac{1}{\beta(t,T)}\bigg[&M_1 + \frac{1}{2}\gamma\epsilon^2(M_2 - M_1^2) \\ &+ \frac{1}{3!}\gamma^2\epsilon^4(M_3 - 3M_1M_2 + 2M_1^3) \\ &+ \frac{1}{4!}\gamma^3\epsilon^6(M_4 - 3M_2^2 + 12M_1^2M_2 \\ &- 4M_1M_3 - 6M_1^4) + \mathrm{O}(\epsilon^8)\bigg], \end{aligned} \tag{37}$$

where we have introduced the notation

$$M_k := \mathbb{E}^0_{t,y}h^k(Y_T), \qquad k \in \mathbb{N}. \tag{38}$$

Then, in view of the identities (33) and (34), the proof is complete. □

4.1. Explicit results for a put option

Suppose $h(y) = (K - y)^+$ for a positive constant K. Then it is a straightforward, though lengthy, process to establish explicit results for $p^a(t, y)$ and $p^a_y(t, y)$. We use the fact that under $\mathbb{Q}^0 \in \mathcal{M}$, and conditional on $Y_t = y$, $\log Y_T$ is normally distributed with mean m and variance s^2, given by

$$m = \log y + (r - q - \sigma_0^2/2)(T - t), \tag{39}$$

$$s^2 = \sigma_0^2(T - t), \tag{40}$$

where we have defined the 'dividend yield' q by

$$q = r - (\mu_0 - \sigma_0\rho\lambda). \tag{41}$$

We make extensive use of the (easily verifiable) integrals

$$\begin{aligned} &\mathbb{E}^0_{t,y}[Y_T^k I_{Y_T\leqslant K}] = \mathrm{e}^{k(m+ks^2/2)}N(-d_1 - (k-1)s) \\ &\quad = y^k \mathrm{e}^{k(r-q+(k-1)\sigma_0^2/2)(T-t)}N(-d_1 - (k-1)s) \\ &\quad (k \in \{0, 1, 2, 3, 4\}). \end{aligned} \tag{42}$$

In (42), I_A denotes the indicator function of event A, $N(\cdot)$ denotes the standard cumulative normal distribution function and we have defined the variable d_1 by

$$d_1 = \frac{\log(y/K) + (r - q + \sigma_0^2/2)(T - t)}{\sigma_0\sqrt{T-t}}. \tag{43}$$

This is the familiar argument of $N(\cdot)$ which appears in the BS formula.

As an illustration, the zeroth-order term in the expansion for $p^a(t, y)$ is $p^{a,0}(t, y)$ given by

$$\begin{aligned} p^{a,0}(t, y) &= \mathrm{e}^{-r(T-t)}\mathbb{E}^0_{t,y}h(Y_T) \\ &= \mathrm{e}^{-r(T-t)}\mathbb{E}^0_{t,y}[(K - Y_T)I_{Y_T\leqslant K}]. \end{aligned} \tag{44}$$

Using (42) this becomes

$$\begin{aligned} &p^{a,0}(t, y) \\ &\quad = K\mathrm{e}^{-r(T-t)}N(-d_1 + \sigma_0\sqrt{T-t}) - y\mathrm{e}^{-q(T-t)}N(-d_1) \\ &\quad = \mathrm{BS}^p(y, K, q, \sigma_0, T - t), \end{aligned} \tag{45}$$

where $\mathrm{BS}^p(y, K, q, \sigma_0, T - t)$ denotes the Black–Scholes put option formula with underlying asset price y, strike K, dividend yield q, volatility σ_0 and time to expiration $T - t$.

In a similar manner we establish all other necessary results. The essential formulae are summarized below:

$$\begin{aligned} \mathbb{E}^0_{t,y}h(Y_T) = M_1 &= KN(-d_1 + s) \\ &\quad - y\mathrm{e}^{(r-q)(T-t)}N(-d_1), \end{aligned} \tag{46}$$

$$\begin{aligned} \mathbb{E}^0_{t,y}h^2(Y_T) = M_2 &= K^2N(-d_1 + s) \\ &\quad - 2Ky\mathrm{e}^{(r-q)(T-t)}N(-d_1) \\ &\quad + y^2\mathrm{e}^{(2(r-q)+\sigma_0^2)(T-t)}N(-d_1 - s), \end{aligned} \tag{47}$$

Table 1. Put ask prices $p^a(0, Y_0)$ and 'deltas' $p^a_y(0, Y_0)$ from the perturbative expansion and from simulation. The parameters are those in table 2. The exception to this is the case $\rho = 1$, in which case no-arbitrage considerations fix $\mu_0 = \mu - \sigma_0\lambda = 0.11$, and the option value is the BS value with volatility σ_0 and dividend yield 0. Figures in parentheses are standard deviations of the observations that were averaged for the simulation results.

Put option asking prices, $\gamma = 0.001$, 2×10^6 simulations					
ρ	$o(\epsilon^0)$	$o(\epsilon^2)$	$o(\epsilon^4)$	$o(\epsilon^6)$	Simulation
−0.95	5.3914	5.4016	5.4016	5.4016	5.4001 (0.0111)
−0.75	5.6320	5.6566	5.6567	5.6567	5.6564 (0.0023)
−0.50	6.0493	6.0944	6.0946	6.0946	6.0970 (0.0246)
−0.25	6.4870	6.5471	6.5474	6.5474	6.5465 (0.0131)
0	6.9451	7.0133	7.0138	7.0138	7.0113 (0.0034)
0.25	7.4238	7.4917	7.4922	7.4922	7.4913 (0.0020)
0.50	7.9231	7.9806	7.9809	7.9809	7.9791 (0.0128)
0.75	8.4428	8.4783	8.4784	8.4784	8.4806 (0.0241)
0.95	8.8733	8.8815	8.8815	8.8815	8.8790 (0.0136)
1	9.3542	9.3542	9.3542	9.3542	9.3514 (0.0180)
Put option deltas					
−0.95	−0.2634	−0.2639	−0.2639	−0.2639	−0.2632
−0.75	−0.2715	−0.2726	−0.2726	−0.2726	−0.2723
−0.50	−0.2850	−0.2870	−0.2870	−0.2870	−0.2866
−0.25	−0.2986	−0.3011	−0.3012	−0.3012	−0.3006
0	−0.3123	−0.3151	−0.3151	−0.3151	−0.3145
0.25	−0.3260	−0.3287	−0.3287	−0.3287	−0.3280
0.50	−0.3397	−0.3418	−0.3419	−0.3419	−0.3411
0.75	−0.3533	−0.3546	−0.3546	−0.3546	−0.3540
0.95	−0.3641	−0.3644	−0.3644	−0.3644	−0.3644
1	−0.3757	−0.3757	−0.3757	−0.3757	−0.3752

$$
\begin{aligned}
\mathbb{E}^0_{t,y} h^3(Y_T) = M_3 = {} & K^3 N(-d_1 + s) \\
& - 3K^2 y \mathrm{e}^{(r-q)(T-t)} N(-d_1) \\
& + 3K y^2 \mathrm{e}^{(2(r-q)+\sigma_0^2)(T-t)} N(-d_1 - s) \\
& - y^3 \mathrm{e}^{3(r-q+\sigma_0^2)(T-t)} N(-d_1 - 2s),
\end{aligned} \tag{48}
$$

$$
\begin{aligned}
\mathbb{E}^0_{t,y} h^4(Y_T) = M_4 = {} & K^4 N(-d_1 + s) \\
& - 4K^3 y \mathrm{e}^{(r-q)(T-t)} N(-d_1) \\
& + 6K^2 y^2 \mathrm{e}^{(2(r-q)+\sigma_0^2)(T-t)} N(-d_1 - s) \\
& - 4K y^3 \mathrm{e}^{3(r-q+\sigma_0^2)(T-t)} N(-d_1 - 2s) \\
& + y^4 \mathrm{e}^{2(2(r-q)+3\sigma_0^2)(T-t)} N(-d_1 - 3s).
\end{aligned} \tag{49}
$$

These results can then be substituted into (35) or (37) for numerical computation of the asking price.

4.1.1. Put option delta.

Differentiating (35) with respect to y gives the following expansion for $p^a_y(t, y)$:

Corollary 1. *The derivative of the asking price $p^a(t, y)$ with respect to y has the perturbative expansion*

$$
\begin{aligned}
\frac{\partial p^a}{\partial y}(t, y) = \frac{1}{\beta(t,T)} \Bigg[& \partial M_1 + \frac{1}{2}\gamma\epsilon^2 (\partial M_2 - 2M_1 \partial M_1) \\
& + \frac{1}{3!}\gamma^2\epsilon^4 (\partial M_3 - 3M_2 \partial M_1 - 3M_1 \partial M_2 + 6M_1^2 \partial M_1) \\
& + \frac{1}{4!}\gamma^3\epsilon^6 (\partial M_4 - 6M_2 \partial M_2 + 12M_1^2 \partial M_2 + 24 M_1 M_2 \partial M_1 \\
& - 4M_1 \partial M_3 - 4M_3 \partial M_1 - 24 M_1^3 \partial M_1) + \mathrm{O}(\epsilon^8) \Bigg],
\end{aligned} \tag{50}
$$

where we have used the notation

$$
\partial M_k \equiv \frac{\partial M_k}{\partial y} = \frac{\partial \mathbb{E}^0_{t,y} h^k(Y_T)}{\partial y}. \tag{51}
$$

The partial derivatives needed to apply the above corollary are obtained by differentiating (46)–(49). This yields the following formulae:

$$
\partial M_1 = -\mathrm{e}^{(r-q)(T-t)} N(-d_1), \tag{52}
$$

$$
\begin{aligned}
\partial M_2 = {} & -2\mathrm{e}^{(r-q)(T-t)} [K N(-d_1) \\
& - y \mathrm{e}^{(r-q+\sigma_0^2)(T-t)} N(-d_1 - s)],
\end{aligned} \tag{53}
$$

$$
\begin{aligned}
\partial M_3 = {} & -3\mathrm{e}^{(r-q)(T-t)} [K^2 N(-d_1) \\
& - 2K y \mathrm{e}^{(r-q+\sigma_0^2)(T-t)} N(-d_1 - s) \\
& + y^2 \mathrm{e}^{(2(r-q)+3\sigma_0^2)(T-t)} N(-d_1 - 2s)],
\end{aligned} \tag{54}
$$

$$
\begin{aligned}
\partial M_4 = {} & -4\mathrm{e}^{(r-q)(T-t)} [K^3 N(-d_1) \\
& - 3K^2 y \mathrm{e}^{(r-q+\sigma_0^2)(T-t)} N(-d_1 - s) \\
& + 3K y^2 \mathrm{e}^{(2(r-q)+3\sigma_0^2)(T-t)} N(-d_1 - 2s) \\
& - y^3 \mathrm{e}^{3(r-q+2\sigma_0^2)(T-t)} N(-d_1 - 3s)].
\end{aligned} \tag{55}
$$

The above recipe is sufficient to give fast computation of the asking price of the put option on the non-traded asset and the associated hedging strategy.

4.2. Numerical results

Using the expectation representation (23) it is a simple matter to produce numerical values for the ask price of the claim, and for its derivative with respect to y, by simulation. This was done for two million samples, and the numerical values compared with those from the perturbation expansions in the last section. The goal is to establish the accuracy (or otherwise) of the expansions across a range of values of the correlation ρ. The simulations were also used to check that the model parameters we used did indeed satisfy the restrictions of

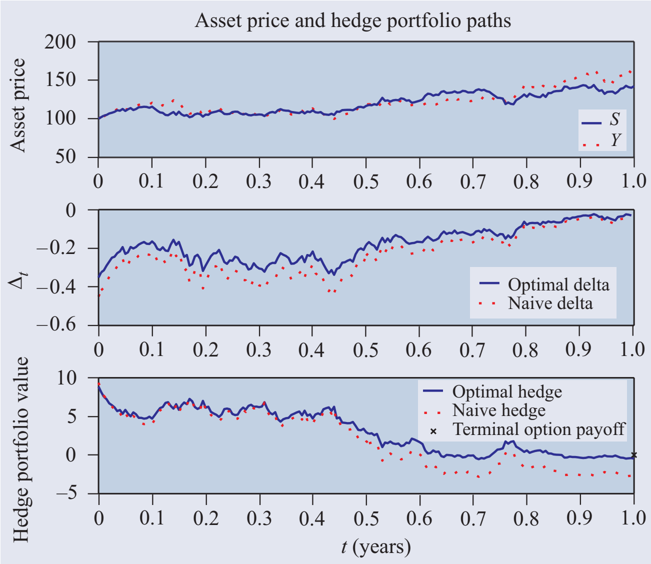

Figure 1. Asset prices (upper graph), hedge ratios (middle graph) and hedge portfolio wealths (lower graph) along a simulated path. The solid curve in the lower two graphs corresponds to the optimal hedge, while the broken curve corresponds to the naive hedge. The parameters are as in table 2, and $\rho = 0.8$, $\gamma = 0.01$.

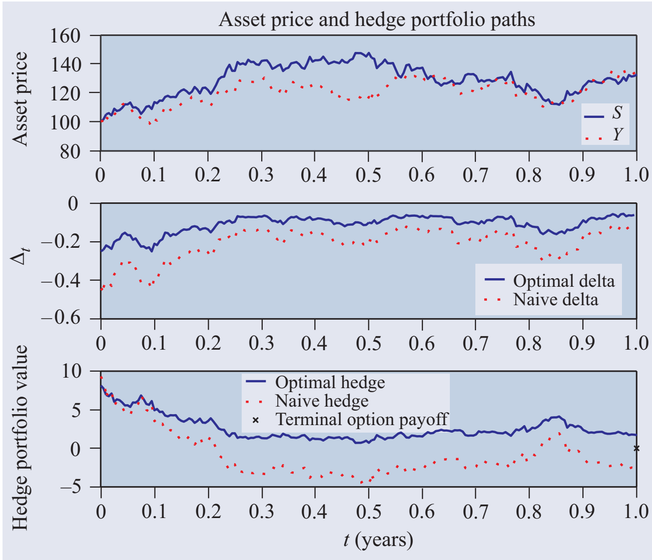

Figure 2. Asset prices (upper graph), hedge ratios (middle graph) and hedge portfolio wealths (lower graph) along a simulated path. The solid curve in the lower two graphs corresponds to the optimal hedge, while the broken curve corresponds to the naive hedge. The parameters are as in table 2, and $\rho = 0.6$, $\gamma = 0.001$.

theorem 2, needed for the perturbation expansions to be valid. All results reported below were for valid model parameters. It was found that risk aversion values γ below about 0.05 guaranteed validity, regardless of other parameter choices. Typical risk aversion parameters for market participants are around 10^{-6} [10], so this is a very mild restriction.

The accuracy of the perturbation expansions is confirmed by the results shown in table 1 for $p^a(t, y)$ and $p^a_y(t, y)$ at time zero, for $\gamma = 0.001$ and various values of ρ. The results produced by the perturbation expansion at order ϵ^2 and beyond are remarkably in line with those from simulation. Further tests, not reported here for the sake of brevity, show that accurate results are obtained across all values of correlation when the risk aversion parameter is below about 0.05, with the accuracy increasing with increasing $|\rho|$ and decreasing γ.

The significance of these results is that we now have a very fast route to computing option prices and hedging strategies. This allows for practical implementation, and for an efficient testing program of the hedging performance of optimal strategies versus the 'naive' strategies which simply use the traded asset as a proxy for the non-traded one. Such a testing procedure is carried out below.

5. Hedging performance of optimal strategies

To analyse hedging performance, we suppose that a put option on asset Y is sold at time zero for price $p^a(0, Y_0)$, defining the initial endowment in our hedging portfolio, and hedged using strategy $(\Delta^a_t)_{0\leqslant t\leqslant T}$ given in theorem 2. Denote the wealth in the hedging portfolio by $(X^a_t)_{0\leqslant t\leqslant T}$, given by (5) with $\pi_t = \Delta^a_t S_t$. The evolution of this wealth in discrete time will be used in the numerical simulations below.

Table 2. Model parameters.

S_0	Y_0	K	r (%)	μ (%)	σ (%)	μ_0 (%)	σ_0 (%)	T (year)
100	100	100	5	10	25	12	30	1

We simulate a path for both asset prices $(S, Y) := (S_t, Y_t)_{0\leqslant t\leqslant T}$ with given correlation ρ, and choose a number of times that the hedge is rebalanced in the option lifetime. The formulae established in the previous section are used to compute the hedge portfolio 'delta' at each rehedging time. Then for each asset price path simulated we compute the terminal tracking error

$$\mathcal{E}_T := X^a_T - (K - Y_T)^+. \tag{56}$$

The above calculation is repeated over a large number M (say, 10 000) of asset price paths.

Finally, we repeat the entire calculation over the same simulated paths, but use a 'naive' approach which assumes we sell the option for $\mathrm{BS}^p(Y_0, K, 0, \sigma_0, T)$ and hedge using the strategy given in (8).

5.1. Results

The results reported below used the parameters shown in table 2 as a base case, and the options were rehedged 200 times during their life.

Figures 1 and 2 illustrate the nature of the simulations. The upper graphs show the traded (solid curve) and non-traded (broken curve) asset prices along a path, while the middle and lower graphs show the hedge ratios and hedge portfolio values along the paths for the optimal (solid curve) and naive (broken curve) strategies. The terminal option payoff is also marked with a cross (×).

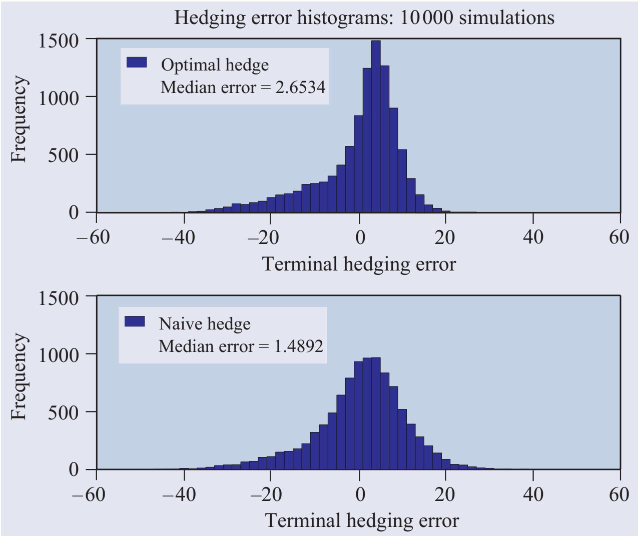

Figure 3. Histograms of terminal hedging error over 10 000 sample paths for the optimal hedging strategy (upper graph) and the naive strategy (lower graph). The parameters are as in table 2, and $\rho = 0.65$, $\gamma = 0.001$.

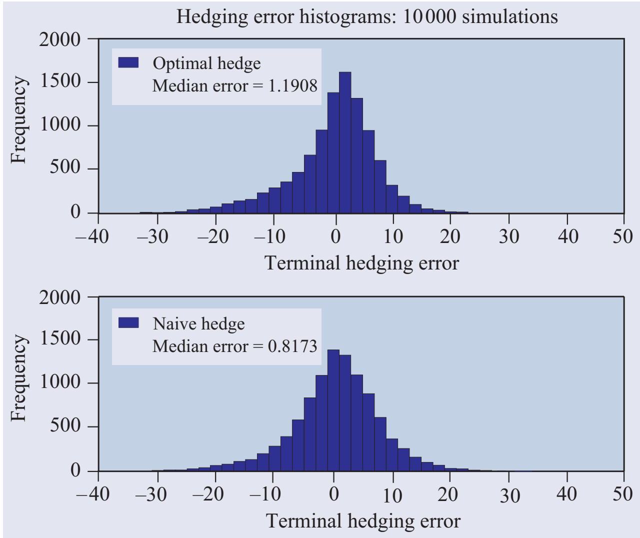

Figure 4. Histograms of terminal hedging error over 10 000 sample paths for the optimal hedging strategy (upper graph) and the naive strategy (lower graph). The parameters are as in table 2, and $\rho = 0.85$, $\gamma = 0.001$.

Table 3. Hedging error statistics for the histograms in figure 3.

	Max	Min	Mean	SD	Median
Optimal hedge	25.65	−48.09	0.1145	9.6342	2.6534
Naive hedge	37.22	−49.68	0.4303	10.3618	1.4892

Table 4. Hedging error statistics for the histograms in figure 4.

	Max	Min	Mean	SD	Median
Optimal hedge	22.24	−32.78	0.1816	6.9951	1.1908
Naive hedge	26.49	−32.27	0.5098	7.0880	0.8173

Figure 3 shows histograms illustrating the distribution of the terminal hedging error produced by the optimal (upper graph) and naive (lower graph) hedging strategies. The results, over 10 000 simulations, are for $\rho = 0.65$ and $\gamma = 0.001$. Both graphs are plotted on the same scales for ease of comparison. It is immediately apparent that the optimal hedging procedure produces a more sharply peaked distribution, with a higher proportion of errors around and just above zero, compared with the naive hedging strategy. The shapes of the histograms show how the optimal method will tolerate small negative errors, but not large losses.

To put some concrete numbers on these visual observations, we give summary statistics for the distributions in table 3. The standard deviation of the naive hedging error distribution is about 7% higher than that of the optimal hedging policy. The really significant statistic, however, is the *median* of the distributions. The median hedging error from the optimal policy is 78% higher than that from the naive hedging policy. In other words, the optimal policy results in positive hedging errors far more frequently than the naive policy. This is precisely what one would require of a good hedging policy. The mean of the distribution is fairly meaningless in this context, as the figures in the table show. Note also how the range of the hedging error is larger with the naive hedging policy. In other words, sometimes one will be lucky and make a large profit, while at other times one will incur a large loss. Systematic improvements are therefore made by the optimal procedure.

Figure 4 shows similar histograms for a higher value of the correlation, namely $\rho = 0.85$. The pattern is similar, as the summary statistics in table 4 show. This time, the median hedging error for the optimal strategy is about 45% higher than that for the naive strategy, and the standard deviation is about 1% higher for the naive strategy. In other words, the optimal strategy is still an improvement over the naive policy, even for a higher correlation.

Figures 5 and 6 show hedging error distributions for $\rho = 0.65$ and 0.85, but now with a larger risk aversion parameter, $\gamma = 0.01$. Summary statistics for these distributions are given in tables 5 and 6 respectively. The results are similar to those reported earlier. For $\rho = 0.65$, the median hedging error for the optimal strategy is about twice (100% higher) that for the naive strategy, and the standard deviation is about 7% higher for the naive strategy. For $\rho = 0.85$, the median hedging error for the optimal strategy is about 75% higher that for the naive strategy, and the standard deviation is about 1% higher for the naive strategy. In other words, the improvements are similar, and in terms of the median, perhaps even greater for the case of a higher risk aversion. This is intuitively correct, of course, as ‘optimality’ should be of greater benefit when one is more sensitive to risk. Similar results, not reported here, hold for other model parameters.

6. An empirical application

In this section we illustrate how the hedging approach we have tested can be applied in a real world situation. We tackle the case of hedging a basket of nine UK stocks using futures contracts on the FTSE100 index. We do not claim to be

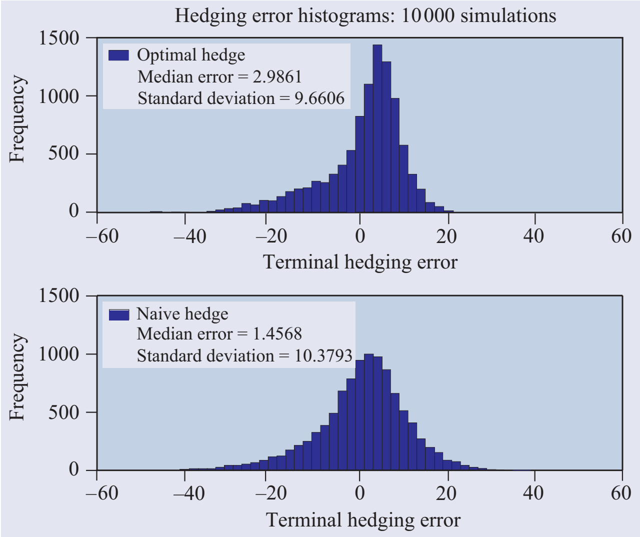

Figure 5. Histograms of terminal hedging error over 10 000 sample paths for the optimal hedging strategy (upper graph) and the naive strategy (lower graph). The parameters are as in table 2, and $\rho = 0.65$, $\gamma = 0.01$.

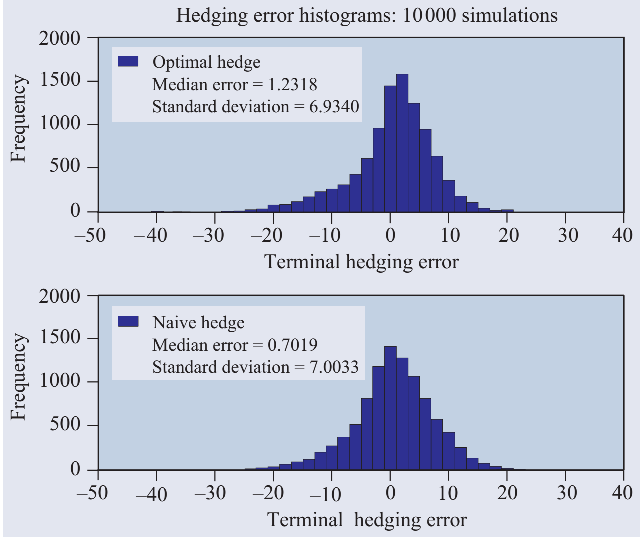

Figure 6. Histograms of terminal hedging error over 10 000 sample paths for the optimal hedging strategy (upper graph) and the naive strategy (lower graph). The parameters are as in table 2, and $\rho = 0.85$, $\gamma = 0.01$.

Table 5. Hedging error statistics for the histograms in figure 5.

	Max	Min	Mean	SD	Median
Optimal hedge	28.28	−47.46	0.5155	9.6606	2.9861
Naive hedge	40.13	−57.04	0.4808	10.3793	1.4568

Table 6. Hedging error statistics for the histograms in figure 6.

	Max	Min	Mean	SD	Median
Optimal hedge	24.70	−34.17	0.3879	6.9340	1.2318
Naive hedge	28.53	−35.94	0.5183	7.0033	0.7019

Table 7. Stocks comprising the non-traded basket.

Abbey National	British Airports Authority	BAE Systems
British Gas	Boots PLC	British Telecom
Shell	Tesco	Vodafone

carrying out an exhaustive empirical testing procedure, but our preliminary results indicate that the method shows promise. An in-depth empirical evaluation of optimal strategies is planned for future papers, and for different applications, such as weather derivatives. This may well require a modification of the theoretical framework, involving a departure from the log-normal diffusion assumption for the asset processes.

We obtained daily (closing price) data from 1 January 1990 to 30 August 2003, on the closest to maturity futures contract on the FTSE100 index, and on nine stocks (listed in table 7) used to construct an equally weighted basket. Overnight interbank rates were obtained for the same period. All data were obtained from Datastream.

Let $(F_t)_{0 \leqslant t \leqslant T}$ denote the futures price process, and assume this follows

$$\mathrm{d}F_t = \mu_{\mathrm{F}} F_t \,\mathrm{d}t + \sigma F_t \,\mathrm{d}w_t, \tag{57}$$

with μ_{F}, σ constants. To adapt the hedging technology developed earlier to the case where the traded asset is a futures contract, we note that if we hold Δ_t futures contracts plus cash at time $t \in [0, T]$, then since it costs nothing to enter a futures contract the wealth process X_t follows

$$\begin{aligned} \mathrm{d}X_t &= \Delta_t \,\mathrm{d}F_t + r X_t \,\mathrm{d}t \\ &= r X_t \,\mathrm{d}t + \pi_t (\mu_{\mathrm{F}} \,\mathrm{d}t + \sigma \,\mathrm{d}w_t) \\ &= r X_t \,\mathrm{d}t + \pi_t ((\mu - r) \,\mathrm{d}t + \sigma \,\mathrm{d}w_t), \end{aligned} \tag{58}$$

where $\mu = \mu_{\mathrm{F}} + r$ and $\pi_t = \Delta_t F_t$. We observe that (58) is of the same form as (5). This means we can use the formulae developed earlier provided we simply add the interest rate to our estimate of the futures price growth rate and use this as an estimate of the parameter μ in all our formulae. (The conscientious reader can confirm this by going through the derivation from first principles. Derive the position needed in the index itself, taking account of the dividend yield on the index, then adjust the required position in the index to a position in futures contracts.)

Consider a put option on the basket, written on 3 September 2002 (time 0) and maturing on 29 March 2003 (time T). We estimate the parameters $\mu_{\mathrm{F}}, \sigma, \mu_0, \sigma_0, \rho$ from the logarithmic returns of a selected time period ending at time 0 (e.g. the previous six months). Extending the time period used to estimate the parameters is, in principle, desirable. However, one must then take into consideration the possibility of structural breaks and other potential deviations from the geometric Brownian motion hypothesis, so we leave this analysis to future papers. Our main concern here is to show how the hedging programs would be applied over a real data set and to compare the optimal and naive hedges.

The parameters used to price and hedge the option are given in table 8 along with the selected values of the strike K and risk aversion γ. We also show (for comparison) the estimates of the price process parameters obtained from the actual price paths that were subsequently realized over

Table 8. Empirical parameters used to value and hedge a put option on a basket of stocks from September 2002 to March 2003. The parameter $\mu = \mu_F + r$, where μ_F is the futures price growth rate. Figures in parentheses indicate the values of parameters estimated from the actual price paths that subsequently ensued over the option's life.

F_0	Y_0	K	r	μ_F	σ	μ_0	σ_0	ρ	γ
4197	369.3	300	0.396	−0.415 (−0.235)	0.325 (0.361)	−0.446 (−0.549)	0.309 (0.347)	0.927 (0.922)	0.01

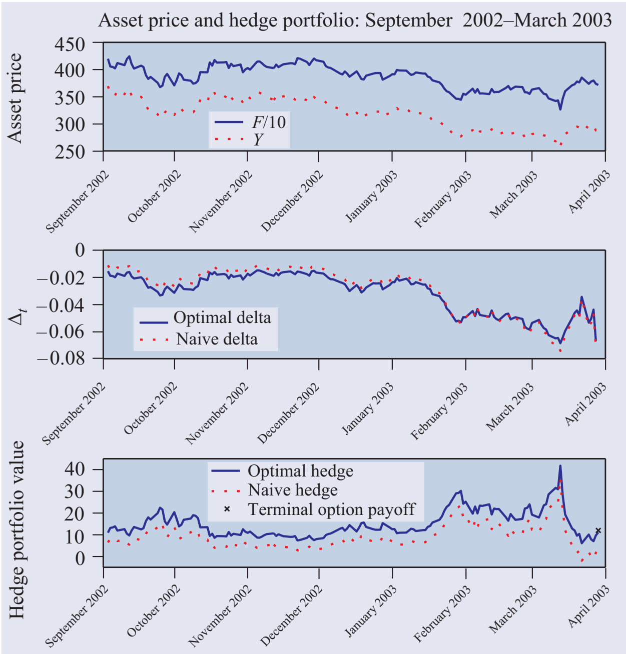

Figure 7. Asset price paths and hedge portfolio from September 2002 to March 2003, using both optimal and naive hedges. The parameters are as in table 8.

the option life. These turn out to be broadly in line with our parameter estimates from the six months prior to the option being written. The interest rate given is the average overnight rate during the option life $[0, T]$. We value and hedge the option using these parameters, assuming daily portfolio rebalancing, and compute the hedge portfolio over the *real* asset price paths that subsequently ensued over $[0, T]$. The terminal hedging error for both the optimal and naive hedging programs is then computed.

Figure 7 shows the futures price (scaled down by a factor of 10) and the basket price paths over the option life, along with the hedge ratios and hedge portfolios over these paths. The terminal hedging errors are −0.87 for the optimal hedge and −8.74 for the naive hedging method, so that over the particular data path used the optimal method did indeed perform better than the naive method. Of course, a natural topic for future research is to repeat these calculations over many real segments of price data, and to compute some suitably normalized hedging error, whose distribution can then be computed, in a manner analogous to that used for the simulated paths in the previous section. This topic will be the subject of future investigations.

7. Conclusions

Using a nonlinear expectation representation for the asking price of a claim on a non-traded asset we have derived analytic perturbation expansions for the price and hedging strategy of the claim. These formulae were used to show how optimal risk management, arising from the embedding of the pricing problem in a utility maximization framework, gives marked improvement in hedging performance over naive policies which use a traded asset as a proxy for the non-traded one. This improvement was measured by computing the distribution of terminal hedging error, and noting the increased frequency of profits over losses, as measured by the median hedging error.

The tests initiated here could be carried out using different risk measures and utility functions, as it would be interesting to see what sort of hedging strategies offer the greatest improvement. The issue of formalizing appropriate metrics to measure risk management performance enters the fray here, and there are presumably links with the coherent measures of risk in [1].

In general, the computation of hedging error distributions is a task that has not received much attention, despite being a natural way to assess the merits of a risk management program. Most studies have simply taken a 'snapshot' of the hedging error over a limited number of scenarios [12]. The application of the methods advocated here to other incomplete markets scenarios, such as stochastic volatility models, is certainly feasible and desirable.

It would also be interesting to add features such as transaction costs to the model analysed in this paper. If one could develop suitable analytic formulae for prices and hedging strategies, along the lines of [21], then it would become feasible to determine which market imperfection (basis risk or transaction costs) is the most severe, in terms of the hedging errors that must be tolerated.

Acknowledgment

Thanks to M H A Davis for suggestions and many helpful comments.

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