COMPUTING THE DISTORTION FUNCTION

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Abstract. A crib sheet which shows how to compute a function of the kind which arises in a distortion solution [2, 3] of an optimal investment problem with an OU MPR, involving the solution of a Riccati ODE and two further ODEs.

Contents

1. The function

Let Y be an OU process:

$$\mathrm{d}Y_t = a(b - Y_t)\,\mathrm{d}t + c\,\mathrm{d}W_t.$$

The function we are interested in is given by

$$F(t,y) := \mathbb{E}\left[\exp\left(-\frac{1}{2}q\int_t^T Y_u^2\,\mathrm{d}u\right)\bigg| Y_t = y\right]. \tag{1.1}$$

The PDE satisfied by $F(\cdot,\cdot)$ is

$$F_t + a(b-y)F_y + \frac{1}{2}c^2F_{yy} - \frac{1}{2}qy^2F = 0, \quad F(T,y) = 1.$$

1.1. Closed form for $F(\cdot,\cdot)$.

One can obtain a closed form for $F(\cdot,\cdot)$ by making the ansatz

$$F(t,y) = \exp\left(-(A(t)y^2 + B(t)y + C(t))\right),$$

for deterministic functions $A(\cdot), B(\cdot), C(\cdot)$. Using this form in the PDE for $F(\cdot,\cdot)$ yields ODEs satisfied by $A(\cdot), B(\cdot), C(\cdot)$:

$$\begin{aligned}
\dot{A}(t) &= 2c^2A^2(t) + 2aA(t) - \frac{1}{2}q, \quad A(T) = 0,\\
\dot{B}(t) &= \left(a + 2c^2A(t)\right)B(t) - 2abA(t), \quad B(T) = 0,\\
\dot{C}(t) &= \frac{1}{2}c^2(B^2(t) - 2A(t)) - abB(t), \quad C(T) = 0.
\end{aligned}$$

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1.1.1. *ODE for $A(\cdot)$.* The ODE for $A(\cdot)$ is a Riccati ODE, solved as follows. Define the function

$$\varphi(t) := \exp\left(2c^2 \int_t^T A(u)\,\mathrm{d}u\right), \quad t \in [0,T].$$

We have $\dot{\varphi}(t) = -2c^2 A(t)\varphi(t)$, so that

$$A(t) = -\frac{1}{2c^2}\frac{\dot{\varphi}(t)}{\varphi(t)}. \tag{1.2}$$

The defintion of $\varphi(\cdot)$ and the terminal condition on $A(\cdot)$ give terminal conditions on $\varphi(\cdot)$ of $\varphi(T) = 1$ and $\dot{\varphi}(T) = 0$.

The first order non-linear ODE for $A(\cdot)$ thus translates into the following second order liner ODE for $\varphi(\cdot)$:

$$\ddot{\varphi} - 2a\dot{\varphi} - qc^2\varphi = 0, \quad \varphi(T) = 1, \quad \dot{\varphi}(T) = 0.$$

The auxiliary equation

$$m^2 - 2am - qc^2 = 0$$

has roots

$$m_\pm = a \pm \gamma, \quad \gamma := \sqrt{a^2 + qc^2}$$

and using the terminal conditions on $\varphi(\cdot)$ we obtain

$$\varphi(t) = \frac{\mathrm{e}^{-a(T-t)}}{2\gamma}\left((\gamma - a)\mathrm{e}^{-\gamma(T-t)} + (\gamma + a)\mathrm{e}^{\gamma(T-t)}\right),$$

which we can re-write as

$$\varphi(t) = \frac{\mathrm{e}^{-a(T-t)}}{\gamma}\left(\gamma\cosh(\gamma(T-t)) + a\sinh(\gamma(T-t))\right) =: \mathrm{e}^{-a(T-t)}\frac{\psi\left(\gamma(T-t)\right)}{\gamma},$$

where, for notational brevity, we have defined the function $\psi(\cdot)$ by

$$\psi(\tau) := \gamma\cosh(\tau) + a\sinh(\tau).$$

Using (1.2) we compute that

$$A(t) = \frac{q\sinh(\gamma(T-t))}{2\psi(\gamma(T-t))}, \quad t \in [0,T].$$

1.1.2. *ODE for $B(\cdot)$.* Multiply the ODE for $B(\cdot)$ by the integrating factor

$$I(t) := \exp\left(\int_t^T (a + 2c^2 A(u))\,\mathrm{d}u\right) = \mathrm{e}^{a(T-t)}\varphi(t) = \frac{\psi(\gamma(T-t))}{\gamma}, \quad t \in [0,T].$$

Noting that

$$\dot{I}(t) = -(a + 2c^2 A(t))I(t).$$

The ODE for $B(\cdot)$ becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(B(t)I(t)\right) = -2abA(t)I(t) = -\frac{qab}{\gamma}\sinh(\gamma(T-t)),$$

or, equivalently,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(B(t)\psi(\gamma(T-t))\right) = -qab\sinh(\gamma(T-t)).$$

Integrating over $[t,T]$ and using the terminal condition for $B(\cdot)$ we arrive at

$$B(t) = \frac{qab}{\gamma}\left(\frac{\cosh(\gamma(T-t)) - 1}{\psi(\gamma(T-t))}\right), \quad t \in [0,T].$$

1.1.3. *ODE for $C(\cdot)$.* To solve for $C(\cdot)$ we integrate the ODE directly over $[t,T]$ and use the terminal condition for $C(\cdot)$, to give

$$C(t) = c^2 \int_t^T A(u)\,\mathrm{d}u + ab \int_t^T B(u)\,\mathrm{d}u - \frac{1}{2}c^2 \int_t^T B^2(u)\,\mathrm{d}u, \quad t \in [0,T].$$

So we need to integrate each of $A(\cdot)$, $B(\cdot)$ and $B^2(\cdot)$.

For the integral involving $A(\cdot)$, we have

$$c^2 \int_t^T A(u)\,\mathrm{d}u = \frac{1}{2}\log\varphi(t) = \frac{1}{2}\left(\log\left(\frac{\psi(\gamma(T-t))}{\gamma}\right) - a(T-t)\right), \quad t \in [0,T].$$

For the integral involving $B(\cdot)$, we have

$$ab \int_t^T B(u)\,\mathrm{d}u = \frac{qa^2b^2}{\gamma} \int_t^T \frac{\cosh(\gamma(T-u)) - 1}{\psi(\gamma(T-u))}\,\mathrm{d}u.$$

Now we use some standard integrals. Using (A.1) and (A.2) we obtain

$$ab \int_t^T B(u)\,\mathrm{d}u = \left(\frac{ab}{\gamma c}\right)^2 \left(\gamma(T-t) - a\log\left(\frac{\psi(\gamma(T-t))}{\gamma}\right) - c\sqrt{q}I(t)\right), \quad t \in [0,T],$$

where

$$I(t) = \left(\tan^{-1}\left(\frac{\psi'(\gamma(T-t))}{c\sqrt{q}}\right) - \tan^{-1}\left(\frac{a}{c\sqrt{q}}\right)\right).$$

For the integral involving $B^2(\cdot)$, we have

$$\frac{1}{2}c^2 \int_t^T B^2(u)\,\mathrm{d}u = \frac{1}{2}\left(\frac{qabc}{\gamma}\right)^2 \int_t^T \frac{\cosh^2(\gamma(T-u)) - 2\cosh(\gamma(T-u)) + 1}{\psi^2(\gamma(T-u))}\,\mathrm{d}u.$$

Using (A.3), (A.4) and (A.5) we obtain, after some work,

$$\begin{aligned}\frac{1}{2}c^2 \int_t^T B^2(u)\,\mathrm{d}u &= \frac{1}{2}\left(\frac{ab}{\gamma c}\right)^2 \left\{\left(\frac{\gamma^2+a^2}{\gamma}\right)\gamma(T-t) - 2c\sqrt{q}I(t) - 2a\log\left(\frac{\psi(\gamma(T-t))}{\gamma}\right)\right. \\ &- \left.\frac{2aqc^2}{\gamma}\left(\frac{1}{\psi(\gamma(T-t))} - \frac{1}{\gamma}\right) + \frac{qc^2}{\gamma^2}(qc^2 - a^2)\frac{\sinh(\gamma(T-t))}{\psi(\gamma(T-t))}\right\}.\end{aligned}$$

Thus, we can compute $C(\cdot)$, and after much computation we obtain

$$\begin{aligned}C(t) &= \frac{1}{2}\left(\log\left(\frac{\psi(\gamma(T-t))}{\gamma}\right) - a(T-t)\right) - \frac{1}{2}\left(\frac{ab}{\gamma c}\right)^2 \left[(\gamma^2 + a^2 - 2\gamma)(T-t)\right. \\ &- \left.\frac{2aqc^2}{\gamma}\left(\frac{1}{\psi(\gamma(T-t))} - \frac{1}{\gamma}\right) + \frac{qc^2}{\gamma^2}(qc^2 - a^2)\frac{\sinh(\gamma(T-t))}{\psi(\gamma(T-t))}\right].\end{aligned}$$

Appendix A. Some standard integrals

Let

$$\phi(x) := b\cosh(x) + a\sinh(x), \quad b > a > 0,$$

so that $\phi'(x) = b\sinh(x)(x) + a\cosh(x)$ and $\phi''(x) = \phi(x)$. We then have

$$\int \frac{1}{\phi(x)}\,\mathrm{d}x = \frac{1}{\sqrt{b^2-a^2}}\tan^{-1}\left(\frac{\phi'(x)}{\sqrt{b^2-a^2}}\right). \tag{A.1}$$

Gradshteyn and Ryzhik (GR) [1] state (A.1) in the form

$$\int \frac{1}{\phi(x)}\,\mathrm{d}x = \frac{1}{\sqrt{b^2-a^2}}\tan^{-1}\left|\sinh\left(x + \tanh^{-1}\left(\frac{a}{b}\right)\right)\right|, \quad b > |a|,$$

and it takes a bit of work to get my nicer form above. Wolfram Alpha (WA) state the result in an equally convoluted fashion, as

$$\int \frac{1}{\phi(x)}\,\mathrm{d}x = \frac{2}{\sqrt{b^2-a^2}}\tan^{-1}\left(\frac{a+b\tanh(x/2)}{\sqrt{b^2-a^2}}\right),$$

which presumably can be converted to my nicer form (but I have not managed this at the time of writing). Nevertheless we can use the fact that these two forms must be the same and must match my form, to simplify other convoluted forms generated by (say) WA.

In a similar vein, we also have

$$\int \frac{\cosh(x)}{\phi(x)}\,\mathrm{d}x = \frac{bx - a\log\phi(x)}{b^2-a^2}. \tag{A.2}$$

Gradshteyn and Ryzhik state this in the form

$$\int \frac{\cosh(x)}{\phi(x)}\,\mathrm{d}x = \frac{1}{b^2-a^2}\left[bx - a\log\cosh\left(x+\tanh^{-1}\left(\frac{a}{b}\right)\right)\right], \quad b>|a|,$$

and it takes a bit of work to get my nicer form above.

Here are some other integrals in a similar spirit:

$$\int \frac{\sinh(x)}{\phi(x)}\,\mathrm{d}x = \frac{b\log\phi(x) - ax}{b^2-a^2},$$

$$\int \frac{1}{\phi^2(x)}\,\mathrm{d}x = \frac{\sinh(x)}{b\phi(x)}, \tag{A.3}$$

$$\int \frac{\cosh(x)}{\phi^2(x)}\,\mathrm{d}x = \frac{1}{b^2-a^2}\left(\frac{b}{\sqrt{b^2-a^2}}\tan^{-1}\left(\frac{\phi'(x)}{\sqrt{b^2-a^2}}\right)+\frac{a}{\phi(x)}\right), \tag{A.4}$$

$$\int \frac{\cosh^2(x)}{\phi^2(x)}\,\mathrm{d}x = \frac{1}{(b^2-a^2)^2}\left((a^2+b^2)x - \frac{a^2(b^2-a^2)\sinh(x)}{b\phi(x)} - 2ab\log\phi(x)\right). \tag{A.5}$$

I got the last two results by carefully manipulating the extremely convoluted forms given by Wolfram Alpha, utilising the known equivalence between some of the terms in the WA and GR forms.

Though we won't need them here, we list some results in the case $a>b>0$:

$$\int \frac{1}{\phi(x)}\,\mathrm{d}x = \frac{1}{\sqrt{a^2-b^2}}\left(\log(\phi(x)) - \log\left(\phi'(x)+\sqrt{a^2-b^2}\right)\right), \tag{A.6}$$

$$\int \frac{\cosh(x)}{\phi(x)}\,\mathrm{d}x = \frac{a\log\phi(x) - bx}{a^2-b^2},$$

$$\int \frac{\sinh(x)}{\phi(x)}\,\mathrm{d}x = \frac{ax - b\log\phi(x)}{a^2-b^2},$$

$$\int \frac{\cosh^2(x)}{\phi^2(x)}\,\mathrm{d}x = \frac{1}{(a^2-b^2)^2}\left((a^2+b^2)x + \frac{a^2}{b}(a^2-b^2)\frac{\sinh(x)}{\phi(x)} - 2ab\log(\phi(x))\right).$$

Gradshteyn and Ryzhik state (A.6) in the form

$$\int \frac{1}{\phi(x)}\,\mathrm{d}x = \frac{1}{\sqrt{a^2-b^2}}\log\left|\tanh\left(\frac{x+\tanh^{-1}(b/a)}{2}\right)\right|,$$

and it takes a bit of work to get my nicer form above.

Wolfram Alpha state these results in equally convoluted ways.

References

[1] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products*, Elsevier/Academic Press, Amsterdam, eighth ed., 2015. Translated from the Russian, Translation edited and with a preface by Daniel Zwillinger and Victor Moll, Revised from the seventh edition [MR2360010].

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[3] T. Zariphopoulou, *A solution approach to valuation with unhedgeable risks*, Finance Stoch., 5 (2001), pp. 61–82.

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