

INFINITE HORIZON UTILITY MAXIMISATION FROM INTER-TEMPORAL WEALTH

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ABSTRACT. We develop a duality theory for the problem of maximising expected lifetime utility from inter-temporal wealth over an infinite horizon, under the minimal no-arbitrage assumption of No Unbounded Profit with Bounded Risk (NUPBR). We use only deflators, with no arguments involving equivalent martingale measures, so do not require the stronger condition of No Free Lunch with Vanishing Risk (NFLVR). Our formalism also works without alteration for the finite horizon version of the problem. As well as extending work of Bouchard and Pham [2] to any horizon and to a weaker no-arbitrage setting, we obtain a stronger duality statement, because we do not assume by definition that the dual domain is the polar set of the primal space. Instead, we adopt a method akin to that used for inter-temporal consumption problems, developing a supermartingale property of the deflated wealth and its path that yields an infinite horizon budget constraint and serves to define the correct dual variables. The structure of our dual space allows us to show that it is convex, without forcing this property by assumption. We proceed to enlarge the primal and dual domains to confer solidity to them, and use supermartingale convergence results which exploit Fatou convergence, to establish that the enlarged dual domain is the bipolar of the original dual space. The resulting duality theorem shows that all the classical tenets of convex duality hold. Moreover, at the optimum, the deflated wealth process is a potential converging to zero. We work out examples, including a case with a stock whose market price of risk is a three-dimensional Bessel process, so satisfying NUPBR but not NFLVR.

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1. INTRODUCTION

Let $U : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a classical utility function and $(X_t)_{t \geq 0}$ a non-negative wealth process generated from self-financing investment in a semimartingale incomplete market on a complete stochastic basis $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t \in [0, \infty)}, \mathbb{P})$, with the filtration \mathbb{F} satisfying the usual hypotheses of right-continuity and augmentation with \mathbb{P} -null sets of \mathcal{F} . Under the minimal no-arbitrage assumption of No Unbounded Profit with Bounded Risk (NUPBR), we develop a duality theory for a problem in which utility is derived from inter-temporal wealth over the infinite horizon:

$$(1.1) \quad \mathbb{E} \left[\int_0^\infty U(X_t) d\kappa_t \right] \rightarrow \max!$$

In (1.1), $\kappa : [0, \infty) \rightarrow \mathbb{R}_+$ is a non-decreasing càdlàg adapted process that will act as a finite measure to assign a weight to utility of wealth at each time. We focus on the infinite horizon case, but our approach also works without alteration for the finite horizon version of (1.1), as we re-iterate in Remark 3.2.

Problems of the type in (1.1) can arise when traditional utility of terminal wealth problems have a random horizon date, as we shall illustrate by some examples in Section 2.2.1, but can just as well be considered in their own right as one possible objective for a long-lived investment fund. A duality theory for such problems was developed by Bouchard and Pham [2] over a finite horizon, with a no-arbitrage assumption that allowed for the existence of equivalent local martingale measures (ELMMs), so tantamount to assuming No Free Lunch with Vanishing Risk (NFLVR) in the terminology of Delbaen and Schachermayer [6]. Here, the underlying assumptions as well as the approach and construction of the dual space are different to those in [2], as we now describe.

First, as indicated above, we relax the no-arbitrage assumption from NFLVR to NUPBR, so we do not rely on the existence of ELMMs, only on the existence of a class of deflators that multiply admissible wealth processes to create supermartingales. It was first made explicit by Karatzas and Kardaras [15] (though was implicit in the terminal wealth problem of Karatzas et al [17] in an incomplete Itô process market, in which which ELMMs were not invoked at all) that all one needs for well-posed utility maximisation problems is the existence of a suitable class of deflators to act as dual variables. In particular, ELMMs are not needed. This is a first reason for adopting NUPBR as our no-arbitrage condition.

Aside from weakening the no-arbitrage assumption, there are other sound reasons for avoiding the use of ELMMs. It is well known that ELMMs will typically not exist over the infinite horizon, because the candidate change of measure density process is not a uniformly integrable martingale. This is the case for the Black-Scholes model for example, as discussed in Karatzas and Shreve [18, Section 1.7]. Moreover, even if ELMMs might exist when restricted to a finite horizon, one needs to proceed with some care in invoking them in an infinite horizon model, by ensuring that events in the tail σ -algebra $\mathcal{F}_\infty := \sigma \left(\bigcup_{t \geq 0} \mathcal{F}_t \right)$ have been excluded in a consistent way. We discuss this issue further in Section 2.1.1. Irrespective of such subtleties, since deflators are the key ingredient for establishing a duality for utility maximisation problems, it is natural to construct a theory which uses only deflators, and makes no use whatsoever of constructions involving ELMMs, and this is what we do. A key step in this approach will be the use of the Stricker and Yan [33] version of the Optional Decomposition Theorem (ODT) to establish bipolarity relations between the primal and dual domains, as opposed to variants of the ODT which state the result in terms of ELMMs.

Second, our approach to establishing the duality between the primal problem in (1.1) and an appropriately defined dual problem differs quite markedly from that in Bouchard and Pham [2], and our basic duality statement is strengthened compared to that in [2], in essence because

we are able to prove, as opposed to assume by definition, that the dual domain is the polar of the primal domain, as we now describe.

The approach taken in [2], over a finite horizon time $[0, T]$, is to define the dual domain (in the case where the initial value of the dual variables is unity) as the set of processes Y such that $\mathbb{E} \left[\int_0^T X_t Y_t d\kappa_t \right] \leq 1$ for all admissible wealth processes with unit initial capital. In other words, the dual domain was explicitly defined in [2] as the polar of the primal domain. This automatically renders the dual domain convex and closed, so bypasses some steps in establishing bipolarity relations between the primal and dual spaces, and hence the duality theorem, but at the expense of weakening the final statement to some degree. This is also the reason that the bulk of the remaining analysis in [2] takes place in the primal domain.

In our method, by contrast, we find the form of the dual problem and the associated dual domain by seeking a supermartingale property satisfied by the pair $(X_t, (X_s)_{0 \leq s \leq t})_{t \geq 0}$, that is, the value of an admissible wealth process at any time, as well as the wealth path up to that time, as follows. Let S be any classical supermartingale deflator, so XS is a supermartingale for all admissible wealth processes, and let β be a non-negative process such that $\int_0^\cdot \beta_s d\kappa_s$ is almost surely finite. We define associated supermartingales R and processes Y by

$$(1.2) \quad R := \exp \left(- \int_0^\cdot \beta_s d\kappa_s \right) S, \quad Y := \beta R = \beta \exp \left(- \int_0^\cdot \beta_s d\kappa_s \right) S.$$

With these processes in place, we show that $M := XR + \int_0^\cdot X_s Y_s d\kappa_s$ is a supermartingale for all admissible wealth processes. The wealth-path deflators Y are then the appropriate dual variables for the problem in (1.1). They involve the auxiliary dual control β above and beyond that implicit in the choice of supermartingale deflator, a typical feature of wealth path dependent utility maximisation problems.

This program yields an infinite horizon budget constraint satisfied by the wealth path, similar to that in Bouchard and Pham [2], over our infinite horizon: $\mathbb{E} \left[\int_0^\infty X_t Y_t d\kappa_t \right] \leq 1$ for all admissible wealth processes with unit initial capital and all deflators with unit initial value. The budget constraint so formed acts (at this point) as a necessary condition for admissible wealth processes and serves to define the appropriate dual variables Y . The form of the dual problem then emerges as

$$\mathbb{E} \left[\int_0^\infty V(Y_t) d\kappa_t \right] \rightarrow \min!$$

over deflators with initial value $Y_0 = y > 0$, where $V : \mathbb{R}_+ \rightarrow \mathbb{R}$ is the convex conjugate of the utility function.

The particular structure in (1.2) of the inter-temporal wealth deflators, involving the supermartingale deflators and the auxiliary dual control β , is crucial, as it allows us to show that the dual space which emerges is convex. We then enlarge the primal domain to encompass processes dominated by admissible wealths (similar in spirit to the procedure used by Kramkov and Schachermayer [21, 22] for the terminal wealth utility maximisation problem), and show that the budget constraint is also a sufficient condition for admissible primal processes, using the Stricker and Yan [33] version of the Optional Decomposition Theorem. Finally, we enlarge the dual domain in a similar manner, to encompass processes dominated by the deflators, show that the resulting dual domain is closed in an appropriate topology (that of convergence in measure $\mu := \kappa \times \mathbb{P}$) by exploiting Fatou convergence of supermartingales, and obtain perfect bipolarity relations between the enlarged primal and dual domains. This bipolarity underlies the subsequent duality results.

We thus prove (as opposed to impose, by definition) that our dual domain has the required convexity and closedness properties needed to establish bipolarity and hence duality, with a supermartingale constraint involving the admissible wealths as a starting point. Put another way, the procedure developed by Kramkov and Schachermayer [21, 22] for the terminal wealth

problem is adapted and made to work for an inter-temporal wealth problem under NUPBR and over the infinite (or indeed, finite) horizon.

The main duality result (Theorem 3.1) shows that all the tenets of the theory hold in our scenario: the marginal utility of optimal wealth is equal to the optimal deflator with initial value equal to the derivative of the primal value function, and the primal and dual value functions are mutually conjugate. Moreover, at the optimum, the supermartingale M becomes a uniformly integrable martingale \widehat{M} , leading to an interesting additional representation of the optimal wealth process:

$$(1.3) \quad \widehat{X}_t \widehat{R}_t = \mathbb{E} \left[\int_t^\infty \widehat{X}_s \widehat{Y}_s d\kappa_s \middle| \mathcal{F}_t \right], \quad t \geq 0,$$

where $\widehat{X}, \widehat{R}, \widehat{Y}$ are the optimal manifestations of the processes X, R, Y . The supermartingale $\widehat{X}\widehat{R}$ becomes a potential (satisfying $\lim_{t \rightarrow \infty} \mathbb{E}[\widehat{X}_t \widehat{R}_t] = 0$) at the optimum, and also converges almost surely to $\widehat{X}_\infty \widehat{R}_\infty = 0$.

The philosophy of our approach can thus be summarised as: the use of a natural supermartingale property to derive an inter-temporal budget constraint as a means of identifying the dual space, then suitably enlarging that space and using supermartingale convergence techniques to get the bipolarity relations. A similar philosophy was applied recently by Monoyios [26] to the infinite horizon optimal consumption problem under NUPBR. There, using an appropriate class of consumption deflators (differing from the wealth-path deflators used here), deflated wealth plus cumulative deflated consumption at the optimum becomes a uniformly integrable martingale, while deflated wealth becomes a potential converging to zero. The final results in [26] thus have a similar flavour to those here (as they must, since classical tenets of duality theory are shown to hold in both cases), but the problem studied in [26] is quite distinct from the one here, involving a different primal variable (consumption, as opposed to the wealth path), and a different set of concomitant deflators. It turns out that the primal domain in [26] is $L^1(\mu)$ -bounded, while here it is the dual space that has this property. We thus do not use any results from [26] here, as we need to formulate a separate, complete proof.

Inter-temporal utility maximisation problems have a long history, usually treating the problem of maximising utility of consumption, as opposed to the less standard objective here, which involves aggregate expected utility from inter-temporal wealth, in the absence of a consumption stream. The consumption literature begins with Merton's [25] dynamic programming solution of the problem in a constant coefficient complete Brownian model, extended to cover issues such as non-negativity constraints on consumption, and bankruptcy, by Karatzas et al [16] using similar methods. The infinite horizon problem for utility from consumption in a complete Itô market was treated via duality methods by Huang and Pagès [13], while Foldes [9, 10] characterised optimal consumption plans in semimartingale markets with well-defined "shadow prices" (local martingales that characterise marginal utility of consumption processes). These correspond to some class of deflators in the modern mathematical finance terminology. The set-up of [9, 10] is very much rooted in traditional economic phraseology, so certain contemporary notions such as the underlying no-arbitrage condition, and the completeness or otherwise of the market, are not entirely transparent. Karatzas and Žitković [19] and Žitković [36] treated problems of optimal consumption with an additional random endowment, in incomplete semimartingale markets on a finite horizon and under the classical no-arbitrage assumption of the existence of ELMMs, equivalent to NFLVR. The consumption problem in an infinite horizon semimartingale incomplete market under NFLVR was treated by Mostovyi [27], and later by Chau et al [4] under NUPBR, with the recent treatment in [26] establishing the duality without recourse to any arguments involving ELMMs.

In contrast to the consumption problems analysed by these papers, the problem studied here, of maximising utility from inter-temporal wealth, has received much less attention. As

indicated earlier, a dual theory for such problems, over a finite horizon and under NFLVR, was developed by Bouchard and Pham [2]. Aside from the duality developed in [2], wealth-path-dependent utility maximisation problems have arisen in models which consider investment and consumption with a random horizon, such as Blanchet-Scalliet et al [1] (in complete Brownian markets with deterministic parameters), or Vellekoop and Davis [35] (who consider a Merton-type problem of optimal consumption in a Black-Scholes model, but with randomly terminating income). Federico et al [8] analyse wealth-path-dependent problems from the dynamic programming and Hamilton-Jacobi-Bellman (HJB) equation viewpoint, using viscosity solution methods to establish regularity of the value functions in Markovian market scenarios driven by Brownian motions. As well as arising from a random termination date, the problem studied here can also be viewed in its own right as describing an objective for a long-lived investment fund, looking to build, as opposed to consume, wealth.

The rest of the paper is structured as follows. In Section 2 we describe the financial market, introduce various classes of deflators and the primal problem, list some examples which fit into our set-up, then derive the budget constraint and formulate the dual problem. In Section 3 we give the main duality theorem (Theorem 3.1), and describe how the result may be re-cast in the case when κ is absolutely continuous with respect to Lebesgue measure (Remark 3.3). In Section 4 we formulate the primal and dual problems in abstract notation on a finite measure space with product measure $\mu := \kappa \times \mathbb{P}$. We re-cast the optimisation problems over suitably enlarged primal and dual domains, and present the bipolarity relations between these spaces (Proposition 4.4) as well as the abstract version of the duality theorem (Theorem 4.5). In Section 5 we prove Proposition 4.4. In many respects this is the heart of the paper. We use the Stricker and Yan [33] optional decomposition results to show that the budget constraint is also a sufficient condition for primal admissibility, then show that the dual domain we have constructed is convex and closed, and make comparisons with the approach of Bouchard and Pham [2]. In Section 6 we prove the abstract duality theorem in the classical manner of Kramkov and Schachermayer [21, 22], from which the concrete duality theorem follows, and also prove the novel representation (1.3) of the optimal wealth process (Proposition 6.13). In Section 7 we work out two examples with power and logarithmic utility: a model whose market price of risk is a three-dimensional Bessel process (so satisfying NUPBR but not NFLVR) with stochastic volatility and correlation, and a Black-Scholes market.

2. FINANCIAL MARKET AND PROBLEM FORMULATION

2.1. The financial market. We have an infinite horizon financial market on a complete stochastic basis $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t \in [0, \infty)}, \mathbb{P})$, with the filtration \mathbb{F} satisfying the usual hypotheses of right-continuity and augmentation with \mathbb{P} -null sets of \mathcal{F} . The market contains d stocks and a cash asset, the latter with strictly positive price process. We shall use the cash asset as numéraire, so without loss of generality (as we shall affirm in Remark 2.1) its price is normalised to unity and we work with discounted quantities throughout. The (discounted) price processes of the stocks are given by a non-negative càdlàg vector semimartingale $P = (P^1, \dots, P^d)$.

The σ -algebra \mathcal{F} can contain more information than that generated by the asset prices, so can include, for example, a random time at which investment ceases, as this is one scenario where inter-temporal wealth utility maximisation can arise. Bouchard and Pham [2] had a similar feature in a finite horizon version of our utility maximisation problem under NFLVR. Note that our formalism and results can be transferred with no alteration to the finite horizon setting, as we re-iterate in Remark 3.2.

A financial agent can trade a self-financing portfolio of the stocks and cash. The agent has initial capital $x > 0$, with the trading strategy represented by a d -dimensional predictable P -integrable process $H = (H^1, \dots, H^d)$, with H^i , $i = 1, \dots, d$ the process for the number of

shares of the i^{th} stock in the portfolio. The agent's wealth process X is given by

$$X_t := x + (H \cdot P)_t, \quad t \geq 0, \quad x > 0,$$

where $(H \cdot P) := \int_0^\cdot H_s dP_s$ denotes the stochastic integral. Let $\mathcal{X}(x)$ denote the set of non-negative wealth processes with initial wealth $x > 0$:

$$\mathcal{X}(x) := \{X : X = x + (H \cdot P) \geq 0, \text{ almost surely}\}, \quad x > 0.$$

We write $\mathcal{X} \equiv \mathcal{X}(1)$ and we have $\mathcal{X}(x) = x\mathcal{X} = \{xX : X \in \mathcal{X}\}$ for $x > 0$. The set \mathcal{X} is a convex (and hence so is $\mathcal{X}(x)$, $x > 0$).

For $y > 0$, let $\mathcal{S}(y)$ denote the set of *supermartingale deflators* (SMDs), positive càdlàg processes S with $S_0 = y$ such that the deflated wealth SX is a supermartingale for all $X \in \mathcal{X}$:

$$(2.1) \quad \mathcal{S}(y) := \{S > 0, \text{ càdlàg}, S_0 = y : SX \text{ is a supermartingale for all } X \in \mathcal{X}\}.$$

We write $\mathcal{S} \equiv \mathcal{S}(1)$, and we have $\mathcal{S}(y) = y\mathcal{S}$ for $y > 0$. The set \mathcal{S} is clearly convex. Since the constant process $X \equiv 1$ lies in \mathcal{X} , each $S \in \mathcal{S}$ is a supermartingale. The supermartingale deflators are the processes used as dual variables by Kramkov and Schachermayer [21, 22] in their treatment of the terminal wealth utility maximisation problem. The dual domain for the forthcoming inter-temporal wealth problem will be based on $\mathcal{S}(y)$ but will not coincide with this space, as we shall see shortly.

Let \mathcal{Z} denote the set of *local martingale deflators* (LMDs), positive càdlàg local martingales Z with unit initial value such that deflated wealth XZ is a local martingale for all $X \in \mathcal{X}$:

$$(2.2) \quad \mathcal{Z} := \{Z > 0, \text{ càdlàg}, Z_0 = 1 : XZ \text{ is a local martingale for all } X \in \mathcal{X}\}.$$

Since the local martingale $XZ \geq 0$ for all $X \in \mathcal{X}$, it is also a supermartingale and, since $X \equiv 1$ lies in \mathcal{X} , each $Z \in \mathcal{Z}$ is also a supermartingale, and we have the inclusion

$$(2.3) \quad \mathcal{S} \supseteq \mathcal{Z}.$$

The set \mathcal{Z} is convex, and contains the density processes of equivalent local martingale measures (ELMMs) in situations where such measures would exist. A feature of our approach is that we shall not be using any constructions involving ELMMs, even restricted to a finite horizon, as we discuss further below in Section 2.1.1.

The standing no-arbitrage assumption we shall make is that the set of supermartingale deflators is non-empty:

$$(2.4) \quad \mathcal{S}(y) \neq \emptyset.$$

The condition (2.4) is equivalent to the condition of no unbounded profit with bounded risk (NUPBR) (also referred to as no arbitrage of the first kind, NA_1), weaker than the no free lunch with vanishing risk (NFLVR) condition, the latter being equivalent to the existence of equivalent local martingale measures (ELMMs), as established by Delbaen and Schachermayer [6] for the case of a locally bounded semimartingale stock price process. There are various characterisations of NUPBR, including that the set \mathcal{Z} of LMDs is non-empty: see Karatzas and Kardaras [15], Kardaras [20], Takaoka and Schweizer [34] and Chau et al [4], as well as the recent overview by Kabanov, Kardaras and Song [14].

2.1.1. Completion of the stochastic basis and equivalent measures. As indicated earlier, we shall not use equivalent local martingale measures (ELMMs), even restricted to a finite horizon. This is partly for aesthetic reasons: since we work under NUPBR and assume only the existence of various classes of deflators, which is the minimal requirement for well posed utility maximisation problems, it is natural to seek proofs which use only deflators.

There is also a mathematical rationale for avoiding ELMMs. We are working on an infinite horizon and have assumed the usual conditions. Thus, each element of the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ includes all the \mathbb{P} -null sets of $\mathcal{F} := \sigma(\bigcup_{t \geq 0} \mathcal{F}_t) =: \mathcal{F}_\infty$, the tail σ -algebra. So,

ultimate events (as time $t \uparrow \infty$) of \mathbb{P} -measure zero are included in any finite time σ -field \mathcal{F}_T , $T < \infty$.

It is well-known that in such a scenario many financial models will not admit an equivalent martingale measure over the infinite horizon, because the candidate change of measure density is not a uniformly integrable martingale. (This is true of the Black-Scholes model, see Karatzas and Shreve [18, Section 1.7].) One then has to proceed with caution when invoking arguments which utilise equivalent measures, by finding a consistent way to eliminate the tail σ -algebra from the picture when restricting to a finite horizon $T < \infty$.

One possible way forward is to not complete the space. This route was taken by Huang and Pagès [13] in an infinite horizon consumption model in a complete Brownian market. This is sound, though care is needed to ensure that no results are used which require the usual hypotheses to hold.

Another way to proceed, if one wishes to consider equivalent measures restricted to a finite horizon $T < \infty$, is to augment the space with null events of a σ -field generated over a finite horizon at least as big as T , that is by $\sigma\left(\bigcup_{0 \leq t \leq T'} \mathcal{F}_t\right)$, for some $0 \leq T \leq T' < \infty$. This can be done in a consistent way, and relies on an application of Carathéodory's extension theorem (Rogers and Williams [31, Theorem II.5.1]). One can then obtain equivalent measures in an infinite horizon model when restricting such measures to any finite horizon. This procedure is carried out in a Brownian filtration in Karatzas and Shreve [18, Section 1.7], with a cautionary example [18, Example 1.7.6], showing that augmenting the σ -field generated by Brownian motion over any finite horizon with null sets of the corresponding tail σ -algebra would render invalid the construction of equivalent measures, even over a finite horizon.

The message is that one has to be careful in using any constructions involving equivalent measures, even restricted to a finite horizon, when working in an infinite horizon financial model.

We avoid any such pitfalls, since we avoid ELMs entirely. In particular, in Section 5 we establish bipolarity results between the primal and dual domains using only the Stricker and Yan [33] version of the optional decomposition theorem, relying on deflators rather than martingale measures.

We mention this issue because many papers appear to use a complete stochastic basis on an infinite horizon, and at the same time then use equivalent measures over a finite or infinite horizon, without any statement about the elimination of the tail σ -field. This applies to some proofs in papers tackling the infinite horizon consumption problem (see Mostovyi [27, Lemma 4.2] and Chau et al [4, Lemma 1]). In a similar vein, some celebrated papers working on an infinite horizon, such as the seminal connection between ELMs and NFLVR of Delbaen and Schachermayer [6], and the optional decomposition result of Kramkov [23], invoke ELMs over an infinite horizon, without seeming to address the issue that these will not exist over a perpetual timeframe in even the simplest Brownian model such as the Black-Scholes model, and that care must sometimes be taken to eliminate the tail σ -algebra if invoking ELMs (even restricted to a finite horizon) in an infinite horizon model.

We would suggest that it was taken as implicit in the papers cited above that, when necessary, the tail σ -algebra was eliminated in a consistent way when invoking arguments involving ELMs. But it should be said that no such qualifying statements were made. We conjecture that all the arguments in these and other papers where such potential inconsistencies may arise can be rendered sound by amendments as described above. This is an issue for possible future investigation, though fortunately not one we need to address, as we bypass all these problems by arguments which avoid the use of ELMs entirely.

2.2. The primal problem. Let $U : [0, \infty) \rightarrow \mathbb{R}$ denote the agent's utility function, assumed to be strictly increasing, strictly concave, continuously differentiable and satisfying the Inada

conditions

$$(2.5) \quad U'(0) := \lim_{x \downarrow 0} U'(x) = +\infty, \quad U'(\infty) := \lim_{x \rightarrow \infty} U'(x) = 0.$$

Let $\kappa : [0, \infty) \rightarrow \mathbb{R}_+$ be a non-negative, non-decreasing càdlàg adapted process, which will act as a finite measure that will discount utility from inter-temporal wealth. We assume that κ satisfies

$$\kappa_0 = 0, \quad \mathbb{P}[\kappa_\infty > 0] > 0, \quad \kappa_\infty \leq K,$$

for some finite constant K , so that $\mathbb{E} \left[\int_0^\infty d\kappa_t \right]$ is bounded.

The agent's primal problem is to maximise expected utility from inter-temporal wealth over the infinite horizon. The primal value function $u(\cdot)$ is defined by

$$(2.6) \quad u(x) := \sup_{X \in \mathcal{X}(x)} \mathbb{E} \left[\int_0^\infty U(X_t) d\kappa_t \right], \quad x > 0.$$

To exclude a trivial problem, we shall assume throughout that the primal value function satisfies

$$u(x) > -\infty, \quad \forall x > 0.$$

This is a mild condition, which can be guaranteed by assuming that for all wealth processes $X \in \mathcal{X}(x)$ we have $\mathbb{E} \left[\int_0^\infty \min(0, U(X_t)) d\kappa_t \right] > -\infty$.

Remark 2.1 (Discounted units). There is no loss of generality in working with discounted quantities (so in effect a zero interest rate). To see this, suppose instead that we have a positive interest rate process $r = (r_t)_{t \geq 0}$, so the cash asset with initial value 1 has positive price process $A_t = e^{\int_0^t r_s ds}$, $t \geq 0$. If \tilde{X} is the un-discounted wealth process, then the problem in (2.6) is $\mathbb{E} \left[\int_0^\infty U \left(\tilde{X}_t / A_t \right) d\kappa_t \right] \rightarrow \max!$ We can define another utility function $\tilde{U} : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ such that $\tilde{U}(A_t, \tilde{X}_t) = U(\tilde{X}_t / A_t)$, $t \geq 0$, and the problem in (2.6) can then be transported to one in terms of the raw (un-discounted) wealth process. For example, if $U(\cdot) = \log(\cdot)$ is logarithmic utility, we choose $\tilde{U}(A, \tilde{X}) = \log(\tilde{X}) - \log(A)$. If $U(x) = x^p / p$, $p < 1$, $p \neq 0$ is power utility, then we choose $\tilde{U}(A, \tilde{X}) = A^{-p} \tilde{X}^p / p$.

Remark 2.2 (Stochastic utility). In the problem (2.6) we can allow $U(\cdot)$ to be stochastic, so to also depend on $\omega \in \Omega$ in an optional way. The analysis is unaffected, as the reader can easily verify, so one can read the proofs with a stochastic utility in mind and with dependence on $\omega \in \Omega$ suppressed throughout.

2.2.1. *Some examples.* We list here some examples of inter-temporal wealth utility maximisation problems, illustrating how the measure κ manifests itself in various cases. Further examples can be found in Bouchard and Pham [2, Section 2].

Example 2.3 (Perpetual wealth-path-dependent utility maximisation). Take $d\kappa_t = e^{-\alpha t} dt$ for some positive discount rate $\alpha > 0$, and an infinite horizon, so the objective is

$$(2.7) \quad \mathbb{E} \left[\int_0^\infty \exp(-\alpha t) U(X_t) dt \right] \rightarrow \max!$$

This is the quintessential example we have in mind as our central problem, and can be thought of as an objective of a long-lived investment fund building up wealth. We shall treat this example under power and logarithmic utility in Section 7, to illustrate the application of the duality theorem of the paper, with different market environments: an incomplete market with a stock with whose market price of risk is a three-dimensional Bessel process (so will satisfy NUPBR but not NFLVR) and which has a stochastic volatility, and a Black-Scholes (thus, complete) market.

There are no esoteric ingredients in (2.7) such as a random termination time which generates the wealth-path-dependent objective, but such modifications can be added. Indeed, suppose

we have a random horizon given by $T \sim \text{Exp}(\lambda)$, an exponentially distributed time with parameter $\lambda > 0$, independent of the stock price filtration. The objective can be re-cast with an additional integral over the probability density function of T , so we have

$$\mathbb{E} \left[\int_0^T \exp(-\alpha t) U(X_t) dt \right] = \mathbb{E} \left[\int_0^\infty \lambda \exp(-\lambda s) \int_0^s \exp(-\alpha t) U(X_t) dt ds \right].$$

Integration by parts allows the objective to be re-written as

$$\mathbb{E} \left[\int_0^\infty \exp(-(\alpha + \lambda)t) U(X_t) dt \right] \rightarrow \max!$$

so we recover a problem of the same type as in (2.7) with a modified discount factor.

Example 2.4 (Utility of terminal wealth at a random horizon). The other classical example which yields an inter-temporal wealth objective is where we maximise expected utility of terminal wealth $\mathbb{E}[U(X_T)]$ at some random horizon T , an almost surely finite \mathcal{F} -measurable non-negative random variable.

For instance, let $T \sim \text{Exp}(\lambda)$ be an exponentially distributed random time with parameter $\lambda > 0$, independent of the stock price filtration. As in Example 2.3, we re-write the objective with an integral over the probability density function of T , so we have

$$\mathbb{E} [U(X_T)] = \mathbb{E} \left[\int_0^\infty \lambda \exp(-\lambda t) U(X_t) dt \right],$$

which again yields a problem of the type in Example 2.3. We observe that κ is given by

$$\kappa_t = \int_0^t \lambda \exp(-\lambda s) ds = 1 - \exp(-\lambda t) = \mathbb{P}[T \leq t], \quad t \geq 0.$$

The obvious generalisation is to a general random time T which is independent of the asset price filtration. In this case one has $\kappa_t = \mathbb{P}[T \leq t]$, $t \geq 0$ in the inter-temporal wealth problem (2.6).

In the case where T is a stopping time we have $\kappa_t = \mathbb{1}_{\{t \geq T\}}$, $t \geq 0$, and this includes the case where T is deterministic, so there is no time horizon uncertainty, and we revert to the classical terminal wealth problem.

Further similar examples are given in Bouchard and Pham [2, Examples 1–3], adapted to the case of a finite horizon for the overall problem in (2.6).

2.3. The budget constraint. Our approach to establishing the form of the dual to the primal utility maximisation problem (2.6) is to determine an appropriate supermartingale constraint satisfied by the pair $(X_t, (X_s)_{0 \leq s \leq t})_{t \geq 0}$, that is, the value of an admissible wealth process at any time as well as the wealth path up to that point. This gives an infinite horizon budget constraint on the wealth path. Using a supermartingale constraint in this way is analogous to the procedure followed in consumption problems, where one considers the wealth process at any time as well the consumption plan up to that time.

Let \mathcal{B} denote the set of all non-negative càdlàg adapted processes β satisfying $\int_0^t \beta_s d\kappa_s < \infty$ almost surely for all $t \geq 0$:

$$(2.8) \quad \mathcal{B} := \left\{ \beta \geq 0 : \text{càdlàg, adapted, such that } \int_0^\cdot \beta_s d\kappa_s < \infty \text{ almost surely} \right\}.$$

The processes in \mathcal{B} will act as an additional dual control, above and beyond that implied in the classical supermartingale deflators, as we shall see in due course.

For any $\beta \in \mathcal{B}$ and for any supermartingale deflator $S \in \mathcal{S}(y)$, define a process R by

$$(2.9) \quad R_t := \exp \left(- \int_0^t \beta_s d\kappa_s \right) S_t, \quad t \geq 0, \quad \beta \in \mathcal{B}, S \in \mathcal{S}(y), \quad y > 0.$$

Denote the set of such processes with initial value $y > 0$ by $\mathcal{R}(y)$:

$$(2.10) \quad \mathcal{R}(y) := \{R : R \text{ is defined by (2.9)}\}, \quad y > 0.$$

We write $\mathcal{R} \equiv \mathcal{R}(1)$ and we have $\mathcal{R}(y) = y\mathcal{R}$ for $y > 0$. We shall prove in Section 5.2 that the set \mathcal{R} is convex (see Lemma 5.5), which will lead to the corresponding property for the dual domain to the primal problem (2.6), to be defined shortly.

Since $\beta \in \mathcal{B}$ is almost surely non-negative, the supermartingale property of the deflated wealth SX in (2.1) also holds for RX , for any $R \in \mathcal{R}(y)$, so we have the inclusion $\mathcal{S}(y) \supseteq \mathcal{R}(y)$, $y > 0$, and each $R \in \mathcal{R}(y)$ is also a supermartingale.

For each $R \in \mathcal{R}(y)$ and for the same $\beta \in \mathcal{B}$ appearing in the definition (2.9), define a process Y by

$$(2.11) \quad Y_t := \beta_t R_t = \beta_t \exp\left(-\int_0^t \beta_s d\kappa_s\right) S_t, \quad t \geq 0, \quad \beta \in \mathcal{B}, S \in \mathcal{S}(y), \quad y > 0.$$

Denote the set of such processes by $\mathcal{Y}(y)$:

$$(2.12) \quad \mathcal{Y}(y) := \{Y : Y \text{ is defined by (2.11)}\}, \quad y > 0.$$

The set $\mathcal{Y}(y)$ will form the domain of the dual problem to the inter-temporal wealth problem (2.6), as we shall see shortly. We shall refer to processes $Y \in \mathcal{Y}(y)$ as *wealth-path deflators* or inter-temporal wealth deflators or, simply, as deflators, when no confusion can arise. We write $\mathcal{Y} \equiv \mathcal{Y}(1)$, with $\mathcal{Y}(y) = y\mathcal{Y}$ for $y > 0$. The set \mathcal{Y} turns out to be convex, as we shall show in Section 5.2. This is an important ingredient in our approach to establishing certain bipolarity relations between the primal and dual domains, which underlie the duality results of the paper. As we shall see, the convexity of \mathcal{Y} will stem from the particular structure of the dual variables as given in (2.11). This structure seems to have eluded some previous studies of inter-temporal wealth utility maximisation problems. We shall say more on this structure and make comparisons with the approach of Bouchard and Pham [2] in Section 5.3, after we prove the bipolarity relations.

The following lemma gives the supermartingale constraint and the resultant infinite horizon budget constraint on admissible wealth processes, which will lead to the form of the dual problem.

Lemma 2.5 (Supermartingale and budget constraints). *Let $\beta \in \mathcal{B}$ be any non-negative càdlàg adapted process satisfying $\int_0^t \beta_s d\kappa_s < \infty$ almost surely for all $t \geq 0$. Define the processes $R \in \mathcal{R}(y)$ and the wealth-path deflators $Y \in \mathcal{Y}(y)$ by (2.9) and (2.11), respectively. We then have that*

$$(2.13) \quad M := RX + \int_0^\cdot X_s Y_s d\kappa_s \quad \text{is a supermartingale.}$$

As a consequence, we have the infinite horizon budget constraint

$$(2.14) \quad \mathbb{E} \left[\int_0^\infty X_t Y_t d\kappa_t \right] \leq xy, \quad x, y > 0, \quad \forall X \in \mathcal{X}(x), Y \in \mathcal{Y}(y).$$

Proof. For $x, y > 0$ let $X \in \mathcal{X}(x)$ be an admissible wealth process and let $S \in \mathcal{S}(y)$ be any supermartingale deflator. The Itô product rule applied to $XR = XS \exp(-\int_0^\cdot \beta_s d\kappa_s)$ gives

$$(2.15) \quad M_t := X_t R_t + \int_0^t X_s Y_s d\kappa_s = xy + \int_0^t \exp\left(-\int_0^s \beta_u d\kappa_u\right) d(X_s S_s), \quad t \geq 0,$$

where we have used the definition (2.11) of the inter-temporal wealth deflators. Since XS is a supermartingale, it has a Doob-Meyer decomposition $XS = xy + L - A$ for some local martingale L and a non-decreasing process A , with $L_0 = A_0 = 0$. Using this Doob-Meyer decomposition, the integral on the right-hand-side of (2.15) is also seen to be a supermartingale, so we obtain the supermartingale property of $M := XR + \int_0^\cdot X_s Y_s d\kappa_s$ as stated in the lemma.

The supermartingale property gives

$$\mathbb{E} \left[X_t R_t + \int_0^t X_s Y_s d\kappa_s \right] \leq xy, \quad t \geq 0.$$

Since XR is non-negative, we thus also have

$$\mathbb{E} \left[\int_0^t X_s Y_s d\kappa_s \right] \leq xy, \quad t \geq 0.$$

Letting $t \uparrow \infty$ and using monotone convergence we obtain the infinite horizon budget constraint (2.14). □

Note that for $\beta \equiv 0$ the supermartingale property in (2.13) is simply the statement that XS is a supermartingale for all $X \in \mathcal{X}$ and $S \in \mathcal{S}$. This is the basic sense in which we are extending the starting point of the methodology of Kramkov and Schachermayer [21, 22] towards duality: begin with a supermartingale constraint to build a budget constraint. The presence of the supermartingales $S \in \mathcal{S}, R \in \mathcal{R}$ in these arguments will ultimately be exploited to invoke supermartingale convergence results involving Fatou convergence of processes, in proving that an abstract dual domain \mathcal{D} (an enlargement of the domain \mathcal{Y} to encompass processes dominated by some $Y \in \mathcal{Y}$) is closed (see Lemmata 5.6 and 5.9).

2.4. The dual problem. Let $V : \mathbb{R}_+ \rightarrow \mathbb{R}$ denote the convex conjugate of the utility function, defined by

$$V(y) := \sup_{x>0} [U(x) - xy], \quad y > 0.$$

The map $y \mapsto V(y)$, $y > 0$, is strictly convex, strictly decreasing, continuously differentiable on \mathbb{R}_+ , $-V(\cdot)$ satisfies the Inada conditions, and we have the bi-dual relation

$$U(x) := \inf_{y>0} [V(y) + xy], \quad x > 0,$$

as well as $V'(\cdot) = -I(\cdot) = -(U')^{-1}(\cdot)$, where $I(\cdot)$ denotes the inverse of marginal utility. In particular, we have the inequality

$$(2.16) \quad V(y) \geq U(x) - xy, \quad \forall x, y > 0, \quad \text{with equality iff } U'(x) = y.$$

From the budget constraint (2.14) we can motivate the form of the dual problem to (2.6) by bounding the achievable utility in the familiar way. For any $X \in \mathcal{X}(x)$ and $Y \in \mathcal{Y}(y)$ we have

$$\begin{aligned} \mathbb{E} \left[\int_0^\infty U(X_t) d\kappa_t \right] &\leq \mathbb{E} \left[\int_0^\infty U(X_t) d\kappa_t \right] + xy - \mathbb{E} \left[\int_0^\infty X_t Y_t d\kappa_t \right] \\ &= \mathbb{E} \left[\int_0^\infty (U(X_t) - X_t Y_t) d\kappa_t \right] + xy \\ (2.17) \quad &\leq \mathbb{E} \left[\int_0^\infty V(Y_t) d\kappa_t \right] + xy, \quad x, y > 0, \end{aligned}$$

the last inequality a consequence of (2.16). This motivates the definition of the dual problem associated with the primal problem (2.6), with dual value function $v : \mathbb{R}_+ \rightarrow \mathbb{R}$ defined by

$$(2.18) \quad v(y) := \inf_{Y \in \mathcal{Y}(y)} \mathbb{E} \left[\int_0^\infty V(Y_t) d\kappa_t \right], \quad y > 0.$$

We shall assume that the dual problem is finitely valued:

$$(2.19) \quad v(y) < \infty, \quad \text{for all } y > 0.$$

Remark 2.6 (Reasonable asymptotic elasticity). As is known from Kramkov and Schachermayer [22], (2.19) is a mild condition that will guarantee a well-posed primal problem. It is also well known that one can alternatively impose the reasonable asymptotic elasticity condition of Kramkov and Schachermayer [21] on the utility function:

$$(2.20) \quad \text{AE}(U) := \limsup_{x \rightarrow \infty} \frac{xU'(x)}{U(x)} < 1,$$

along with the assumption that $u(x) < \infty$ for some $x > 0$. Then, as in Kramkov and Schachermayer [22, Note 2] or Bouchard and Pham [2, Remark 5.1], these conditions can be shown to yield (2.19).

3. THE MAIN DUALITY

Here is the central duality statement of the paper.

Theorem 3.1 (Perpetual inter-temporal wealth duality under NUPBR). *Define the primal inter-temporal wealth utility maximisation problem by (2.6) and the corresponding dual problem by (2.18). Assume (2.4), (2.5) and that*

$$u(x) > -\infty, \forall x > 0, \quad v(y) < \infty, \forall y > 0.$$

Then:

- (i) $u(\cdot)$ and $v(\cdot)$ are conjugate:

$$v(y) = \sup_{x > 0} [u(x) - xy], \quad u(x) = \inf_{y > 0} [v(y) + xy], \quad x, y > 0.$$

- (ii) *The primal and dual optimisers $\widehat{X}(x) \in \mathcal{X}(x)$ and $\widehat{Y}(y) \in \mathcal{Y}(y)$ exist and are unique, so that*

$$u(x) = \mathbb{E} \left[\int_0^\infty U(\widehat{X}_t(x)) d\kappa_t \right], \quad v(y) = \mathbb{E} \left[\int_0^\infty V(\widehat{Y}_t(y)) d\kappa_t \right], \quad x, y > 0,$$

with $\widehat{Y}(y) = \widehat{\beta} \widehat{R}(y) = \widehat{\beta} \exp \left(- \int_0^\cdot \widehat{\beta}_s d\kappa_s \right) \widehat{S}(y)$, for an optimal $\widehat{\beta} \in \mathcal{B}$ and optimal supermartingales $\widehat{R}(y) \in \mathcal{R}(y)$ and $\widehat{S}(y) \in \mathcal{S}(y)$.

- (iii) *With $y = u'(x)$ (equivalently, $x = -v'(y)$), the primal and dual optimisers are related by*

$$U'(\widehat{X}_t(x)) = \widehat{Y}_t(y), \quad \text{equivalently,} \quad \widehat{X}_t(x) = -V'(\widehat{Y}_t(y)), \quad t \geq 0,$$

and satisfy

$$(3.1) \quad \mathbb{E} \left[\int_0^\infty \widehat{X}_t(x) \widehat{Y}_t(y) d\kappa_t \right] = xy.$$

Moreover, the associated optimal wealth process $\widehat{X}(x)$ satisfies

$$(3.2) \quad \widehat{X}_t(x) \widehat{R}_t(y) = \mathbb{E} \left[\int_t^\infty \widehat{X}_s(x) \widehat{Y}_s(y) d\kappa_s \middle| \mathcal{F}_t \right], \quad t \geq 0,$$

and the process $\widehat{M} := \widehat{X}(x) \widehat{R}(y) + \int_0^\cdot \widehat{X}_s(x) \widehat{Y}_s(y) d\kappa_s$ is a uniformly integrable martingale.

- (iv) *The functions $u(\cdot)$ and $-v(\cdot)$ are strictly increasing, strictly concave, satisfy the Inada conditions, and for all $x, y > 0$ their derivatives satisfy*

$$xu'(x) = \mathbb{E} \left[\int_0^\infty U'(\widehat{X}_t(x)) \widehat{X}_t(x) d\kappa_t \right], \quad yv'(y) = \mathbb{E} \left[\int_0^\infty V'(\widehat{Y}_t(y)) \widehat{Y}_t(y) d\kappa_t \right].$$

Remark 3.2 (The finite horizon case). As the analysis in the sequel will show, it is easy to verify that all our methodology works without alteration for the finite horizon version of (2.6), with some terminal time $T < \infty$. The budget constraint is altered to have an upper limit of T as are all the results of Theorem 3.1. We thus extend the problem studied in Bouchard and Pham [2] to the NUPBR scenario, in addition to the strengthening of the basic duality statement as described below, where we do not have to assume *a priori* that the dual domain is the polar of the primal domain.

The proof of Theorem 3.1 will be given in Section 6, and will rely on bipolarity results and an abstract version of the duality theorem in Section 4, with the bipolarity results proven in Section 5. Duality results akin to items (i)–(iii) of the theorem (but not the additional novel characterisation (3.2) of the optimal wealth process) were obtained by Bouchard and Pham [2] over a finite horizon and under NFLVR. Compared to [2], Theorem 3.1 makes a stronger statement in other ways. We describe this strengthening briefly here, and will give further details in Section 5.3 after we prove bipolarity relations between the primal and dual domains, as some of the features are directly concerned with such polarity results.

First, we strengthen the duality for inter-temporal wealth utility maximisation to the weaker no-arbitrage assumption of NUPBR, compared to the NFLVR assumption in [2]. Second, we avoid having to define the dual domain as the polar of the primal domain. As indicated in the Introduction, the dual domain in [2] was directly defined as the set of deflators for which a finite horizon version of the budget constraint holds. In the language of the polar of a set (defined in Section 4, see Definition 4.1) the dual space is set equal to the polar of the primal space, by definition. This automatically renders the dual domain convex and closed, but the statement of the duality result is then somewhat weaker, because one half of the perfect bipolarity between the primal and dual domains (as given in Proposition 4.4) has been achieved by definition.

In our approach, the dual space arises from the budget constraint (2.14), itself derived from the supermartingale property (2.13) of the process M . This renders the budget constraint a necessary condition for admissibility. On enlarging the primal domain to include processes dominated by some admissible wealth, we show in Lemma 5.2 that the budget constraint is also a sufficient condition for admissibility. This uses the Stricker and Yan [33] version of the optional decomposition theorem, avoiding martingale measures in favour of local martingale deflators. This equivalence between primal admissibility and the budget constraint establishes that the enlarged primal set \mathcal{C} is the polar of the dual space \mathcal{Y} .

We then show that our dual space is convex, relying on the particular structure of the wealth path deflators in (2.11). An enlargement of the dual domain (in a similar vein to the primal enlargement), combined with supermartingale convergence results which exploit Fatou convergence of processes, culminates in Lemma 5.6, which shows that the enlarged dual domain \mathcal{D} is closed (in an appropriate topology). This, along with convexity and solidity, yields that the enlarged dual domain \mathcal{D} is the bipolar of the original domain \mathcal{Y} . Thus gives us the perfect bipolarity we need between \mathcal{C} and \mathcal{D} .

The above procedure is in essence the Kramkov and Schachermayer [21, 22] program for bipolarity and duality, adapted to an inter-temporal wealth framework. We shall describe these features of the bipolarity derivations in more detail in Section 5.3, and compare the program to that of Bouchard and Pham [2], after we have proven the bipolarity relations.

Remark 3.3 (The case where $\kappa \ll \text{Leb}$). Theorem 3.1 holds true regardless of whether the measure κ admits a density with respect to Lebesgue measure. However, when $\kappa \ll \text{Leb}$ there is a natural change of variable which one would use in computations, as we shall see in the course of some examples in Section 7, so we highlight here how the Theorem 3.1 is slightly re-cast in that case. The scenario to keep in mind is the case where $d\kappa_t = e^{-\alpha t} dt$ for a positive impatience rate $\alpha > 0$.

In the definition (2.8) of the set \mathcal{B} , one replaces κ by Lebesgue measure. With an abuse of notation, to use the same symbol for this set of auxiliary dual controls, \mathcal{B} now denotes the set of non-negative càdlàg processes β such that $\int_0^\cdot \beta_s ds < \infty$ almost surely. With similar abuse of notation, the set $\mathcal{R}(y)$ is composed of processes $R := \exp(-\int_0^\cdot \beta_s ds) S$, for supermartingale deflators $S \in \mathcal{S}(y)$. The wealth-path deflators are then given by $Y := \beta R$, and once again we denote the set of such processes by $\mathcal{Y}(y)$. The supermartingale property (2.13) converts to the statement that the process $M := XR + \int_0^\cdot X_s Y_s ds$ is a supermartingale. The budget constraint (2.14) becomes $\mathbb{E}[\int_0^\infty X_t Y_t dt] \leq xy$.

With this notation, define the positive process $\gamma = (\gamma_t)_{t \geq 0}$ as the reciprocal of $(d\kappa_t/dt)_{t \geq 0}$:

$$\gamma_t := \left(\frac{d\kappa_t}{dt} \right)^{-1}, \quad t \geq 0.$$

The dual problem then takes the form

$$(3.3) \quad v(y) := \inf_{Y \in \mathcal{Y}(y)} \mathbb{E} \left[\int_0^\infty V(\gamma_t Y_t) d\kappa_t \right], \quad y > 0,$$

as can be confirmed by repeating the computation that led to (2.17) in this altered set-up.

With these changes, items (ii)–(iv) of Theorem 3.1 are altered to:

- (ii)' The primal and dual optimisers $\widehat{X}(x) \in \mathcal{X}(x)$ and $\widehat{Y}(y) \in \mathcal{Y}(y)$ exist and are unique, so that

$$u(x) = \mathbb{E} \left[\int_0^\infty U(\widehat{X}_t(x)) d\kappa_t \right], \quad v(y) = \mathbb{E} \left[\int_0^\infty V(\gamma_t \widehat{Y}_t(y)) d\kappa_t \right], \quad x, y > 0,$$

with $\widehat{Y}(y) = \widehat{\beta} \widehat{R}(y) = \widehat{\beta} \exp(-\int_0^\cdot \widehat{\beta}_s ds) \widehat{S}(y)$, for an optimal $\widehat{\beta} \in \mathcal{B}$ and optimal supermartingales $\widehat{R}(y) \in \mathcal{R}(y)$ and $\widehat{S}(y) \in \mathcal{S}(y)$.

- (iii)' With $y = u'(x)$ (equivalently, $x = -v'(y)$), the primal and dual optimisers are related by

$$(3.4) \quad U'(\widehat{X}_t(x)) = \gamma_t \widehat{Y}_t(y), \quad \text{equivalently,} \quad \widehat{X}_t(x) = -V'(\gamma_t \widehat{Y}_t(y)), \quad t \geq 0,$$

and satisfy

$$(3.5) \quad \mathbb{E} \left[\int_0^\infty \widehat{X}_t(x) \widehat{Y}_t(y) dt \right] = xy.$$

Moreover, the associated optimal wealth process $\widehat{X}(x)$ satisfies

$$(3.6) \quad \widehat{X}_t(x) \widehat{R}_t(y) = \mathbb{E} \left[\int_t^\infty \widehat{X}_s(x) \widehat{Y}_s(y) ds \middle| \mathcal{F}_t \right], \quad t \geq 0,$$

and the process $\widehat{M} := \widehat{X}(x) \widehat{R}(y) + \int_0^\cdot \widehat{X}_s(x) \widehat{Y}_s(y) ds$ is a uniformly integrable martingale.

- (iv)' The functions $u(\cdot)$ and $-v(\cdot)$ are strictly increasing, strictly concave, satisfy the Inada conditions, and for all $x, y > 0$ their derivatives satisfy

$$xu'(x) = \mathbb{E} \left[\int_0^\infty U'(\widehat{X}_t(x)) \widehat{X}_t(x) d\kappa_t \right], \quad yv'(y) = \mathbb{E} \left[\int_0^\infty V'(\gamma_t \widehat{Y}_t(y)) \widehat{Y}_t(y) dt \right].$$

4. ABSTRACT BIPOLARITY AND DUALITY

In this section we specify a finite measure space which allows us to write the primal and dual problems in abstract notation, over suitably enlarged primal and dual domains. We then state the bipolarity relations between the abstract primal and dual domains in Proposition 4.4, which forms the basis for the subsequent abstract duality of Theorem 4.5.

Set

$$\mathbf{\Omega} := [0, \infty) \times \Omega.$$

Let \mathcal{G} denote the optional σ -algebra on $\mathbf{\Omega}$, that is, the sub- σ -algebra of $\mathcal{B}([0, \infty)) \otimes \mathcal{F}$ generated by evanescent sets and stochastic intervals of the form $\llbracket T, \infty \rrbracket$ for arbitrary stopping times T . Define the measure

$$(4.1) \quad \mu := \kappa \times \mathbb{P}$$

on $(\mathbf{\Omega}, \mathcal{G})$. On the resulting finite measure space $(\mathbf{\Omega}, \mathcal{G}, \mu)$, denote by $L_+^0(\mu)$ the space of non-negative μ -measurable functions, corresponding to non-negative infinite horizon processes.

The primal and dual domains for our optimisation problems (2.6) and (2.18) are now considered as subsets of $L_+^0(\mu)$. The abstract primal and dual domains will be enlargements of $\mathcal{X}(x)$ and $\mathcal{Y}(y)$ to accommodate processes dominated by some element of the original domain in question.

The abstract primal domain is $\mathcal{C}(x)$, defined by

$$(4.2) \quad \mathcal{C}(x) := \{g \in L_+^0(\mu) : g \leq X, \mu\text{-a.e.}, \text{ for some } X \in \mathcal{X}(x)\}, \quad x > 0.$$

We write $\mathcal{C} \equiv \mathcal{C}(1)$, with $\mathcal{C}(x) = x\mathcal{C}$ for $x > 0$, and the set \mathcal{C} is convex. Since $U(\cdot)$ is increasing, the primal value function of (2.6) is now written in the abstract notation as an optimisation over $g \in \mathcal{C}(x)$:

$$(4.3) \quad u(x) := \sup_{g \in \mathcal{C}(x)} \int_{\mathbf{\Omega}} U(g) d\mu, \quad x > 0.$$

The abstract dual domain is obtained by a similar enlargement of the original dual domain. Define the set $\mathcal{D}(y)$ by

$$(4.4) \quad \mathcal{D}(y) := \{h \in L_+^0(\mu) : h \leq Y, \mu\text{-a.e.}, \text{ for some } Y \in \mathcal{Y}(y)\}, \quad y > 0.$$

We write $\mathcal{D} \equiv \mathcal{D}(1)$, we have $\mathcal{D}(y) = y\mathcal{D}$ for $y > 0$, and the set \mathcal{D} is convex, inheriting this property from \mathcal{Y} . This is a crucial feature, and relies on our demonstration of the convexity of \mathcal{Y} in Section 5.2 (see Lemma 5.5), which in turn relies on the inter-temporal wealth deflators having the particular structure in (2.11).

With this notation, and since $V(\cdot)$ is decreasing, the dual problem (2.18) takes the form

$$(4.5) \quad v(y) := \inf_{h \in \mathcal{D}(y)} \int_{\mathbf{\Omega}} V(h) d\mu, \quad y > 0.$$

4.1. Abstract bipolarity. The abstract duality theorem relies on the abstract bipolarity result in Proposition 4.4 below which connects the sets \mathcal{C} and \mathcal{D} . The result is of course in the spirit of Kramkov and Schachermayer [21, Proposition 3.1].

We shall sometimes employ the notation

$$(4.6) \quad \langle g, h \rangle := \int_{\mathbf{\Omega}} gh d\mu, \quad g, h \in L_+^0(\mu).$$

Let us recall the concepts of set solidity and the polar of a set.

Definition 4.1 (Solid set, closed set). A subset $A \subseteq L_+^0(\mu)$ is called *solid* if $f \in A$ and $0 \leq g \leq f$, μ -a.e. implies that $g \in A$.

A set is *closed in μ -measure*, or simply *closed*, if it is closed with respect to the topology of convergence in measure μ .

Definition 4.2 (Polar of a set). The *polar*, A° , of a set $A \subseteq L_+^0(\mu)$, is defined by

$$A^\circ := \{h \in L_+^0(\mu) : \langle g, h \rangle \leq 1, \text{ for each } g \in A\}.$$

For clarity and for later use, we state here the bipolar theorem of Brannath and Schachermayer [3, Theorem 1.3], originally proven in a probability space, and adapted here to the measure space $(\Omega, \mathcal{G}, \mu)$.

Theorem 4.3 (Bipolar theorem, Brannath and Schachermayer [3], Theorem 1.3). *On the finite measure space $(\Omega, \mathcal{G}, \mu)$:*

- (i) *For a set $A \subseteq L_+^0(\mu)$, its polar A° is a closed, convex, solid subset of $L_+^0(\mu)$.*
- (ii) *The bipolar $A^{\circ\circ}$, defined by*

$$A^{\circ\circ} := \{g \in L_+^0(\mu) : \langle g, h \rangle \leq 1, \text{ for each } h \in A^\circ\},$$

is the smallest closed, convex, solid set in $L_+^0(\mu)$ containing A .

Proposition 4.4 (Abstract bipolarity). *Under the condition (2.4), the abstract primal and dual sets \mathcal{C} and \mathcal{D} satisfy the following properties:*

- (i) *\mathcal{C} and \mathcal{D} are both closed with respect to convergence in measure μ , convex and solid;*
- (ii) *\mathcal{C} and \mathcal{D} satisfy the bipolarity relations*

$$(4.7) \quad g \in \mathcal{C} \iff \langle g, h \rangle \leq 1, \quad \forall h \in \mathcal{D}, \quad \text{that is, } \mathcal{C} = \mathcal{D}^\circ,$$

$$(4.8) \quad h \in \mathcal{D} \iff \langle g, h \rangle \leq 1, \quad \forall g \in \mathcal{C}, \quad \text{that is, } \mathcal{D} = \mathcal{C}^\circ;$$

- (iii) *\mathcal{C} and \mathcal{D} are bounded in $L^0(\mu)$, and \mathcal{D} is also bounded in $L^1(\mu)$.*

The proof of Proposition 4.4 will be given in Section 5, where we shall establish that the infinite horizon budget constraint is also a sufficient condition for admissibility, once the primal domain is enlarged to accommodate processes dominated by admissible wealths. This culminates in the full bipolarity relations once we enlarge dual domain in a similar manner. The derivations in Section 5 are quite distinct from previous approaches, and are the bedrock of the mathematical results. As indicated earlier, we shall establish the bipolarity results without any recourse whatsoever to constructions involving ELMMs, by exploiting ramifications of the Stricker and Yan [33] version of the optional decomposition theorem.

4.2. The abstract duality. Armed with the abstract bipolarity in Proposition 4.4, we have the following abstract version of the convex duality relations between the primal problem (4.3) and its dual (4.5). The theorem shows that all the natural tenets of utility maximisation theory, as established by Kramkov and Schachermayer [21, 22] in the terminal wealth problem under NFLVR, extend to the infinite horizon inter-temporal wealth problem under NUPBR, with weak underlying assumptions on the primal and dual domains.

Theorem 4.5 (Abstract duality theorem). *Define the primal value function $u(\cdot)$ by (4.3) and the dual value function by (4.5). Assume that the utility function satisfies the Inada conditions (2.5) and that*

$$(4.9) \quad u(x) > -\infty, \quad \forall x > 0, \quad v(y) < \infty, \quad \forall y > 0.$$

Then, with Proposition 4.4 in place, we have:

- (i) *$u(\cdot)$ and $v(\cdot)$ are conjugate:*

$$(4.10) \quad v(y) = \sup_{x>0} [u(x) - xy], \quad u(x) = \inf_{y>0} [v(y) + xy], \quad x, y > 0.$$

- (ii) *The primal and dual optimisers $\hat{g}(x) \in \mathcal{C}(x)$ and $\hat{h}(y) \in \mathcal{D}(y)$ exist and are unique, so that*

$$u(x) = \int_{\Omega} U(\hat{g}(x)) \, d\mu, \quad v(y) = \int_{\Omega} V(\hat{h}(y)) \, d\mu, \quad x, y > 0.$$

- (iii) With $y = u'(x)$ (equivalently, $x = -v'(y)$), the primal and dual optimisers are related by

$$U'(\widehat{g}(x)) = \widehat{h}(y), \quad \text{equivalently,} \quad \widehat{g}(x) = -V'(\widehat{h}(y)),$$

and satisfy

$$\langle \widehat{g}(x), \widehat{h}(y) \rangle = xy.$$

- (iv) $u(\cdot)$ and $-v(\cdot)$ are strictly increasing, strictly concave, satisfy the Inada conditions, and their derivatives satisfy

$$xu'(x) = \int_{\Omega} U'(\widehat{g}(x))\widehat{g}(x) \, d\mu, \quad yv'(y) = \int_{\Omega} V'(\widehat{h}(y))\widehat{h}(y) \, d\mu, \quad x, y > 0.$$

The proof of Theorem 4.5 will be given in Section 6, and uses as its starting point the bipolarity result in Proposition 4.4.

The duality proof itself follows some of the classical steps (with adaptations) of Kramkov and Schachermayer [21, 22]. For completeness and clarity we shall give a full, self-contained treatment.

5. BIPOLARITY RELATIONS

In this section we prove Proposition 4.4, which establishes in particular the bipolarity relations (4.7) and (4.8) between the enlarged primal and dual domains \mathcal{C} and \mathcal{D} in (4.2) and (4.4).

5.1. Sufficiency of the budget constraint. The budget constraint (2.14), as derived in Lemma 2.5, constitutes a necessary condition for admissible inter-temporal wealth processes. Setting $x = y = 1$ in (2.14), we thus have the implications

$$(5.1) \quad X \in \mathcal{X} \implies \mathbb{E} \left[\int_0^\infty X_t Y_t \, d\kappa_t \right] \leq 1, \quad \forall Y \in \mathcal{Y},$$

and

$$(5.2) \quad Y \in \mathcal{Y} \implies \mathbb{E} \left[\int_0^\infty X_t Y_t \, d\kappa_t \right] \leq 1, \quad \forall X \in \mathcal{X}.$$

We wish to establish the reverse implications in some form, if need be by enlarging the primal and dual domains.

Recall the enlarged primal domain $\mathcal{C} \equiv \mathcal{C}(1)$ in (4.2) of processes dominated by admissible wealths with initial capital 1. The budget constraint (2.14) clearly holds with $g \in \mathcal{C}$ in place of $X \in \mathcal{X}$, so the implication (5.1) extends from \mathcal{X} to \mathcal{C} :

$$(5.3) \quad g \in \mathcal{C} \implies \mathbb{E} \left[\int_0^\infty g_t Y_t \, d\kappa_t \right] \leq 1, \quad \forall Y \in \mathcal{Y}.$$

We establish the reverse implication to (5.3) in Lemma 5.2 below. This requires some version of the Optional Decomposition Theorem (ODT), originally formulated by El Karoui and Quenez [7] in a Brownian setting. This was generalised to markets with locally bounded semimartingale stock prices by Kramkov [23], extended to the non-locally bounded case by Föllmer and Kabanov [11], and to models with constraints by Föllmer and Kramkov [12]. The relevant version of the ODT for us is the one due to Stricker and Yan [33], which uses local martingale deflators, rather than ELMMs. We shall use a result from [33] which applies to the super-hedging of American claims, so is designed to construct a process which can super-replicate a payoff at an arbitrary time. The salient observation is that this result can also be used to dominate a process over all times, and this is how we shall employ it.

For clarity we state here the ODT results we need, and specify afterwards precisely which results from [33] we have taken.

For $t \geq 0$, let $\mathcal{T}(t)$ denote the set of \mathbb{F} -stopping times with values in $[t, \infty)$. For $t = 0$, write $\mathcal{T} \equiv \mathcal{T}(0)$, and recall the set \mathcal{Z} of local martingale deflators in (2.2).

Theorem 5.1 (Stricker and Yan [33] ODT). (i) *Let W be an adapted non-negative process. The process ZW is a supermartingale for each $Z \in \mathcal{Z}$ if and only if W admits a decomposition of the form*

$$W = W_0 + (\phi \cdot P) - A,$$

where ϕ is a predictable P -integrable process such that $Z(\phi \cdot P)$ is a local martingale for each $Z \in \mathcal{Z}$, A is an adapted increasing process with $A_0 = 0$, and for all $Z \in \mathcal{Z}$ and $T \in \mathcal{T}$, $\mathbb{E}[Z_T A_T] < \infty$. In this case, moreover, we have $\sup_{Z \in \mathcal{Z}, T \in \mathcal{T}} \mathbb{E}[Z_T A_T] \leq W_0$.

(ii) *Let $b = (b_t)_{t \geq 0}$ be a non-negative càdlàg process such that $\sup_{Z \in \mathcal{Z}, T \in \mathcal{T}} \mathbb{E}[Z_T b_T] < \infty$. Then there exists an adapted càdlàg process W that dominates b : $W_t \geq b_t$ almost surely for all $t \geq 0$, ZW is a supermartingale for each $Z \in \mathcal{Z}$, and the smallest such process W is given by*

$$(5.4) \quad W_t = \operatorname{ess\,sup}_{Z \in \mathcal{Z}, T \in \mathcal{T}(t)} \frac{1}{Z_t} \mathbb{E}[Z_T b_T | \mathcal{F}_t], \quad t \geq 0.$$

Part (i) of Theorem 5.1 is taken from [33, Theorem 2.1]. Part (ii) is a combination of [33, Lemma 2.4 and Remark 2].

The following lemma establishes the reverse implication to (5.3).

Lemma 5.2. *Suppose g is a non-negative càdlàg process satisfying*

$$(5.5) \quad \mathbb{E} \left[\int_0^\infty g_t Y_t \, d\kappa_t \right] \leq 1, \quad \forall Y \in \mathcal{Y}.$$

Then, $g \in \mathcal{C}$.

Proof. Since g is assumed to satisfy (5.5) for all $Y \in \mathcal{Y}$ and because we have the inclusion (2.3), we see that (5.5) is satisfied for $Y = \beta \exp(-\int_0^\cdot \beta_s \, d\kappa_s) Z$, for any non-negative càdlàg $\beta \in \mathcal{B}$ and for any local martingale deflator $Z \in \mathcal{Z}$.

Fix a stopping time $T \in \mathcal{T}$, and for each $n \in \mathbb{N}$ choose β according to

$$\beta_t = \frac{\mathbb{1}_{\{T \leq t < T+1/n\}}}{\kappa_{T+1/n} - \kappa_T}, \quad t \geq 0, \quad n \in \mathbb{N}.$$

Define the process $\nu^{(n)}$ by

$$\begin{aligned} \nu_t^{(n)} &:= \beta_t \exp \left(-C(\kappa_{T+1/n} - \kappa_T) \int_0^t \beta_s \, d\kappa_s \right) \\ &= \frac{\mathbb{1}_{\{T \leq t < T+1/n\}}}{\kappa_{T+1/n} - \kappa_T} \exp \left(-C \int_0^t \mathbb{1}_{\{T \leq s < T+1/n\}} \, d\kappa_s \right), \quad t \geq 0, \end{aligned}$$

for a constant $C > 0$ large enough to ensure that $C \int_0^t \mathbb{1}_{\{T \leq s < T+1/n\}} \, d\kappa_s \geq \int_0^t \beta_s \, d\kappa_s$, $t \geq 0$, so that $\nu^{(n)} \leq \beta \exp(-\int_0^\cdot \beta_s \, d\kappa_s)$ almost surely. We then have, for each $n \in \mathbb{N}$ and $Z \in \mathcal{Z}$,

$$\begin{aligned} 1 &\geq \mathbb{E} \left[\int_0^\infty g_t Y_t \, d\kappa_t \right] \\ &\geq \mathbb{E} \left[\int_0^\infty g_t \nu_t^{(n)} Z_t \, d\kappa_t \right] \\ &= \mathbb{E} \left[\frac{1}{\kappa_{T+1/n} - \kappa_T} \int_T^{T+1/n} g_t Z_t \exp(C(\kappa_t - \kappa_T)) \, d\kappa_t \right]. \end{aligned}$$

Letting $n \rightarrow \infty$, and using Fatou's lemma and the right-continuity of Zg , we obtain

$$\mathbb{E}[Z_T g_T] \leq 1, \quad \forall Z \in \mathcal{Z}, T \in \mathcal{T}.$$

Since $Z \in \mathcal{Z}$ and $T \in \mathcal{T}$ were arbitrary, we have

$$\sup_{Z \in \mathcal{Z}, T \in \mathcal{T}} \mathbb{E}[Z_T g_T] \leq 1 < \infty.$$

Thus, from part (ii) of the Stricker-Yan version of optional decomposition, Theorem 5.1, there exists a càdlàg process W that dominates g , so $W_t \geq g_t$, a.s., $\forall t \geq 0$, and ZW is a supermartingale for each $Z \in \mathcal{Z}$. From (5.4), the smallest such W given by

$$W_t = \operatorname{ess\,sup}_{Z \in \mathcal{Z}, T \in \mathcal{T}(t)} \frac{1}{Z_t} \mathbb{E}[Z_T g_T | \mathcal{F}_t], \quad t \geq 0,$$

so that $W_0 \leq 1$. Further, by part (i) of Theorem (5.1), there exists a predictable P -integrable process H and an adapted increasing process A , with $A_0 = 0$, such that W has decomposition $W = W_0 + (H \cdot P) - A$, with $Z(H \cdot P)$ a local martingale for each $Z \in \mathcal{Z}$, and $\mathbb{E}[Z_T A_T] < \infty$ for all $Z \in \mathcal{Z}$ and $T \in \mathcal{T}$.

Since W dominates g , we can define a process X by

$$X_t := 1 + (H \cdot P)_t, \quad t \geq 0,$$

which also dominates g , since its initial value is no smaller than W_0 and we have dispensed with the increasing process A . We observe that X corresponds to the value of a self-financing wealth process with initial capital 1 which dominates g , so that $g \in \mathcal{C}$. \square

We can now assemble consequences of the budget constraint and of Lemma 5.2 which, combined with the bipolar theorem, gives the following polarity properties of the set \mathcal{C} .

Lemma 5.3 (Polarity properties of \mathcal{C}). *The set $\mathcal{C} \equiv \mathcal{C}(1)$ of admissible wealth processes with initial capital $x = 1$ is a closed, convex and solid subset of $L_+^0(\mu)$. It is equal to the polar of the set $\mathcal{Y} \equiv \mathcal{Y}(1)$ of (2.12) with respect to measure μ :*

$$(5.6) \quad \mathcal{C} = \mathcal{Y}^\circ,$$

so that

$$(5.7) \quad \mathcal{C}^\circ = \mathcal{Y}^{\circ\circ},$$

and \mathcal{C} is equal to its bipolar:

$$(5.8) \quad \mathcal{C}^{\circ\circ} = \mathcal{C}.$$

Proof. Lemma 5.2, combined with the implication in (5.3), gives the equivalence

$$g \in \mathcal{C} \iff \mathbb{E} \left[\int_0^\infty g_t Y_t d\kappa_t \right] \leq 1, \quad \forall Y \in \mathcal{Y}.$$

Equivalently, in terms of the measure μ of (4.1), we have

$$(5.9) \quad g \in \mathcal{C} \iff \int_\Omega g Y d\mu \leq 1, \quad \forall Y \in \mathcal{Y}.$$

The characterisation (5.9) is the dual representation of \mathcal{C} :

$$\mathcal{C} = \{g \in L_+^0(\mu) : \langle g, Y \rangle \leq 1, \quad \text{for each } Y \in \mathcal{Y}\}.$$

This says that \mathcal{C} is the polar of \mathcal{Y} , establishing (5.6) and thus (5.7).

Part (i) of the bipolar theorem, Theorem 4.3, along with (5.6), imply that \mathcal{C} is a closed, convex and solid subset of $L_+^0(\mu)$ (since it is equal to the polar of a set) as claimed. Part (ii) of Theorem 4.3 gives $\mathcal{C}^{\circ\circ} \supseteq \mathcal{C}$ with $\mathcal{C}^{\circ\circ}$ the smallest closed, convex, solid set containing \mathcal{C} . But since \mathcal{C} is itself closed, convex and solid, we have (5.8).

□

Remark 5.4. There are other ways to obtain the closed, convex and solid properties of \mathcal{C} . First, the equivalence (5.9) along with Fatou's lemma yields that the set \mathcal{C} is closed with respect to the topology of convergence in measure μ . To see this, let $(g^n)_{n \in \mathbb{N}}$ be a sequence in \mathcal{C} which converges μ -a.e. to an element $g \in L_+^0(\mu)$. For arbitrary $Y \in \mathcal{Y}$ we obtain, via Fatou's lemma and the fact that $g^n \in \mathcal{C}$ for each $n \in \mathbb{N}$,

$$\int_{\Omega} gY \, d\mu \leq \liminf_{n \rightarrow \infty} \int_{\Omega} g^n Y \, d\mu \leq 1,$$

so by (5.9), $g \in \mathcal{C}$, and thus \mathcal{C} is closed. Further, it is straightforward to establish the convexity of \mathcal{C} (inherited from the convexity of \mathcal{X}) from its definition. Finally, solidity of \mathcal{C} is also clear: if one can dominate an element $g \in \mathcal{C}$ with a self-financing wealth process, then one can also dominate any smaller process with the same portfolio.

5.2. Convexity of the dual domain. We now turn to the dual side of the analysis. The first step is to establish convexity properties of the sets \mathcal{R} and \mathcal{Y} . Here, the particular structure of the dual variables in (2.9) and (2.11) comes into play.

Lemma 5.5. *The sets \mathcal{R} and \mathcal{Y} of (2.10) and (2.12) are convex.*

Proof. Take two elements $S^1, S^2 \in \mathcal{S}$ and two elements $\beta^1, \beta^2 \in \mathcal{B}$, and define $R^1, R^2 \in \mathcal{R}$ and $Y^1, Y^2 \in \mathcal{Y}$ by

$$R^i := \exp\left(-\int_0^\cdot \beta_s^i \, d\kappa_s\right) S^i, \quad Y^i := \beta^i R^i, \quad i = 1, 2.$$

For two constants $\lambda_1, \lambda_2 \geq 0$ such that $\lambda_1 + \lambda_2 = 1$, define the convex combinations

$$\bar{S} := \lambda_1 S^1 + \lambda_2 S^2, \quad \bar{R} := \lambda_1 R^1 + \lambda_2 R^2, \quad \bar{Y} := \lambda_1 Y^1 + \lambda_2 Y^2.$$

Observe that $\bar{S} \in \mathcal{S}$ because the set \mathcal{S} of supermartingale deflators is convex.

Since $\beta^i \geq 0$, $i = 1, 2$ and the set \mathcal{S} is convex, we have

$$\bar{R} \leq \lambda_1 S^1 + \lambda_2 S^2 = \bar{S} \in \mathcal{S}.$$

We can therefore define a non-negative process $\tilde{\beta} \in \mathcal{B}$ by the relation

$$(5.10) \quad \bar{R} = \exp\left(-\int_0^\cdot \tilde{\beta}_s \, d\kappa_s\right) \bar{S},$$

This shows that $\bar{R} \in \mathcal{R}$, so that \mathcal{R} is convex, as claimed.

Define a non-negative process $\hat{\beta} \in \mathcal{B}$ by

$$(5.11) \quad \bar{Y} = \hat{\beta} \bar{R}.$$

To establish that \mathcal{Y} is convex, we need to show the existence of a process $\bar{\beta} \in \mathcal{B}$ such that

$$(5.12) \quad \bar{Y} = \bar{\beta} \exp\left(-\int_0^\cdot \bar{\beta}_s \, d\kappa_s\right) \bar{S}.$$

From (5.10), (5.11) and (5.12) we thus require $\bar{\beta}$ to satisfy the relation

$$\bar{\beta} \exp\left(-\int_0^\cdot \bar{\beta}_s \, d\kappa_s\right) = \hat{\beta} \exp\left(-\int_0^\cdot \tilde{\beta}_s \, d\kappa_s\right),$$

which, given processes $\tilde{\beta}$ and $\hat{\beta}$, does have a unique solution for $\bar{\beta}$, due to the monotonicity of the exponential function. Thus \mathcal{Y} is convex.

□

The next step is to attempt to reach some form of reverse polarity result to (5.6). It is here that the enlargement of the dual domain from \mathcal{Y} to the set \mathcal{D} of (4.4) comes into play.

To see why this enlargement is needed, we first observe that the implication (5.2) extends from \mathcal{X} to \mathcal{C} , so we have

$$(5.13) \quad Y \in \mathcal{Y} \implies \langle g, Y \rangle \leq 1, \quad \forall g \in \mathcal{C},$$

which implies that

$$(5.14) \quad \mathcal{Y} \subseteq \mathcal{C}^\circ.$$

We do not have the reverse inclusion, because we do not have the reverse implication to (5.13), so cannot write a full bipolarity relation between sets \mathcal{C} and \mathcal{Y} . The enlargement from \mathcal{Y} to the set \mathcal{D} resolves the issue, yielding the inter-temporal wealth bipolarity of Lemma 5.7 below. This procedure, in the spirit of Kramkov and Schachermayer [21], requires us to establish that the enlarged domain is closed in an appropriate topology. Here is the relevant result.

Lemma 5.6. *The enlarged dual domain $\mathcal{D} \equiv \mathcal{D}(1)$ of (4.4) is closed with respect to the topology of convergence in measure μ .*

The proof of Lemma 5.6 will be given further below. First, we use the result of the lemma to establish the bipolarity result below.

Lemma 5.7 (Inter-temporal wealth bipolarity). *Given Lemma 5.6, the set \mathcal{D} is a closed, convex and solid subset of $L_+^0(\mu)$, and the sets \mathcal{C} and \mathcal{D} satisfy the bipolarity relations*

$$(5.15) \quad \mathcal{C} = \mathcal{D}^\circ, \quad \mathcal{D} = \mathcal{C}^\circ.$$

Proof. For any $h \in \mathcal{D}$ there will exist an element $Y \in \mathcal{Y}$ such that $h \leq Y$, μ -almost everywhere. Hence, the implication (5.13) holds true with \mathcal{D} in place of \mathcal{Y} :

$$h \in \mathcal{D} \implies \langle g, h \rangle \leq 1, \quad \forall g \in \mathcal{C},$$

which yields the analogue of (5.14):

$$(5.16) \quad \mathcal{D} \subseteq \mathcal{C}^\circ.$$

Combining (5.7) and (5.16) we have

$$(5.17) \quad \mathcal{D} \subseteq \mathcal{Y}^{\circ\circ}.$$

Part (ii) of the bipolar theorem, Theorem 4.3, says that $\mathcal{Y}^{\circ\circ} \supseteq \mathcal{Y}$ and that $\mathcal{Y}^{\circ\circ}$ is the smallest closed, convex, solid set which contains \mathcal{Y} . But \mathcal{D} is also closed, convex and solid (closed due to Lemma 5.6, convexity following easily from the convexity of \mathcal{Y} , and solidity is obvious), and by definition $\mathcal{D} \supseteq \mathcal{Y}$, so we also have

$$(5.18) \quad \mathcal{D} \supseteq \mathcal{Y}^{\circ\circ}.$$

Thus, (5.17) and (5.18) give

$$(5.19) \quad \mathcal{D} = \mathcal{Y}^{\circ\circ}.$$

In other words, in enlarging from \mathcal{Y} to \mathcal{D} we have succeeded in reaching the bipolar of the former.

Combining (5.19) and (5.7) we see that \mathcal{D} is the polar of \mathcal{C} ,

$$(5.20) \quad \mathcal{D} = \mathcal{C}^\circ,$$

so we have the second equality in (5.15). From (5.20) we get $\mathcal{D}^\circ = \mathcal{C}^{\circ\circ}$ which, combined with (5.8), yields the first equality in (5.15), and the proof is complete. \square

It remains to prove Lemma 5.6, which we used above. We recall the concept of Fatou convergence of stochastic processes from Föllmer and Kramkov [12], that will be needed.

Definition 5.8 (Fatou convergence). Let $(Y^n)_{n \in \mathbb{N}}$ be a sequence of processes on a stochastic basis $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, uniformly bounded from below, and let τ be a dense subset of \mathbb{R}_+ . The sequence $(Y^n)_{n \in \mathbb{N}}$ is said to be *Fatou convergent on τ* to a process Y if

$$Y_t = \limsup_{s \downarrow t, s \in \tau} \limsup_{n \rightarrow \infty} Y_s^n = \liminf_{s \downarrow t, s \in \tau} \liminf_{n \rightarrow \infty} Y_s^n, \quad \text{a.s. } \forall t \geq 0.$$

If $\tau = \mathbb{R}_+$, the sequence is simply called *Fatou convergent*.

The relevant consequence for our purposes is Föllmer and Kramkov [12, Lemma 5.2], that for a sequence $(S^n)_{n \in \mathbb{N}}$ of supermartingales, uniformly bounded from below, with $S_0^n = 0$, $n \in \mathbb{N}$, there is a sequence $(Y^n)_{n \in \mathbb{N}}$ of supermartingales, with $Y^n \in \text{conv}(S^n, S^{n+1}, \dots)$, and a supermartingale Y with $Y_0 \leq 0$, such that $(Y^n)_{n \in \mathbb{N}}$ is Fatou convergent on a dense subset τ of \mathbb{R}_+ to Y . Here, $\text{conv}(S^n, S^{n+1}, \dots)$ denotes a convex combination $\sum_{k=n}^{N(n)} \lambda_k S^k$ for $\lambda_k \in [0, 1]$ with $\sum_{k=n}^{N(n)} \lambda_k = 1$. The requirement that $S_0^n = 0$ is of course no restriction, since for a supermartingale with (say) $S_0^n = 1$ (as we shall have when we apply these results below for supermartingales in \mathcal{Y}), we can always subtract the initial value 1 to reach a process which starts at zero.

To prove Lemma 5.6 we shall need the following lemma on Fatou convergence of convex combinations of elements in \mathcal{R}, \mathcal{S} and, as a consequence, \mathcal{Y} . This result could instead have been developed in the course of proving Lemma 5.6, but it simplifies the proof of the latter a great deal to establish it separately.

Lemma 5.9. *Let τ be a dense subset of \mathbb{R}_+ . Let $(\tilde{R}^n)_{n \in \mathbb{N}}$ be a sequence in \mathcal{R} , so given by*

$$\tilde{R}^n = \exp \left(- \int_0^\cdot \tilde{\beta}_s^n d\kappa_s \right) \tilde{S}^n, \quad n \in \mathbb{N},$$

for a sequence $(\tilde{\beta}^n)_{n \in \mathbb{N}}$ in \mathcal{B} and a sequence of supermartingale deflators $(\tilde{S}^n)_{n \in \mathbb{N}}$ in \mathcal{S} . Then for each $n \in \mathbb{N}$ there exist convex combinations $R^n \in \text{conv}(\tilde{R}^n, \tilde{R}^{n+1}, \dots) \in \mathcal{R}$, $S^n \in \text{conv}(\tilde{S}^n, \tilde{S}^{n+1}, \dots) \in \mathcal{S}$, and a process $\beta^n \in \mathcal{B}$ such that

$$(5.21) \quad R^n = \exp \left(- \int_0^\cdot \beta_s^n d\kappa_s \right) S^n, \quad n \in \mathbb{N},$$

and such that the sequence $(R^n)_{n \in \mathbb{N}}$ (respectively, $(S^n)_{n \in \mathbb{N}}$) is Fatou convergent on τ to a supermartingale $R \in \mathcal{R}$ (respectively, $S \in \mathcal{S}$), with

$$(5.22) \quad R = \exp \left(- \int_0^\cdot \beta_s d\kappa_s \right) S,$$

for a process $\beta \in \mathcal{B}$. As a consequence, the sequence of inter-temporal wealth deflators $(Y^n)_{n \in \mathbb{N}} \in \mathcal{Y}$ given by $Y^n = \beta^n R^n$ is Fatou convergent on τ to the element $Y = \beta R \in \mathcal{Y}$.

Proof. Since \mathcal{R} and \mathcal{S} are both convex sets, the convex combinations R^n, S^n of the lemma lie in \mathcal{R}, \mathcal{S} , respectively. Indeed, by similar reasoning as in the proof of Lemma 5.5, for non-negative constants $(\lambda_k)_{k=n}^{N(n)}$ such that $\sum_{k=n}^{N(n)} \lambda_k = 1$, we have

$$(5.23) \quad R^n := \sum_{k=n}^{N(n)} \lambda_k \tilde{R}^k = \sum_{k=n}^{N(n)} \lambda_k \exp \left(- \int_0^\cdot \tilde{\beta}_s^k d\kappa_s \right) \tilde{S}^k \leq \sum_{k=n}^{N(n)} \lambda_k \tilde{S}^k =: S^n,$$

which shows that $R^n \leq S^n$, implying $R^n \in \mathcal{R}$ and $S^n \in \mathcal{S}$, and implying the existence of $\beta^n \in \mathcal{B}$ such that (5.21) holds. From Föllmer and Kramkov [12, Lemma 5.2] there exist supermartingales R and S such that the sequences $(R^n)_{n \in \mathbb{N}}$ and $(S^n)_{n \in \mathbb{N}}$ Fatou converge on τ to R and S respectively.

Define a supermartingale sequence $(\tilde{V}^n)_{n \in \mathbb{N}}$ by $\tilde{V}^n := X \tilde{S}^n$, for $X \in \mathcal{X}$. Once again from [12, Lemma 5.2] there exists a sequence $(V^n)_{n \in \mathbb{N}}$ of supermartingales with each $V^n \in$

$\text{conv}(\tilde{V}^n, \tilde{V}^{n+1}, \dots) = X \text{conv}(\tilde{S}^n, \tilde{S}^{n+1}, \dots)$, and a supermartingale V , such that $(V^n)_{n \in \mathbb{N}}$ is Fatou convergent on τ to V . Since $V^n \in X \text{conv}(\tilde{S}^n, \tilde{S}^{n+1}, \dots)$ for each $n \in \mathbb{N}$, we have $V^n = XS^n$, for $S^n \in \text{conv}(\tilde{S}^n, \tilde{S}^{n+1}, \dots)$. Because the sequence $(S^n)_{n \in \mathbb{N}}$ is Fatou convergent on τ to the supermartingale S , the sequence $(V^n)_{n \in \mathbb{N}} = (XS^n)_{n \in \mathbb{N}}$ is Fatou convergent on τ to the supermartingale $V = XS$. Since XS is a supermartingale and $X \in \mathcal{X}$, we have $S \in \mathcal{S}$.

The same argument as in the last paragraph, now applied to the supermartingale sequence $(\tilde{W}^n)_{n \in \mathbb{N}}$ defined by $\tilde{W}^n := X\tilde{R}^n$, establishes that $R \in \mathcal{S}$. But because $R^n \leq S^n$, μ -a.e., we have $R \leq S$, μ -a.e., so there exists a process $\beta \in \mathcal{B}$ such that (5.22) holds, and thus in fact we have $R \in \mathcal{R} \subseteq \mathcal{S}$. We have thus established that the sequence in (5.21) Fatou converges to the process R in (5.22), and this implies that the sequence $(Y^n)_{n \in \mathbb{N}}$ defined by $Y^n := \beta^n R^n$ must Fatou converge to a process $\beta R =: Y \in \mathcal{Y}$, since the same process $\beta^n \in \mathcal{B}$ appears in the sequence in (5.21) as well as in the sequence $(Y^n)_{n \in \mathbb{N}}$, and the proof is complete. \square

With this preparation, we can now prove Lemma 5.6.

Proof of Lemma 5.6. Let $(h^n)_{n \in \mathbb{N}}$ be a sequence in \mathcal{D} , converging μ -a.e. to some $h \in L_+^0(\mu)$. We want to show that $h \in \mathcal{D}$.

Since $h^n \in \mathcal{D}$, for each $n \in \mathbb{N}$ we have $h^n \leq \hat{Y}^n$, μ -a.e for some element $\hat{Y}^n \in \mathcal{Y}$ given by $\hat{Y}^n = \hat{\beta}^n \hat{R}^n$, for a non-negative process $\hat{\beta}^n \in \mathcal{B}$ and a supermartingale $\hat{R}^n \in \mathcal{R}$ given by $\hat{R}^n = \exp\left(-\int_0^\cdot \hat{\beta}_s^n d\kappa_s\right) \hat{S}^n$, for a supermartingale deflator $\hat{S}^n \in \mathcal{S}$.

Consider a convex combination

$$(5.24) \quad Y^n = \sum_{k=n}^{N(n)} \lambda_k \hat{Y}^k \geq \sum_{k=n}^{N(n)} \lambda_k h^k, \quad n \in \mathbb{N},$$

for non-negative constants $(\lambda_k)_{k=n}^{N(n)}$ such that $\sum_{k=n}^{N(n)} \lambda_k = 1$.

By convexity of the set \mathcal{Y} , we have $Y^n \in \mathcal{Y}$ for each $n \in \mathbb{N}$, so there exist processes $\beta^n \in \mathcal{B}$, $R^n \in \mathcal{R}$, $S^n \in \mathcal{S}$ such that

$$Y^n = \beta^n R^n = \beta^n \exp\left(-\int_0^\cdot \beta_s^n d\kappa_s\right) S^n, \quad n \in \mathbb{N}.$$

By convexity of the sets \mathcal{R} and \mathcal{S} there will exist sequences $(\tilde{R}^n)_{n \in \mathbb{N}}$ in \mathcal{R} and $(\tilde{S}^n)_{n \in \mathbb{N}}$ in \mathcal{S} , such that $R^n \in \text{conv}(\tilde{R}^n, \tilde{R}^{n+1}, \dots) \in \mathcal{R}$, and $S^n \in \text{conv}(\tilde{S}^n, \tilde{S}^{n+1}, \dots) \in \mathcal{S}$, and these convex combinations will in general differ from that in (5.24). We thus have the analogue of (5.23):

$$R^n = \sum_{k=n}^{\tilde{N}(n)} \tilde{\lambda}_k \tilde{R}^k = \sum_{k=n}^{\tilde{N}(n)} \tilde{\lambda}_k \exp\left(-\int_0^\cdot \tilde{\beta}_s^k d\kappa_s\right) \tilde{S}^k \leq \sum_{k=n}^{\tilde{N}(n)} \tilde{\lambda}_k \tilde{S}^k = S^n, \quad n \in \mathbb{N},$$

for some sequence $(\tilde{\beta}^n)_{n \in \mathbb{N}}$ in \mathcal{B} and non-negative constants $(\tilde{\lambda}_k)_{k=n}^{\tilde{N}(n)}$ such that $\sum_{k=n}^{\tilde{N}(n)} \tilde{\lambda}_k = 1$. By Lemma 5.9, the sequences $(R^n)_{n \in \mathbb{N}}$ in \mathcal{R} and $(S^n)_{n \in \mathbb{N}}$ in \mathcal{S} Fatou converge on a dense subset τ of \mathbb{R}_+ to supermartingales $R \in \mathcal{R}$ and $S \in \mathcal{S}$, respectively, and such that (5.22) holds for some process $\beta \in \mathcal{B}$. Then, again by Lemma 5.9, the sequence $(Y^n)_{n \in \mathbb{N}}$ Fatou converges on τ to $Y = \beta R \in \mathcal{Y}$. So the first term in (5.24) converges to $Y \in \mathcal{Y}$ while the last term converges to h as $n \rightarrow \infty$, so the inequality in (5.24) gives $h \leq Y$, and thus $h \in \mathcal{D}$. \square

With the inter-temporal wealth bipolarity of Lemma 5.7, we can establish Proposition 4.4.

Proof of Proposition 4.4. From the properties of \mathcal{C} established in Lemma 5.3, we have all the claimed properties of \mathcal{C} in items (i) and (ii). The corresponding assertions for \mathcal{D} follow from Lemma 5.7.

For item (iii), consider first the set \mathcal{D} . Since the wealth process $X \equiv 1 \in \mathcal{X}$, the constant function $g \equiv 1 \in \mathcal{C}$, and the budget constraint (equivalently, the polar relation (4.7)) in this case gives $\int_{\Omega} h d\mu \leq 1$, so \mathcal{D} is bounded in $L^1(\mu)$ and hence in $L^0(\mu)$.

For the L^0 -boundedness of \mathcal{C} , we shall find a positive element $\bar{h} \in \mathcal{D}$ and show that \mathcal{C} is bounded in $L^1(\bar{h} d\mu)$, and hence bounded in $L^0(\mu)$. Since the constant supermartingale $S \equiv 1 \in \mathcal{S}$ and since the constant process $\beta \equiv \alpha > 0$ for some positive constant α , lies in \mathcal{B} , we can take $\mathcal{Y} \ni \bar{Y}_t := \alpha \exp(-\alpha \kappa_t)$, $t \geq 0$, and then choose $\mathcal{D} \ni \bar{h} \equiv \bar{Y}$. We see that $\bar{h} \in \mathcal{D}$ is strictly positive except on a set of μ -measure zero. Then, the budget constraint (equivalently, the polar relation (4.8)) gives $\int_{\Omega} g \bar{h} d\mu \leq 1$ for any $g \in \mathcal{C}$. Thus, \mathcal{C} is bounded in $L^1(\bar{h} d\mu)$ and hence bounded in $L^0(\mu)$. \square

5.3. On approaches to establishing bipolarity. In this section we compare the approach we have taken to establishing the polar relations (4.7) and (4.8) in Proposition 4.4, between the enlarged primal and dual domains \mathcal{C} and \mathcal{D} , with the approach taken by Bouchard and Pham [2]. This is instructive and will indicate how we have been able to strengthen the statement of the final duality result, in essence by proving, as opposed to partially assuming, the polar relations, which is what Bouchard and Pham [2] were compelled to do.

5.3.1. The Kramkov-Schachermayer approach. Our approach is in the spirit of the recipe created by Kramkov and Schachermayer [21, 22] for the terminal wealth utility maximisation problem, adapted to an inter-temporal framework. One begins with a supermartingale property linking the elements of the primal and dual domains. (In the terminal wealth problem one has the admissible wealth processes $X \in \mathcal{X}$ and the supermartingale deflators $S \in \mathcal{S}$, with XS a supermartingale for each $X \in \mathcal{X}$ and $S \in \mathcal{S}$.) Here, we invoke the additional dual controls $\beta \in \mathcal{B}$, and from these and the supermartingale deflators we construct the supermartingales $R \in \mathcal{R}$ and the inter-temporal wealth deflators $Y \in \mathcal{Y}$ according to the relations in (2.9) and (2.11), repeated below for the case $y = 1$, so for $S \in \mathcal{S}$:

$$(5.25) \quad R := \exp\left(-\int_0^\cdot \beta_s d\kappa_s\right) S, \quad Y := \beta R, \quad \beta \in \mathcal{B}, S \in \mathcal{S}.$$

Observe that the deflators $Y \in \mathcal{Y}$ are given by $Y = \nu S$, $S \in \mathcal{S}$, with the process ν given by

$$(5.26) \quad \nu_t := \beta_t \exp\left(-\int_0^t \beta_s d\kappa_s\right), \quad t \geq 0, \quad \beta \in \mathcal{B}.$$

We see that ν satisfies

$$\int_0^\infty \nu_t d\kappa_t = 1 - \exp\left(-\int_0^\infty \beta_t d\kappa_t\right) \leq 1, \quad \text{almost surely,}$$

and hence also $\mathbb{E}\left[\int_0^\infty \nu_t d\kappa_t\right] \leq 1$ or, in the notation of (4.6),

$$(5.27) \quad \langle \nu, 1 \rangle \leq 1.$$

This structure of dual variables for wealth-path-dependent utility maximisation problems, namely a multiplicative auxiliary control which augments the classical deflators and which satisfies a constraint of the form in (5.27), is not uncommon, and we shall see a similar feature shortly when we describe the Bouchard and Pham [2] approach. The key insight that arises in our approach is that this auxiliary control must have the very specific structure in (5.26), which confers convexity to the dual domain.

From (5.25) and the properties of $S \in \mathcal{S}$, we get that the process M in (2.13) is a supermartingale, and in turn this gives the budget constraint (2.14), repeated below for the case

$x = y = 1$, as a necessary condition for admissibility of a wealth process:

$$\mathbb{E} \left[\int_0^\infty X_t Y_t d\kappa_t \right] \leq 1, \quad \forall X \in \mathcal{X}, Y \in \mathcal{Y}.$$

Then, enlarging the primal domain from \mathcal{X} to \mathcal{C} , Lemma 5.2 establishes that the budget constraint is also a sufficient condition for admissibility, so we obtain the polar properties of Lemma 5.3 for \mathcal{C}

$$\mathcal{C} = \mathcal{Y}^\circ, \quad \mathcal{C}^\circ = \mathcal{Y}^{\circ\circ}, \quad \mathcal{C}^{\circ\circ} = \mathcal{C},$$

which imply that \mathcal{C} is a closed, convex and solid (CCS) subset of $L_+^0(\mu)$.

Now to the dual side of the story. Using the particular form of the dual variables in (5.25) we established in Lemma 5.5 that the dual domain \mathcal{Y} is convex. This convexity is passed on to the enlarged dual domain \mathcal{D} . Then, again using the structure in (5.25), and in particular that the deflators $Y \in \mathcal{Y}$ contain the supermartingales $R \in \mathcal{R}, S \in \mathcal{S}$, we are able to exploit Fatou convergence of supermartingales to show that \mathcal{D} is closed with respect to the topology of convergence in μ -measure. This, along with the convexity and (obvious) solidity of \mathcal{D} , shows that \mathcal{D} is also a CCS subset of $L_+^0(\mu)$, matching the property we obtained for \mathcal{C} . In particular, we obtain the key result that the enlargement from \mathcal{Y} to \mathcal{D} has taken as to the bipolar of the original dual domain:

$$\mathcal{D} = \mathcal{Y}^{\circ\circ}.$$

This result then readily combines with the earlier polarity properties of \mathcal{C} to establish the perfect bipolarity relations (4.7) and (4.8).

The message is that we have made the Kramkov and Schachermayer [21, 22] prescription for obtaining bipolarity work: begin with a supermartingale property to arrive at the correct definition of the dual variables, make no assumptions regarding convexity and closed properties of either the primal or dual domains, show that with a natural enlargement of these domains to obtain solid sets, all the required CCS properties of the domains, and hence bipolarity, follows. This bipolarity is then the bedrock of the subsequent program for the proof of the duality theorem, as we shall see in Section 6.

This methodology is to be contrasted with the approach in [2], which we now describe.

5.3.2. The Bouchard-Pham approach. The first difference between our methodology and that of Bouchard and Pham [2] is that in [2], the dual domain (let us call it \mathcal{D}^{BP}) is *defined* as the polar of the primal domain. Over a finite horizon $T < \infty$, the dual variables Y^{BP} and dual domain are thus defined according to

$$\mathcal{D}^{\text{BP}} := \left\{ Y^{\text{BP}} \geq 0 : \mathbb{E} \left[\int_0^T X_t Y_t^{\text{BP}} d\kappa_t \right] \leq 1, \forall X \in \mathcal{X} \right\},$$

(see the definition of the set $\mathcal{D}(y)$ in [2, Page 584]). In other words,

$$(5.28) \quad \mathcal{D}^{\text{BP}} := \mathcal{X}^0,$$

by assumption. This automatically confers the CCS property to the dual domain, but the statement of the result is weakened, having been obtained by definition. The reason that this approach had to be adopted, we conjecture, is that the authors of [2] did not have to hand the specific structure of the dual variables in (5.25) that emerges in our approach.

This conjecture is reinforced by the reasoning which now follows. In a subsequent refinement Bouchard and Pham [2] show that, under an assumption called **(Hf)** (namely, that κ decomposes into a continuous density plus a linear combination of indicator functions of the form $\mathbb{1}_{\{\tau \leq t\}}$, $t \in [0, T]$, for any \mathbb{F} -stopping time τ), processes of the form $\nu^{\text{BP}} Z^{\mathcal{M}}$ lie in their dual domain, where $Z^{\mathcal{M}}$ is the density process of an ELMM, and ν^{BP} is any process satisfying $\langle \nu^{\text{BP}}, 1 \rangle_T := \mathbb{E} \left[\int_0^T \nu_t^{\text{BP}} d\kappa_t \right] \leq 1$. The similarity with the structure we have in (5.27) is clear. If we denote the set of processes $\nu^{\text{BP}} Z^{\mathcal{M}}$ by \mathcal{Z}^{BP} , then under their additional assumption

(Hf), Bouchard and Pham [2] are able to re-cast their dual problem as a minimisation over the convex hull of \mathcal{Z}^{BP} . This, therefore, is the analogue, under NFLVR and over a finite horizon, of the dual structure we have used, but with two caveats. First, they have to use the convex hull of \mathcal{Z}^{BP} , because the set \mathcal{Z}^{BP} is not known to be convex in general. Second, this lack of convexity is due to the fact that the authors of [2] do not have the particular structure of the auxiliary dual control ν^{BP} that we have found in (5.26), a structure that was crucial in our establishing the convexity of our dual domain. All that is known about the processes ν^{BP} is that they satisfy $\langle \nu^{\text{BP}}, 1 \rangle_T \leq 1$, and this is not enough to afford a proof of convexity of \mathcal{Z}^{BP} .

Finally, the discussion above also explains why the bulk of the analysis in [2] is carried out on the primal side of the problem. Since the definition in (5.28) confers the CCS property to the dual domain by assumption, the remaining work in [2] is concerned with enlarging the primal domain to confer solidity and proving the remaining polarity relation, as can be verified by examining [2, Section 5].

In summary, we are able to strengthen the duality statement in [2] to any horizon and under NUPBR, by making the broad pattern of the Kramkov and Schachermayer [21, 22] program for bipolarity work, without having to assume the associated properties of either the primal or dual domain. Instead, we begin with a natural supermartingale property linking the primal and dual elements, thus identifying the natural dual space for the problem, along with its particular structure, so that the closed and convex features of the domains, from which the existence and uniqueness of the optimisers are ultimately deduced, are demonstrated, as opposed to being assumed.

6. PROOFS OF THE DUALITY THEOREMS

In this section we prove the abstract duality of Theorem 4.5, from which the concrete duality of Theorem 3.1 is then deduced. Throughout this section, we have in place the result of Proposition 4.4, as this bipolarity is the starting point of the duality proof. The proof of Theorem 4.5 proceeds via a series of lemmas. The procedure has a similar flavour to that of Kramkov and Schachermayer [21, 22] for an abstract duality proof in the context of the terminal wealth utility maximisation problem, with variations where appropriate, and with an additional result, Proposition 6.13, which gives the additional characterisation (3.2) of the optimal wealth process as well as the uniformly integrable martingale property of the process $\widehat{M} := \widehat{X}(x)\widehat{R}(y) + \int_0^\cdot \widehat{X}_s(x)\widehat{Y}_s(y) d\kappa_s$. This proposition also establishes that the process $\widehat{X}(x)\widehat{R}(y)$ is a potential, and that its limiting value is $\lim_{t \rightarrow \infty} \widehat{X}_t(x)\widehat{R}_t(y) = 0$ almost surely.

Let us state the basic properties that are taken as given throughout this section.

Fact 6.1. Throughout this section, assume that the utility function satisfies the Inada conditions (2.5), that the sets \mathcal{C} and \mathcal{D} satisfy all the properties in Proposition 4.4, and that the abstract primal and dual value functions in (4.3) and (4.5) satisfy the minimal conditions in (4.9).

All subsequent lemmata and propositions in this section implicitly take Fact 6.1 as given.

The first step is to establish weak duality.

Lemma 6.2 (Weak duality). *The primal and dual value functions $u(\cdot)$ and $v(\cdot)$ of (4.3) and (4.5) satisfy the weak duality bounds*

$$(6.1) \quad v(y) \geq \sup_{x>0} [u(x) - xy], \quad y > 0, \quad \text{equivalently} \quad u(x) \leq \inf_{y>0} [v(y) + xy], \quad x > 0.$$

As a result, $u(x)$ is finitely valued for all $x > 0$. Moreover, we have the limiting relations

$$(6.2) \quad \limsup_{x \rightarrow \infty} \frac{u(x)}{x} \leq 0, \quad \liminf_{y \rightarrow \infty} \frac{v(y)}{y} \geq 0.$$

Proof. Recall the inequality (2.17). By the same argument carried out in the measure space $(\Omega, \mathcal{G}, \mu)$ we have, for any $g \in \mathcal{C}(x)$ and $h \in \mathcal{D}(y)$, using the polarity relations in (4.7) and (4.8),

$$\begin{aligned}
 \int_{\Omega} U(g) \, d\mu &\leq \int_{\Omega} U(g) \, d\mu + xy - \int_{\Omega} gh \, d\mu \\
 &= \int_{\Omega} (U(g) - gh) \, d\mu + xy \\
 (6.3) \qquad &\leq \int_{\Omega} V(h) \, d\mu + xy, \quad x, y > 0,
 \end{aligned}$$

the last inequality a consequence of (2.16). Maximising the left-hand-side of (6.3) over $g \in \mathcal{C}(x)$ and minimising the right-hand-side over $h \in \mathcal{D}(y)$ gives $u(x) \leq v(y) + xy$ for all $x, y > 0$, and (6.1) follows.

The assumption that $v(y) < \infty$ for all $y > 0$ immediately yields that $u(x)$ is finitely valued for some $x > 0$. Since $U(\cdot)$ is strictly increasing and strictly concave, and given the convexity of \mathcal{C} , these properties are inherited by $u(\cdot)$, which is therefore finitely valued for all $x > 0$. Finally, the relations in (6.1) easily lead to those in (6.2). \square

Above, we obtained concavity and monotonicity of $u(\cdot)$ by using convexity of \mathcal{C} and the properties of $U(\cdot)$. Similar arguments show that $v(\cdot)$ is strictly decreasing and strictly convex. We shall see these properties reproduced in proofs of existence and uniqueness of the optimisers for $u(\cdot), v(\cdot)$.

The next step is to give a compactness lemma for the dual domain.

Lemma 6.3 (Compactness lemma for \mathcal{D}). *Let $(\tilde{h}^n)_{n \in \mathbb{N}}$ be a sequence in \mathcal{D} . Then there exists a sequence $(h^n)_{n \in \mathbb{N}}$ with $h^n \in \text{conv}(\tilde{h}^n, \tilde{h}^{n+1}, \dots)$, which converges μ -a.e. to an element $h \in \mathcal{D}$ that is μ -a.e. finite.*

Proof. Delbaen and Schachermayer [6, Lemma A1.1] (adapted from a probability space to the finite measure space $(\Omega, \mathcal{G}, \mu)$) implies the existence of a sequence $(h^n)_{n \in \mathbb{N}}$, with $h^n \in \text{conv}(\tilde{h}^n, \tilde{h}^{n+1}, \dots)$, which converges μ -a.e. to an element h that is μ -a.e. finite because \mathcal{D} is bounded in $L^0(\mu)$ (the finiteness also from [6, Lemma A1.1]). By convexity of \mathcal{D} , each $h^n, n \in \mathbb{N}$ lies in \mathcal{D} . Finally, by Fatou's lemma, for every $g \in \mathcal{C}$ we have

$$\int_{\Omega} gh \, d\mu = \int_{\Omega} \liminf_{n \rightarrow \infty} gh^n \, d\mu \leq \liminf_{n \rightarrow \infty} \int_{\Omega} gh^n \, d\mu \leq 1,$$

so that $h \in \mathcal{D}$. \square

Results in the style of Lemma 6.3 are standard in these duality proofs. We will see a similar result for the primal domain \mathcal{C} shortly.

The next step in the chain of results we need is a uniform integrability result for the family $(V^-(h))_{h \in \mathcal{D}(y)}$. This will facilitate a proof of existence and uniqueness of the dual minimiser, and of the conjugacy for the value functions by establishing the first relation in (4.10).

Lemma 6.4 (Uniform integrability of $(V^-(h))_{h \in \mathcal{D}(y)}$). *The family $(V^-(h))_{h \in \mathcal{D}(y)}$ is uniformly integrable, for any $y > 0$.*

The style of the proof is along identical lines to Kramkov and Schachermayer [21, Lemma 3.2], but we give the proof for completeness.

Proof of Lemma 6.4. Since $V(\cdot)$ is decreasing, we need only consider the case where $V(\infty) := \lim_{y \rightarrow \infty} V(y) = -\infty$ (otherwise there is nothing to prove). Let $\varphi : (-V(0), -V(\infty)) \mapsto (0, \infty)$

denote the inverse of $-V(\cdot)$. Then $\varphi(\cdot)$ is strictly increasing. For any $h \in \mathcal{D}(y)$ (so $\int_{\Omega} h \, d\mu \leq y$) we have, for all $y > 0$,

$$\int_{\Omega} \varphi(V^-(h)) \, d\mu \leq \varphi(0) + \int_{\Omega} \varphi(-V(h)) \, d\mu = \varphi(0) + \int_{\Omega} h \, d\mu \leq \varphi(0) + y.$$

Then, using l'Hôpital's rule and the change of variable $\varphi(x) = y \iff x = -V(y)$, and recalling the function $I(\cdot) = -V'(\cdot)$ (the inverse of marginal utility $U'(\cdot)$), we have

$$(6.4) \quad \lim_{x \rightarrow -V(\infty)} \frac{\varphi(x)}{x} = \lim_{x \rightarrow \infty} \frac{\varphi(x)}{x} = \lim_{y \rightarrow \infty} \frac{y}{-V(y)} = \lim_{y \rightarrow \infty} \frac{1}{I(y)} = +\infty,$$

on using the Inada conditions (2.5). The $L^1(\mu)$ -boundedness of $\mathcal{D}(y)$ means we can apply the de la Vallée-Poussin theorem (Pham [28, Theorem A.1.2]) which, combined with (6.4), implies the uniform integrability of the family $(V^-(h))_{h \in \mathcal{D}(y)}$. \square

One can now proceed to prove either existence of a unique optimiser in the dual problem, or conjugacy of the value functions. We proceed first with the former, followed by conjugacy.

Lemma 6.5 (Dual existence). *The optimal solution $\hat{h}(y) \in \mathcal{D}(y)$ to the dual problem (4.5) exists and is unique, so that $v(\cdot)$ is strictly convex.*

Proof. Fix $y > 0$. Let $(h^n)_{n \in \mathbb{N}}$ be a minimising sequence in $\mathcal{D}(y)$ for $v(y) < \infty$. That is

$$(6.5) \quad \lim_{n \rightarrow \infty} \int_{\Omega} V(h^n) \, d\mu = v(y) < \infty.$$

By the compactness lemma for \mathcal{D} (and thus also for $\mathcal{D}(y) = y\mathcal{D}$), Lemma 6.3, we can find a sequence $(\hat{h}^n)_{n \in \mathbb{N}}$ of convex combinations, so $\mathcal{D}(y) \ni \hat{h}^n \in \text{conv}(h^n, h^{n+1}, \dots)$, $n \in \mathbb{N}$, which converges μ -a.e. to some element $\hat{h}(y) \in \mathcal{D}(y)$. We claim that $\hat{h}(y)$ is the dual optimiser. That is, that we have

$$(6.6) \quad \int_{\Omega} V(\hat{h}(y)) \, d\mu = v(y).$$

From convexity of $V(\cdot)$ and (6.5) we deduce that

$$\lim_{n \rightarrow \infty} \int_{\Omega} V(\hat{h}^n) \, d\mu \leq \lim_{n \rightarrow \infty} \int_{\Omega} V(h^n) \, d\mu = v(y),$$

which, combined with the obvious inequality $v(y) \leq \lim_{n \rightarrow \infty} \int_{\Omega} V(\hat{h}^n) \, d\mu$ means that we also have, further to (6.5),

$$\lim_{n \rightarrow \infty} \int_{\Omega} V(\hat{h}^n) \, d\mu = v(y).$$

In other words

$$(6.7) \quad \lim_{n \rightarrow \infty} \int_{\Omega} V^+(\hat{h}^n) \, d\mu - \lim_{n \rightarrow \infty} \int_{\Omega} V^-(\hat{h}^n) \, d\mu = v(y) < \infty,$$

and note therefore that both integrals in (6.7) are finite.

From Fatou's lemma, we have

$$(6.8) \quad \lim_{n \rightarrow \infty} \int_{\Omega} V^+(\hat{h}^n) \, d\mu \geq \int_{\Omega} V^+(\hat{h}(y)) \, d\mu.$$

From Lemma 6.4 we have uniform integrability of $(V^-(\hat{h}^n))_{n \in \mathbb{N}}$, so that

$$(6.9) \quad \lim_{n \rightarrow \infty} \int_{\Omega} V^-(\hat{h}^n) \, d\mu = \int_{\Omega} V^-(\hat{h}(y)) \, d\mu.$$

Thus, using (6.8) and (6.9) in (6.7), we obtain

$$v(y) \geq \int_{\Omega} V(\widehat{h}(y)) \, d\mu,$$

which, combined with the obvious inequality $v(y) \leq \int_{\Omega} V(\widehat{h}(y)) \, d\mu$, yields (6.6). The uniqueness of the dual optimiser follows from the strict convexity of $V(\cdot)$, as does the strict convexity of $v(\cdot)$. For this last claim, fix $y_1 < y_2$ and $\lambda \in (0, 1)$, note that $\lambda\widehat{h}(y_1) + (1 - \lambda)\widehat{h}(y_2) \in \mathcal{D}(\lambda y_1 + (1 - \lambda)y_2)$ (yet must be sub-optimal for $v(\lambda y_1 + (1 - \lambda)y_2)$ as it is not guaranteed to equal $\widehat{h}(\lambda y_1 + (1 - \lambda)y_2)$) and therefore, using the strict convexity of $V(\cdot)$,

$$v(\lambda y_1 + (1 - \lambda)y_2) \leq \int_{\Omega} V\left(\lambda\widehat{h}(y_1) + (1 - \lambda)\widehat{h}(y_2)\right) \, d\mu < \lambda v(y_1) + (1 - \lambda)v(y_2).$$

□

We now establish conjugacy of the value functions. The method is similar to the classical method of proof in Kramkov and Schachermayer [21, Lemma 3.4], and works by bounding the elements in the primal domain to create a compact set for the weak* topology $\sigma(L^\infty, L^1)$ on $L^\infty(\mu)$,¹ so as to apply the minimax theorem, involving a maximisation over a compact set and a minimisation over a subset of a vector space. This uses the fact that the dual domain is bounded in $L^1(\mu)$.

For the convenience of the reader here is the minimax theorem as we shall apply it (see Strasser [32, Theorem 45.8]).

Theorem 6.6 (Minimax). *Let \mathcal{X} be a $\sigma(E', E)$ -compact convex subset of the topological dual E' of a normed vector space E , and let \mathcal{Y} be a convex subset of E . Assume that $f : \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R}$ satisfies the following conditions:*

- (1) $x \mapsto f(x, y)$ is continuous and concave on \mathcal{X} for every $y \in \mathcal{Y}$;
- (2) $y \mapsto f(x, y)$ is convex on \mathcal{Y} for every $x \in \mathcal{X}$.

Then:

$$\sup_{x \in \mathcal{X}} \inf_{y \in \mathcal{Y}} f(x, y) = \inf_{y \in \mathcal{Y}} \sup_{x \in \mathcal{X}} f(x, y).$$

Here is the conjugacy result for the primal and dual value functions.

Lemma 6.7 (Conjugacy). *The dual value function in (4.5) satisfies the conjugacy relation*

$$v(y) = \sup_{x > 0} [u(x) - xy], \quad \text{for each } y > 0,$$

where $u(\cdot)$ is the primal value function in (4.3).

Proof. For $n \in \mathbb{N}$ denote by \mathcal{B}_n the set of elements in $L_+^0(\mu)$ lying in a ball of radius n :

$$\mathcal{B}_n := \{g \in L_+^0(\mu) : g \leq n, \mu - \text{a.e.}\}.$$

The sets $(\mathcal{B}_n)_{n \in \mathbb{N}}$ are $\sigma(L^\infty, L^1)$ -compact. Because each $h \in \mathcal{D}(y)$ is μ -integrable, $\mathcal{D}(y)$ is a closed, convex subset of the vector space $L^1(\mu)$, so we apply the minimax theorem as given in Theorem 6.6 to the compact set \mathcal{B}_n (n fixed) and the set $\mathcal{D}(y)$, with the function $f(g, h) := \int_{\Omega} (U(g) - gh) \, d\mu$, for $g \in \mathcal{B}_n$, $h \in \mathcal{D}(y)$, to give

$$(6.10) \quad \sup_{g \in \mathcal{B}_n} \inf_{h \in \mathcal{D}(y)} \int_{\Omega} (U(g) - gh) \, d\mu = \inf_{h \in \mathcal{D}(y)} \sup_{g \in \mathcal{B}_n} \int_{\Omega} (U(g) - gh) \, d\mu.$$

¹Recall that a sequence $(g^n)_{n \in \mathbb{N}}$ in $L^\infty(\mu)$ converges to $g \in L^\infty(\mu)$ with respect to the weak* topology $\sigma(L^\infty, L^1)$ if and only if $(\langle g^n, h \rangle)_{n \in \mathbb{N}}$ converges to $\langle g, h \rangle$ for each $h \in L^1(\mu)$.

By the bipolarity relation $\mathcal{C} = \mathcal{D}^\circ$ in (4.7), an element $g \in L_+^0(\mu)$ lies in $\mathcal{C}(x)$ if and only if $\sup_{h \in \mathcal{D}(y)} \int_{\Omega} gh \, d\mu \leq xy$. Thus, the limit as $n \rightarrow \infty$ on the left-hand-side of (6.10) is given as

$$(6.11) \quad \lim_{n \rightarrow \infty} \sup_{g \in \mathcal{B}_n} \inf_{h \in \mathcal{D}(y)} \int_{\Omega} (U(g) - gh) \, d\mu = \sup_{x > 0} \sup_{g \in \mathcal{C}(x)} \left(\int_{\Omega} U(g) \, d\mu - xy \right) = \sup_{x > 0} [u(x) - xy].$$

Now consider the right-hand-side of (6.10). Define

$$V_n(y) := \sup_{0 < x \leq n} [U(x) - xy], \quad y > 0, \quad n \in \mathbb{N}.$$

The right-hand-side of (6.10) is then given as

$$\inf_{h \in \mathcal{D}(y)} \sup_{g \in \mathcal{B}_n} \int_{\Omega} (U(g) - gh) \, d\mu = \inf_{h \in \mathcal{D}(y)} \int_{\Omega} V_n(h) \, d\mu =: v_n(y),$$

so that taking the limit as $n \rightarrow \infty$ and equating this with the limit obtained in (6.11), we have

$$(6.12) \quad \lim_{n \rightarrow \infty} v_n(y) = \sup_{x > 0} [u(x) - xy] \leq v(y),$$

with the inequality due to the weak duality bound in (6.1). Consequently, we will be done if we can now show that we also have

$$\lim_{n \rightarrow \infty} v_n(y) \geq v(y).$$

Evidently, $(v_n(y))_{n \in \mathbb{N}}$ is an increasing sequence satisfying the limiting inequality in (6.12). Let $(\tilde{h}^n)_{n \in \mathbb{N}}$ be a minimising sequence in $\mathcal{D}(y)$ for $\lim_{n \rightarrow \infty} v_n(y)$, so such that

$$\lim_{n \rightarrow \infty} \int_{\Omega} V_n(\tilde{h}^n) \, d\mu = \lim_{n \rightarrow \infty} v_n(y).$$

The compactness lemma for \mathcal{D} , Lemma 6.3, implies the existence of a sequence $(h^n)_{n \in \mathbb{N}}$ in $\mathcal{D}(y)$, with $h^n \in \text{conv}(\tilde{h}^n, \tilde{h}^{n+1}, \dots)$, which converges μ -a.e. to an element $h \in \mathcal{D}(y)$. Now, $V_n(y) = V(y)$ for $y \geq I(n)$, where $I(\cdot) = -V'(\cdot)$ is the inverse of $U'(\cdot)$ (and $V_n(\cdot) \rightarrow V(\cdot)$ as $n \rightarrow \infty$). So we deduce from Lemma 6.4 that the sequence $(V_n^-(h^n))_{n \in \mathbb{N}}$ is uniformly integrable, and hence that

$$(6.13) \quad \lim_{n \rightarrow \infty} \int_{\Omega} V_n^-(h^n) \, d\mu = \int_{\Omega} V^-(h) \, d\mu.$$

On the other hand, from Fatou's lemma, we have

$$(6.14) \quad \lim_{n \rightarrow \infty} \int_{\Omega} V_n^+(h^n) \, d\mu \geq \int_{\Omega} V^+(h) \, d\mu,$$

so (6.13) and (6.14) give

$$(6.15) \quad \lim_{n \rightarrow \infty} \int_{\Omega} V_n(h^n) \, d\mu \geq \int_{\Omega} V(h) \, d\mu.$$

Finally, using convexity of $V_n(\cdot)$ and (6.15), we obtain

$$\lim_{n \rightarrow \infty} v_n(y) = \lim_{n \rightarrow \infty} \int_{\Omega} V_n(\tilde{h}^n) \, d\mu \geq \lim_{n \rightarrow \infty} \int_{\Omega} V_n(h^n) \, d\mu \geq \int_{\Omega} V(h) \, d\mu \geq v(y),$$

and the proof is complete. \square

We now move on to the primal side of the analysis. The first step is an analogous compactness result to Lemma 6.3, this time for the primal domain. The proof is identical to the proof of Lemma 6.3 so is omitted.

Lemma 6.8 (Compactness lemma for \mathcal{C}). *Let $(\tilde{g}^n)_{n \in \mathbb{N}}$ be a sequence in \mathcal{C} . Then there exists a sequence $(g^n)_{n \in \mathbb{N}}$ with $g^n \in \text{conv}(\tilde{g}^n, \tilde{g}^{n+1}, \dots)$, which converges μ -a.e. to an element $g \in \mathcal{C}$ that is μ -a.e. finite.*

To prove existence of a unique primal optimiser we also need a result analogous to Lemma 6.4, on the uniform integrability of a sequence $(U^+(g^n))_{n \in \mathbb{N}}$ for $g^n \in \mathcal{C}(x)$. The proof is in the style of Kramkov and Schachermayer [22, Lemma 1].

Lemma 6.9 (Uniform integrability of $(U^+(g^n))_{n \in \mathbb{N}}$, $g^n \in \mathcal{C}(x)$). *Let $(g^n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{C}(x)$, for any fixed $x > 0$. The sequence $(U^+(g^n))_{n \in \mathbb{N}}$ is uniformly integrable.*

Proof. Fix $x > 0$. If $U(\infty) \leq 0$ there is nothing to prove, so assume $U(\infty) > 0$.

If the sequence $(U^+(g^n))_{n \in \mathbb{N}}$ is not uniformly integrable, then, passing if need be to a subsequence still denoted by $(g^n)_{n \in \mathbb{N}}$, we can find a constant $\alpha > 0$ and a disjoint sequence $(A_n)_{n \in \mathbb{N}}$ of sets of (Ω, \mathcal{G}) (so $A_n \in \mathcal{G}$, $n \in \mathbb{N}$ and $A_i \cap A_j = \emptyset$ if $i \neq j$) such that

$$\int_{\Omega} U^+(g^n) \mathbb{1}_{A_n} d\mu \geq \alpha, \quad n \in \mathbb{N}.$$

(See for example Pham [28, Corollary A.1.1].) Define a sequence $(f^n)_{n \in \mathbb{N}}$ of elements in $L_+^0(\mu)$ by

$$f^n := x_0 + \sum_{k=1}^n g^k \mathbb{1}_{A_k},$$

where $x_0 := \inf\{x > 0 : U(x) \geq 0\}$.

For any $h \in \mathcal{D}$ (so satisfying $\int_{\Omega} h d\mu \leq 1$) we have

$$\int_{\Omega} f^n h d\mu = \int_{\Omega} \left(x_0 + \sum_{k=1}^n g^k \mathbb{1}_{A_k} \right) h d\mu \leq x_0 + \sum_{k=1}^n \int_{\Omega} g^k h \mathbb{1}_{A_k} d\mu \leq x_0 + nx.$$

Thus, $f^n \in \mathcal{C}(x_0 + nx)$, $n \in \mathbb{N}$.

On the other hand, since $U^+(\cdot)$ is non-negative and non-decreasing,

$$\begin{aligned} \int_{\Omega} U(f^n) d\mu &= \int_{\Omega} U^+(f^n) d\mu \\ &= \int_{\Omega} U^+ \left(x_0 + \sum_{k=1}^n g^k \mathbb{1}_{A_k} \right) d\mu \\ &\geq \int_{\Omega} U^+ \left(\sum_{k=1}^n g^k \mathbb{1}_{A_k} \right) d\mu \\ &= \sum_{k=1}^n \int_{\Omega} U^+ \left(g^k \mathbb{1}_{A_k} \right) d\mu \geq \alpha n. \end{aligned}$$

Therefore,

$$\limsup_{z \rightarrow \infty} \frac{u(z)}{z} = \limsup_{n \rightarrow \infty} \frac{u(x_0 + nx)}{x_0 + nx} \geq \limsup_{n \rightarrow \infty} \frac{\int_{\Omega} U(f^n) d\mu}{x_0 + nx} \geq \limsup_{n \rightarrow \infty} \left(\frac{\alpha n}{x_0 + nx} \right) = \frac{\alpha}{x} > 0,$$

which contradicts the limiting weak duality bound in (6.2). This contradiction establishes the result. \square

One can now proceed to prove existence of a unique optimiser in the primal problem. The method of proof is similar to the proof of dual existence, Lemma 6.5, with adjustments for maximisation as opposed to minimisation and concavity of $U(\cdot)$ replacing convexity of $V(\cdot)$, so is included just for completeness.

Lemma 6.10 (Primal existence). *The optimal solution $\widehat{g}(x) \in \mathcal{C}(x)$ to the primal problem (4.3) exists and is unique, so that $u(\cdot)$ is strictly concave.*

Proof. Fix $x > 0$. Let $(g^n)_{n \in \mathbb{N}}$ be a maximising sequence in $\mathcal{C}(x)$ for $u(x) < \infty$ (the finiteness proven in Lemma 6.2). That is

$$(6.16) \quad \lim_{n \rightarrow \infty} \int_{\Omega} U(g^n) \, d\mu = u(x) < \infty.$$

By the compactness lemma for \mathcal{C} (and thus also for $\mathcal{C}(x) = x\mathcal{C}$), Lemma 6.8, we can find a sequence $(\widehat{g}^n)_{n \in \mathbb{N}}$ of convex combinations, so $\mathcal{C}(x) \ni \widehat{g}^n \in \text{conv}(g^n, g^{n+1}, \dots)$, $n \in \mathbb{N}$, which converges μ -a.e. to some element $\widehat{g}(x) \in \mathcal{C}(x)$. We claim that $\widehat{g}(x)$ is the primal optimiser. That is, that we have

$$(6.17) \quad \int_{\Omega} U(\widehat{g}(x)) \, d\mu = u(x).$$

By concavity of $U(\cdot)$ and (6.16) we have

$$\lim_{n \rightarrow \infty} \int_{\Omega} U(\widehat{g}^n) \, d\mu \geq \lim_{n \rightarrow \infty} \int_{\Omega} U(g^n) \, d\mu = u(x),$$

which, combined with the obvious inequality $u(x) \geq \lim_{n \rightarrow \infty} \int_{\Omega} U(\widehat{g}^n) \, d\mu$ means that we also have, further to (6.16),

$$\lim_{n \rightarrow \infty} \int_{\Omega} U(\widehat{g}^n) \, d\mu = u(x).$$

In other words

$$(6.18) \quad \lim_{n \rightarrow \infty} \int_{\Omega} U^+(\widehat{g}^n) \, d\mu - \lim_{n \rightarrow \infty} \int_{\Omega} U^-(\widehat{g}^n) \, d\mu = u(x) < \infty,$$

and note therefore that both integrals in (6.18) are finite.

From Fatou's lemma, we have

$$(6.19) \quad \lim_{n \rightarrow \infty} \int_{\Omega} U^-(\widehat{g}^n) \, d\mu \geq \int_{\Omega} U^-(\widehat{g}(x)) \, d\mu.$$

From Lemma 6.9 we have uniform integrability of $(U^+(\widehat{g}^n))_{n \in \mathbb{N}}$, so that

$$(6.20) \quad \lim_{n \rightarrow \infty} \int_{\Omega} U^+(\widehat{g}^n) \, d\mu = \int_{\Omega} U^+(\widehat{g}(x)) \, d\mu.$$

Thus, using (6.19) and (6.20) in (6.18), we obtain

$$u(x) \leq \int_{\Omega} U(\widehat{g}(x)) \, d\mu,$$

which, combined with the obvious inequality $u(x) \geq \int_{\Omega} U(\widehat{g}(x)) \, d\mu$, yields (6.17). The uniqueness of the primal optimiser follows from the strict concavity of $U(\cdot)$, as does the strict concavity of $u(\cdot)$. For this last claim, fix $x_1 < x_2$ and $\lambda \in (0, 1)$, note that $\lambda\widehat{g}(x_1) + (1 - \lambda)\widehat{g}(x_2) \in \mathcal{C}(\lambda x_1 + (1 - \lambda)x_2)$ (yet must be sub-optimal for $u(\lambda x_1 + (1 - \lambda)x_2)$ as it is not guaranteed to equal $\widehat{g}(\lambda x_1 + (1 - \lambda)x_2)$) and therefore, using the strict concavity of $U(\cdot)$,

$$u(\lambda x_1 + (1 - \lambda)x_2) \geq \int_{\Omega} U(\lambda\widehat{g}(x_1) + (1 - \lambda)\widehat{g}(x_2)) \, d\mu > \lambda u(x_1) + (1 - \lambda)u(x_2).$$

□

We now move on to further characterise the derivatives of the value functions, as well as the primal and dual optimisers. The first result is on the derivative of the primal value function $u(\cdot)$ at zero (equivalently, the derivative of the dual value function $v(\cdot)$ at infinity). The proof of the following lemma is in the style of Kramkov and Schachermayer [21, Lemma 3.5].

Lemma 6.11. *The derivatives of the primal value function in (4.3) at zero and of the dual value function in (4.5) at infinity are given by*

$$(6.21) \quad u'(0) := \lim_{x \downarrow 0} u'(x) = +\infty, \quad -v'(\infty) := \lim_{y \rightarrow \infty} (-v'(y)) = 0.$$

Proof. By the conjugacy result in Lemma 6.7 between the value functions, the assertions in (6.21) are equivalent. We shall prove the second assertion.

The function $-v(\cdot)$ is strictly concave and strictly increasing, so there is a finite non-negative limit $-v'(\infty) := \lim_{y \rightarrow \infty} (-v'(y))$. Because $-V(\cdot)$ is increasing with $\lim_{y \rightarrow \infty} (-V'(y)) = 0$, for any $\epsilon > 0$ there exists a number $C_\epsilon > 0$ such that $-V(y) \leq C_\epsilon + \epsilon y$, $\forall y > 0$. Using this, the $L^1(\mu)$ -boundedness of \mathcal{D} (so that $\int_{\Omega} h \, d\mu \leq y$, $\forall h \in \mathcal{D}(y)$) and l'Hôpital's rule, we have, with $\int_{\Omega} d\mu =: \delta > 0$,

$$\begin{aligned} 0 \leq \lim_{y \rightarrow \infty} -v'(y) &= \lim_{y \rightarrow \infty} \frac{-v(y)}{y} = \lim_{y \rightarrow \infty} \sup_{h \in \mathcal{D}(y)} \int_{\Omega} \frac{-V(h)}{y} \, d\mu \\ &\leq \lim_{y \rightarrow \infty} \sup_{h \in \mathcal{D}(y)} \int_{\Omega} \frac{C_\epsilon + \epsilon h}{y} \, d\mu \\ &\leq \lim_{y \rightarrow \infty} \left(\frac{C_\epsilon \delta}{y} + \epsilon \right) = \epsilon, \end{aligned}$$

and taking the limit as $\epsilon \downarrow 0$ gives the result. \square

The final step in the series of lemmas that will furnish us with the proof of the abstract duality of Theorem 4.5 is to characterise the derivative of the primal value function $u(\cdot)$ at infinity (equivalently, the derivative of the dual value function $v(\cdot)$ at zero) along with a duality characterisation of the primal and dual optimisers.

Lemma 6.12. (1) *The derivatives of the primal value function in (4.3) at infinity and of the dual value function in (4.5) at zero are given by*

$$(6.22) \quad u'(\infty) := \lim_{x \rightarrow \infty} u'(x) = 0, \quad -v'(0) := \lim_{y \downarrow 0} (-v'(y)) = +\infty.$$

(2) *For any fixed $x > 0$, with $y = u'(x)$ (equivalently $x = -v'(y)$), the primal and dual optimisers $\widehat{g}(x), \widehat{h}(y)$ are related by*

$$(6.23) \quad U'(\widehat{g}(x)) = \widehat{h}(y) = \widehat{h}(u'(x)), \quad \mu\text{-a.e.},$$

and satisfy

$$(6.24) \quad \int_{\Omega} \widehat{g}(x) \widehat{h}(y) \, d\mu = xy = xu'(x).$$

(3) *The derivatives of the value functions satisfy the relations*

$$(6.25) \quad xu'(x) = \int_{\Omega} U'(\widehat{g}(x)) \widehat{g}(x) \, d\mu, \quad yv'(y) = \int_{\Omega} V'(\widehat{h}(y)) \widehat{h}(y) \, d\mu, \quad x, y > 0.$$

Proof. Recall the inequality (2.16), which also applies to the value functions because they are also conjugate by Lemma 6.7. We thus have, in addition to (2.16),

$$(6.26) \quad v(y) \geq u(x) - xy, \quad \forall x, y > 0, \quad \text{with equality iff } y = u'(x).$$

With $\widehat{g}(x) \in \mathcal{C}(x)$, $x > 0$ and $\widehat{h}(y) \in \mathcal{D}(y)$, $y > 0$ denoting the primal and dual optimisers, the bipolarity relations (4.7) and (4.8) imply that we have

$$\int_{\Omega} \widehat{g}(x) \widehat{h}(y) \, d\mu \leq xy, \quad x, y > 0.$$

Using this as well as (2.16) and (6.26) we have

$$(6.27) \quad 0 \leq \int_{\Omega} \left(V(\widehat{h}(y)) - U(\widehat{g}(x)) + \widehat{g}(x)\widehat{h}(y) \right) d\mu \leq v(y) - u(x) + xy, \quad x, y > 0,$$

The right-hand-side of (6.27) is zero if and only if $y = u'(x)$, due to (6.26), and the non-negative integrand must then be μ -a.e. zero, which by (2.16) can only happen if (6.23) holds, which establishes that primal-dual relation.

Thus, for any fixed $x > 0$ and with $y = u'(x)$, and hence equality in (6.27), we have

$$\begin{aligned} 0 &= \int_{\Omega} \left(V(\widehat{h}(y)) - U(\widehat{g}(x)) + \widehat{g}(x)\widehat{h}(y) \right) d\mu \\ &= v(y) - u(x) + \int_{\Omega} \widehat{g}(x)\widehat{h}(y) d\mu \\ &= v(y) - u(x) + xy, \quad y = u'(x), \end{aligned}$$

which implies that (6.24) must hold. Inserting the explicit form of $\widehat{h}(y) = U'(\widehat{g}(x))$ into (6.24) yields the first relation in (6.25). Similarly, setting $\widehat{g}(x) = I(\widehat{h}(y)) = -V'(\widehat{h}(y))$ into (6.24), with $x = -v'(y)$ (equivalent to $y = u'(x)$), yields the second relation in (6.25).

It remains to establish the relations in (6.22), which are equivalent assertions. We shall prove the second one. This will use the fact that \mathcal{D} is a subset of $L^1(\mu)$.

From the second relation in (6.25) and the fact that

$$(6.28) \quad \int_{\Omega} gh d\mu \leq xy, \quad \forall g \in \mathcal{C}(x), h \in \mathcal{D}(y), \quad x, y > 0,$$

we see that, for any $y > 0$, we have $-V'(\widehat{h}(y)) \in \mathcal{C}(-v'(y))$. Thus, for any $h \in \mathcal{D}$, (6.28) implies that

$$(6.29) \quad -v'(y) \geq \int_{\Omega} -V'(\widehat{h}(y))h d\mu, \quad \forall h \in \mathcal{D},$$

which we shall make use of shortly.

Since $\mathcal{D}(y)$ is a subset of $L^1(\mu)$, we have $\int_{\Omega} \widehat{h}(y) d\mu \leq y$, and hence

$$(6.30) \quad \int_{\Omega} \frac{\widehat{h}(y)}{y} d\mu \leq 1, \quad \forall y > 0.$$

Using Fatou's lemma in (6.30) we have

$$1 \geq \liminf_{y \downarrow 0} \int_{\Omega} \frac{\widehat{h}(y)}{y} d\mu \geq \int_{\Omega} \liminf_{y \downarrow 0} \left(\frac{\widehat{h}(y)}{y} \right) d\mu,$$

which, given that $\widehat{h}(y)/y$ is non-negative, gives that $\liminf_{y \downarrow 0} (\widehat{h}(y)/y) < \infty$, μ -a.e. Therefore, writing $\widehat{h}(y) =: y\widehat{h}^y$, which defines a unique element $\widehat{h}^y \in \mathcal{D}$, we have

$$\widehat{h}^0 := \liminf_{y \downarrow 0} \widehat{h}^y = \liminf_{y \downarrow 0} \frac{\widehat{h}(y)}{y} < \infty, \quad \mu\text{-a.e.}$$

Using this property and applying Fatou's lemma to (6.29) we obtain, on using $-V'(0) = +\infty$,

$$+\infty \geq \liminf_{y \downarrow 0} (-v'(y)) \geq \liminf_{y \downarrow 0} \int_{\Omega} -V'(y\widehat{h}^y)h d\mu \geq \int_{\Omega} \liminf_{y \downarrow 0} (-V'(y\widehat{h}^y))h d\mu = +\infty,$$

which gives us the second relation in (6.22). □

We have now established all results that give the duality in Theorem 4.5, so let us confirm this.

Proof of Theorem 4.5. Lemma 6.7 implies the relations (4.10) of item (i). The statements in item (ii) are implied by Lemma 6.10 and Lemma 6.5. Items (iii) and (iv) follow from Lemma 6.11 and Lemma 6.12. \square

We are almost ready to prove the concrete duality in Theorem 3.1, because Theorem 4.5 readily implies nearly all of the assertions of Theorem 3.1. The outstanding assertion is the characterisation of the optimal wealth process in (3.2) and the associated uniformly integrable martingale property of the process $\widehat{M} := \widehat{X}(x)\widehat{R}(y) + \int_0^\cdot \widehat{X}_s(x)\widehat{Y}_s(y) d\kappa_s$. So we proceed to establish these assertions in the proposition below, which turns out to be interesting in its own right. We take as given the other assertions of Theorem 3.1, and in particular the optimal budget constraint in (3.1). We shall confirm the proof of Theorem 3.1 in its entirety after the proof of the next result.

Proposition 6.13 (Optimal wealth process). *Given the saturated budget constraint equality in (3.1), the optimal wealth process is characterised by (3.2). The process*

$$\widehat{M}_t := \widehat{X}_t(x)\widehat{R}_t(y) + \int_0^t \widehat{X}_s(x)\widehat{Y}_s(y) d\kappa_s, \quad 0 \leq t < \infty,$$

is a uniformly integrable martingale, converging to an integrable random variable \widehat{M}_∞ , so the martingale extends to $[0, \infty]$. The process $\widehat{X}(x)\widehat{R}(y)$ is a potential, that is, a non-negative supermartingale satisfying $\lim_{t \rightarrow \infty} \mathbb{E}[\widehat{X}_t(x)\widehat{R}_t(y)] = 0$. Moreover, $\widehat{X}_\infty(x)\widehat{R}_\infty(y) = 0$, almost surely.

Proof. It simplifies notation if we take $x = y = 1$, and is without loss of generality: although $y = u'(x)$ in (3.1), one can always multiply the utility function by an arbitrary constant so as to ensure that $u'(1) = 1$. We thus have the optimal budget constraint

$$(6.31) \quad \mathbb{E} \left[\int_0^\infty \widehat{X}_t \widehat{Y}_t d\kappa_t \right] = 1,$$

for $\widehat{X} \equiv \widehat{X}(1) \in \mathcal{X}$ and $\widehat{Y} \equiv \widehat{Y}(1) \in \mathcal{Y}$. Since $\widehat{X} \in \mathcal{X}$, we know there exists an optimal wealth process $\widehat{X} \equiv \widehat{X}(1)$ and an associated optimal trading strategy \widehat{H} , such that $\widehat{X} = 1 + (\widehat{H} \cdot P) \geq 0$, and such that $\widehat{M} := \widehat{X}\widehat{R} + \int_0^\cdot \widehat{X}_s \widehat{Y}_s d\kappa_s$ is a supermartingale over $[0, \infty)$. The supermartingale condition, by the same arguments that led to the derivation of the budget constraint in Lemma 2.5, leads to the inequality $\mathbb{E} \left[\int_0^\infty \widehat{X}_t \widehat{Y}_t d\kappa_t \right] \leq 1$ instead of the equality (6.31). Similarly, if the supermartingale is strict, we get a strict inequality in place of (6.31). We thus deduce that \widehat{M} must be a martingale over $[0, \infty)$. We shall show that this extends to $[0, \infty]$, along with the other claims in the lemma.

Since \widehat{M} is a martingale, the (non-negative càdlàg) deflated wealth process $\widehat{X}\widehat{R}$ is a martingale minus a non-decreasing process, so is a non-negative càdlàg supermartingale, and thus (by Cohen and Elliott [5, Corollary 5.2.2], for example) converges to an integrable limiting random variable $\widehat{X}_\infty \widehat{R}_\infty := \lim_{t \rightarrow \infty} \widehat{X}_t \widehat{R}_t$ (and moreover $\widehat{X}_t \widehat{R}_t \geq \mathbb{E}[\widehat{X}_\infty \widehat{R}_\infty], t \geq 0$). The non-decreasing integral in \widehat{M} clearly also converges to an integrable random variable, by virtue of the budget constraint. Thus, \widehat{M} also converges to an integrable random variable $\widehat{M}_\infty := \widehat{X}_\infty \widehat{R}_\infty + \int_0^\infty \widehat{X}_t \widehat{Y}_t d\kappa_t$. By Protter [30, Theorem I.13], the extended martingale over $[0, \infty]$, $(\widehat{M}_t)_{t \in [0, \infty]}$ is then uniformly integrable, as claimed.

The martingale condition gives

$$\mathbb{E} \left[\widehat{X}_t \widehat{R}_t + \int_0^t \widehat{X}_s \widehat{Y}_s d\kappa_s \right] = 1, \quad 0 \leq t < \infty.$$

Taking the limit as $t \rightarrow \infty$, using monotone convergence in the second term within the expectation and utilising (6.31) yields

$$\lim_{t \rightarrow \infty} \mathbb{E}[\widehat{X}_t \widehat{R}_t] = 0,$$

so that $\widehat{X}\widehat{R}$ is a potential, as claimed.

Using the uniform integrability of \widehat{M} and taking the limit as $t \rightarrow \infty$ in $\mathbb{E}[\widehat{M}_t] = 1$, $t \geq 0$, we have

$$1 = \lim_{t \rightarrow \infty} \mathbb{E}[\widehat{M}_t] = \mathbb{E} \left[\lim_{t \rightarrow \infty} \widehat{M}_t \right] = \mathbb{E}[\widehat{X}_\infty \widehat{R}_\infty] + 1,$$

on using (6.31). Hence, we get $\mathbb{E}[\widehat{X}_\infty \widehat{R}_\infty] = 0$ and, since $\widehat{X}_\infty \widehat{R}_\infty$ is non-negative, we deduce that $\widehat{X}_\infty \widehat{R}_\infty = 0$, almost surely as claimed.

We can now assemble these ingredients to arrive at the optimal wealth process formula (3.2). Applying the martingale condition again, this time over $[t, u]$ for some $t \geq 0$, we have

$$\mathbb{E} \left[\widehat{X}_u \widehat{R}_u + \int_0^u \widehat{X}_s \widehat{Y}_s d\kappa_s \middle| \mathcal{F}_t \right] = \widehat{X}_t \widehat{R}_t + \int_0^t \widehat{X}_s \widehat{Y}_s d\kappa_s, \quad 0 \leq t \leq u < \infty.$$

Taking the limit as $u \rightarrow \infty$ and using the uniform integrability of \widehat{M} we obtain

$$\mathbb{E} \left[\lim_{u \rightarrow \infty} \left(\widehat{X}_u \widehat{R}_u + \int_0^u \widehat{X}_s \widehat{Y}_s d\kappa_s \right) \middle| \mathcal{F}_t \right] = \widehat{X}_t \widehat{R}_t + \int_0^t \widehat{X}_s \widehat{Y}_s d\kappa_s, \quad t \geq 0,$$

which, on using $\widehat{X}_\infty \widehat{R}_\infty = 0$, re-arranges to

$$\widehat{X}_t \widehat{R}_t = \mathbb{E} \left[\int_t^\infty \widehat{X}_s \widehat{Y}_s d\kappa_s \middle| \mathcal{F}_t \right], \quad t \geq 0,$$

which establishes (3.2), and the proof is complete. \square

Proof of Theorem 3.1. Given the definitions of the sets $\mathcal{C}(x)$ and $\mathcal{D}(y)$ in (4.2) and (4.4), respectively, and the identification of the abstract value functions in (4.3) and (4.5) with their concrete counterparts in (2.6) and (2.18), Theorem 4.5 implies all the assertions of Theorem 3.1, with the exception of the optimal wealth process formula (3.2) and the uniform integrability of $\widehat{M} := \widehat{X}(x)\widehat{R}(y) + \int_0^\cdot \widehat{X}_s(x)\widehat{Y}_s(y) d\kappa_s$, which are established by Proposition 6.13. \square

7. EXAMPLES

We end with two examples. The first uses an incomplete market model with strict local martingale deflators, which is covered in our framework. The market features a three-dimensional Bessel process for the market price of risk (MPR) of a stock which also has a stochastic volatility. We consider the problem 2.6 with the measure κ satisfying $d\kappa_t = \exp(-\alpha t) dt$ for a constant discount rate $\alpha > 0$, so that

$$(7.1) \quad \kappa_t = \frac{1}{\alpha} (1 - e^{-\alpha t}), \quad t \geq 0.$$

Since κ is absolutely continuous with respect to Lebesgue measure, we use the formalism in Remark 3.3. We then specialise the example to the Black-Scholes model, to confirm that we obtain results consistent with the example presented by Bouchard and Pham [2, Section 4]. The market is of course complete in this simple case.

We shall use a constant relative risk aversion (CRRA) utility function of the power form:

$$(7.2) \quad U(x) = \frac{x^p}{p}, \quad p < 1, p \neq 0, \quad x \in \mathbb{R}_+.$$

The case $p = 0$ corresponds formally to logarithmic utility, $U(x) = \log(x)$, and setting $p = 0$ in the results for the power utility function does indeed recover the results for logarithmic utility, as can be verified by carrying out the analysis directly for that case.

Example 7.1 (Three-dimensional Bessel process MPR, with stochastic volatility and correlation). Take an infinite horizon complete stochastic basis $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, with \mathbb{F} satisfying the usual hypotheses. Let (W, W^\perp) be a two-dimensional Brownian motion. We take \mathbb{F} to be the augmented filtration generated by (W, W^\perp) .

Let B denote the process which solves the stochastic differential equation

$$dB_t = \frac{1}{B_t} dt + dW_t =: \lambda_t dt + dW_t, \quad B_0 = 1.$$

The process B is the so-called three-dimensional Bessel process. The process $\lambda := 1/B$ will be the market price of risk of a stock with price process P and stochastic volatility process $\sigma > 0$, driven by the correlated Brownian motion $\widetilde{W} := \rho W + \sqrt{1 - \rho^2} W^\perp$, and with $\rho \in [-1, 1]$ some \mathbb{F} -adapted stochastic correlation. We need not specify the dynamics of σ or ρ any further for the purposes of the example. The stock price dynamics are given by

$$dP_t = \sigma_t P_t dB_t = \sigma_t P_t (\lambda_t dt + dW_t).$$

Note that this model satisfies the so-called *structure condition* of Pham et al [29], because P admits the decomposition $P = P_0 + L + A$ with $L \in \mathcal{M}_{0, \text{loc}}^2$ a locally square-integrable local martingale null at zero and A a predictable process of finite variation null at zero, and such that $A = \int_0^\cdot \widehat{\lambda}_s d\langle L \rangle_s$ for a predictable process $\widehat{\lambda}$.

Take a constant relative risk aversion (CRRA) utility function as in (7.2), with the measure κ given by (7.1), so that $\gamma_t = e^{\alpha t}$, $t \geq 0$. The primal value function is

$$u(x) := \sup_{X \in \mathcal{X}(x)} \mathbb{E} \left[\int_0^\infty e^{-\alpha t} U(X_t) dt \right], \quad x > 0.$$

The wealth process satisfies

$$(7.3) \quad dX_t = \sigma_t \pi_t (\lambda_t dt + dW_t), \quad X_0 = x,$$

where $\pi = HS$ is the trading strategy expressed in terms of the wealth placed in the stock, with H the process for the number of shares.

With $\mathcal{E}(\cdot)$ denoting the stochastic exponential, the supermartingale deflators in this model are given by local martingale deflators of the form

$$(7.4) \quad Z := \mathcal{E}(-\lambda \cdot W - \psi \cdot W^\perp),$$

for an arbitrary process ψ satisfying $\int_0^t \psi_s^2 ds < \infty$ almost surely for all $t \geq 0$, with each such ψ leading to a different deflator: this market is of course incomplete. Let Ψ denote the set of such integrands ψ . In the case that σ and ρ are deterministic, the market is complete and there is a unique local martingale deflator $Z^{(0)} := \mathcal{E}(-\lambda \cdot W)$. It is well-known (see for instance Larsen [24, Example 2.2]) that $Z^{(0)}$ is a strict local martingale and, what is more, that $Z^{(0)} = \lambda$ and that λ is square integrable. The strict local martingale property is inherited by Z in (7.4), for any choice of integrand ψ .

The supermartingales $R \in \mathcal{R}$ are given by $R = \exp(-\int_0^\cdot \beta_s ds) Z$ and the inter-temporal wealth deflators $Y \in \mathcal{Y}$ by $Y = \beta R$, that is,

$$(7.5) \quad Y_t = \beta_t \exp\left(-\int_0^t \beta_s ds\right) Z_t, \quad t \geq 0,$$

with $\beta \in \mathcal{B}$, so $\int_0^\cdot \beta_s ds < \infty$ almost surely. The process $M := XR + \int_0^\cdot X_s Y_s ds$ is given as

$$(7.6) \quad M_t := X_t R_t + \int_0^t X_s Y_s ds = x + \int_0^t R_s (\sigma_s \pi_s - \lambda_s X_s) dW_s - \int_0^t X_s R_s \psi_s dW_s^\perp, \quad t \geq 0,$$

which is a non-negative local martingale and thus a supermartingale.

The convex conjugate of the utility function is $V(y) := -y^q/q$, $y > 0$, where $q < 1$, $q \neq 0$ is the conjugate variable to p , satisfying $1 - q = (1 - p)^{-1}$. The dual value function is given by

$$v(y) := \inf_{Y \in \mathcal{Y}} \mathbb{E} \left[\int_0^\infty e^{-\alpha t} V(yY_t e^{\alpha t}) dt \right], \quad y > 0.$$

The dual minimisation involves both an optimisation over the local martingale deflators $Z \in \mathcal{Z}$ as well as over the auxiliary dual control $\beta \in \mathcal{B}$, since the wealth-path deflators $Y \in \mathcal{Y}$ are given by (7.5).

Denote the unique dual minimiser by $\widehat{Y} \in \mathcal{Y}$, given by

$$\widehat{Y} = \widehat{\beta} \exp \left(- \int_0^\cdot \widehat{\beta}_s ds \right) \widehat{Z} = \widehat{\beta} \widehat{R},$$

where $\widehat{\beta} \in \mathcal{B}$ is the optimal auxiliary dual control, $\widehat{R} \in \mathcal{R}$ denotes the optimal incarnation of the supermartingale R and \widehat{Z} denotes the optimal local martingale deflator, given by

$$\widehat{Z} := \mathcal{E}(-\lambda \cdot W - \widehat{\psi} \cdot W^\perp),$$

for some optimal integrand $\widehat{\psi}$ in (7.4). For use below, define the non-negative martingale H by

$$(7.7) \quad H_t := \mathbb{E} \left[\int_0^\infty e^{-\alpha(1-q)s} \widehat{Y}_s^q ds \middle| \mathcal{F}_t \right], \quad t \geq 0.$$

Using (3.4), the optimal wealth process is given by

$$(7.8) \quad (\widehat{X}_t(x))^{-(1-p)} = u'(x) e^{\alpha t} \widehat{Y}_t, \quad t \geq 0.$$

By (3.5) the optimisers satisfy the saturated budget constraint

$$(7.9) \quad \mathbb{E} \left[\int_0^\infty \widehat{X}_t(x) \widehat{Y}_t dt \right] = x.$$

The relations (7.8) and (7.9) yield

$$(7.10) \quad \widehat{X}_t(x) = \frac{x}{H_0} e^{-\alpha(1-q)t} \widehat{Y}_t^{-(1-q)}, \quad t \geq 0.$$

Using the result (7.10) in the right-hand-side of (3.6), the optimal wealth process then also satisfies

$$\widehat{X}_t(x) \widehat{R}_t = \frac{x}{H_0} \mathbb{E} \left[\int_t^\infty e^{-\alpha(1-q)s} \widehat{Y}_s^q ds \middle| \mathcal{F}_t \right], \quad t \geq 0.$$

More pertinently, the optimal martingale \widehat{M} , corresponding to the process in (7.6) at the optimum, is computed as

$$(7.11) \quad \widehat{M}_t := \widehat{X}_t(x) \widehat{R}_t + \int_0^t \widehat{X}_s(x) \widehat{Y}_s ds = \frac{x}{H_0} H_t, \quad t \geq 0,$$

so is indeed a martingale.

By martingale representation, \widehat{M} will have a stochastic integral representation which, without loss of generality, can be written in the form

$$(7.12) \quad \widehat{M}_t = x + \int_0^t \widehat{R}_s \widehat{X}_s(x) (\varphi_s - q\lambda_s) dW_s + \int_0^t \widehat{R}_s \widehat{X}_s(x) \xi_s dW_s^\perp, \quad t \geq 0,$$

for some integrands φ, ξ . Comparing with the representation in (7.6) at the optimum yields the optimal trading strategy in terms of the optimal portfolio proportion $\widehat{\theta} := \widehat{\pi} / \widehat{X}(x)$, and

the optimal integrand $\widehat{\psi}$, as

$$(7.13) \quad \widehat{\theta}_t := \frac{\widehat{\pi}_t}{\widehat{X}_t(x)} = \frac{\lambda_t}{\sigma_t(1-p)} + \frac{\varphi_t}{\sigma_t}, \quad \widehat{\psi}_t = -\xi_t, \quad t \geq 0.$$

In particular, the process φ records the correction to the Merton-type strategy $\lambda/(\sigma(1-p))$ due to the stochastic volatility and correlation.

This is as far as one can go without computing explicitly the dual minimiser \widehat{Y} , which is typically impossible in closed form for power utility, except for some special cases such as a Black-Scholes model (as we shall show further below).

For the special case of logarithmic utility, one can set $p = 0$ and $q = 0$ in the results for power utility, which gives that $H = 1/\alpha$ is constant, and so $\widehat{M} = x$ is also constant, yielding

$$\widehat{\theta}_t = \frac{\lambda_t}{\sigma_t}, \quad \widehat{\psi}_t = 0, \quad t \geq 0,$$

giving the classic myopic trading strategy for logarithmic utility (and the correction to the Merton strategy satisfies $\varphi = q\lambda = 0$ for $q = 0$, as it should).

In particular, since $\widehat{\psi} \equiv 0$, the dual optimiser is given as

$$(7.14) \quad \widehat{Y} = \widehat{\beta} \exp\left(-\int_0^\cdot \widehat{\beta}_s ds\right) Z^{(0)},$$

for some optimal auxiliary dual control $\widehat{\beta} \in \mathcal{B}$, with $Z^{(0)} = \mathcal{E}(-\lambda \cdot W)$ the minimal local martingale deflator. Moreover, setting $q = 0$ in (7.10) and using $H = 1/\alpha$ gives the optimal wealth process in the form

$$(7.15) \quad \widehat{X}_t(x) = \frac{\alpha x e^{-\alpha t}}{\widehat{Y}_t}, \quad t \geq 0.$$

But, using the optimal strategy $\widehat{\pi} = (\lambda/\sigma)\widehat{X}(x)$ in the wealth SDE (7.3), we also compute that

$$(7.16) \quad \widehat{X}_t(x) = \frac{x}{Z_t^{(0)}}, \quad t \geq 0.$$

Equating the two expressions for $\widehat{X}(x)$ in (7.15) and (7.16), and then using (7.14), yields that the optimal auxiliary dual control is also constant, and given by

$$(7.17) \quad \widehat{\beta}_t = \alpha, \quad t \geq 0.$$

These results for logarithmic utility can of course be obtained by going directly through the analysis from scratch in the manner above. Indeed, one can directly compute the dual value function, as follows. Using the definition (3.3) along with $V(y) = -(1 + \log(y))$ for logarithmic utility, one expresses the dual value function as

$$v(y) = \frac{1}{\alpha} (V(y) - 1) + \inf_{\beta \in \mathcal{B}, \psi \in \Psi} \mathbb{E} \left[\int_0^\infty e^{-\alpha t} \left(\int_0^t (\beta_s + \frac{1}{2}(\lambda_s^2 + \psi_s^2)) ds - \log(\beta_t) \right) dt \right].$$

The optimisations over ψ and β can be carried out separately. Clearly, the term involving ψ is minimised by $\widehat{\psi} \equiv 0$, while an integration by parts in the remaining integrals yields

$$v(y) = \frac{1}{\alpha} (V(y) - 1) + \inf_{\beta \in \mathcal{B}} \mathbb{E} \left[\int_0^\infty e^{-\alpha t} \left(\frac{1}{2\alpha} \lambda_t^2 + \frac{\beta_t}{\alpha} - \log(\beta_t) \right) dt \right].$$

The minimisation over β can then be carried out pointwise, yielding (7.17) and giving the dual optimiser for logarithmic utility: $\widehat{Y}_t = \alpha \exp(-\alpha t) Z_t^{(0)}$, $t \geq 0$ as before. Using this dual optimiser in (3.4) gives (7.16).

Example 7.2 (Black-Scholes model, CRRA utility). If we specialise Example 7.1 to the case where λ and σ are constant, we are in a Black-Scholes market and the computations for power utility can be carried out explicitly. We show this in order to verify that our formalism reproduces the results of the example in Bouchard and Pham [2, Section 4]. The market is now complete, and there is a unique local martingale deflator given by $Z = \mathcal{E}(-\lambda W)$. The wealth-path deflators take the form

$$Y = \beta \exp\left(-\int_0^\cdot \beta_s ds\right) Z,$$

for some $\beta \in \mathcal{B}$.

With this structure, the same method as for Example 7.1 yields the same representation (7.10) for the optimal wealth process, where in this case the dual minimiser is given by

$$(7.18) \quad \widehat{Y} = \widehat{\beta} \exp\left(-\int_0^\cdot \widehat{\beta}_s ds\right) \mathcal{E}(-\lambda W),$$

for some optimal auxiliary dual control $\widehat{\beta} \in \mathcal{B}$, and the martingale H in (7.7) has the same representation with the dual minimiser in (7.18) in place.

The process M of (7.6) is this time given by the same expression but without the integral involving ψ , so we have

$$M_t := X_t R_t + \int_0^t X_s Y_s ds = x + \int_0^t R_s (\sigma \pi_s - \lambda X_s) dW_s, \quad t \geq 0.$$

The optimal martingale \widehat{M} once again has the representation in (7.11), and has a stochastic integral representation of the form in (7.12) but without the integral with respect to W^\perp , and we once again find an expression of the form in (7.13) for the optimal trading strategy. Our goal is to now compute the dual minimiser, by computing $\widehat{\beta}$, and to thus show that the correction φ to the Merton strategy is zero in this case.

To compute $\widehat{\beta}$ we examine the dual value function, which is expressed in the form

$$v(y) = \inf_{\beta \in \mathcal{B}} V(y) \mathbb{E} \left[\int_0^\infty \exp\left(-\alpha(1-q)t - q \int_0^t \beta_s ds\right) \beta_t^q Z_t^q dt \right].$$

Given the constant parameters of the model, one now makes the (not unreasonable) ansatz that $\widehat{\beta}$ is deterministic, and in fact constant. With this conjecture, one passes the expectation inside the integral, uses

$$(7.19) \quad \mathbb{E}[Z_u^q | \mathcal{F}_t] = \mathcal{E}(-q\lambda W)_t \exp\left(-\frac{1}{2}q(1-q)\lambda^2 u\right), \quad 0 \leq t \leq u,$$

and computes the resultant expression to arrive at

$$v(y) = \inf_{\beta} V(y) \left(\frac{\beta^q}{q\beta + (1-q)(\alpha + \frac{1}{2}q\lambda^2)} \right).$$

Straightforward differentiation gives the (constant) optimiser as

$$\widehat{\beta} = \alpha + \frac{1}{2}q\lambda^2,$$

and (7.18) then gives the dual minimiser. With this in place, one expresses the martingale H in the form

$$H_t = \left(\alpha + \frac{1}{2}q\lambda^2\right)^q \mathbb{E} \left[\int_0^\infty \exp\left(\left(\alpha + \frac{1}{2}q\lambda^2\right)u\right) Z_u^q | \mathcal{F}_t \right], \quad t \geq 0.$$

Once again, we take the expectation inside the integral and use (7.19), and we arrive at

$$H_t = \left(\alpha + \frac{1}{2}q\lambda^2 \right)^{-(1-q)} \mathcal{E}(-q\lambda W)_t, \quad t \geq 0.$$

This in turn yields that the optimal martingale \widehat{M} is given by

$$\widehat{M}_t = x \frac{H_t}{H_0} = x \mathcal{E}(-q\lambda W)_t, \quad t \geq 0,$$

and the optimal wealth process is given by the representation (7.10) as

$$\widehat{X}_t(x) = x \frac{\mathcal{E}(-q\lambda W)_t}{Z_t}, \quad t \geq 0.$$

Thus, the processes \widehat{M} , $\widehat{X}(x)$ are related according to

$$\widehat{X}_t(x)Z_t = \widehat{M}_t, \quad t \geq 0.$$

We can now compute the optimal trading strategy. Using the dynamics of the wealth process for any strategy π , given by (7.3) with constant parameters, we have that

$$(7.20) \quad X_t Z_t = x + \int_0^t (\sigma \pi_s - \lambda X_s) dW_s, \quad t \geq 0.$$

On the other hand, at the optimum, since $\widehat{X}(x)Z = x\mathcal{E}(-q\lambda W)$, we have

$$(7.21) \quad \widehat{X}_t(x)Z_t = x - q\lambda \int_0^t \widehat{X}_s Z_s dW_s, \quad t \geq 0.$$

Equating (7.20) at the optimum with (7.21) gives the optimal trading strategy as

$$\widehat{\theta}_t \equiv \frac{\widehat{\pi}_t}{\widehat{X}_t(x)} = \frac{\lambda}{\sigma(1-p)}, \quad t \geq 0,$$

so the optimal strategy is the Merton strategy, as expected.

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