

DUALITY FOR OPTIMAL CONSUMPTION UNDER NO UNBOUNDED PROFIT WITH BOUNDED RISK

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We give a definitive treatment of duality for optimal consumption over the infinite horizon, in a semimartingale incomplete market satisfying no unbounded profit with bounded risk (NUPBR). Rather than base the dual domain on (local) martingale deflators, we use a class of supermartingale deflators such that deflated wealth plus cumulative deflated consumption is a supermartingale for all admissible consumption plans. This yields a strong duality, because the enlarged dual domain of processes dominated by deflators is naturally closed, without invoking its closure. In this way, we automatically reach the bipolar of the set of deflators. We complete this picture by proving that the set of processes dominated by local martingale deflators is dense in our dual domain, confirming that we have identified the natural dual space. In addition to the optimal consumption and deflator, we characterise the optimal wealth process. At the optimum, deflated wealth is a supermartingale and a potential, while deflated wealth plus cumulative deflated consumption is a uniformly integrable martingale. This is the natural generalisation of the corresponding feature in the terminal wealth problem, where deflated wealth at the optimum is a uniformly integrable martingale. We use no constructions involving equivalent local martingale measures. This is natural, given that such measures typically do not exist over the infinite horizon and that we are working under NUPBR, which does not require their existence. The structure of the duality proof reveals an interesting feature compared with the terminal wealth problem. There, the dual domain is L^1 -bounded, but here the primal domain has this property, and hence many steps in the duality proof show a marked reversal of roles for the primal and dual domains, compared with the proofs of Kramkov and Schachermayer (*Ann. Appl. Probab.* **9** (1999) 904–950; *Ann. Appl. Probab.* **13** (2003) 1504–1516).

1. Introduction. This paper gives a definitive treatment of duality for the optimal consumption and investment problem for an agent maximising cumulative discounted utility from consumption over an infinite horizon. This problem has a long history, first being solved in a constant coefficient complete Brownian model by Merton [23] using dynamic programming methods. The same model was studied in great detail, considering also issues such as nonnegativity constraints on consumption, and bankruptcy, by Karatzas et al [13] using similar methods. Duality methods for a finite horizon version of the problem to maximise utility from consumption and terminal wealth, in a complete Itô process market, were developed by Karatzas, Lehoczky and Shreve [14]. The infinite horizon problem for utility from consumption in a complete Itô market was treated via duality methods by Huang and Pagès [10]. In an incomplete Itô market, duality methods for the finite horizon problem of maximising utility from terminal wealth were developed in a seminal paper by Karatzas et al. [15]. These methods were extended to finite horizon problems including consumption and portfolio constraints (including market incompleteness) by Cvitanić and Karatzas [5] and Shreve and Xu [33]. Duality methods in an incomplete market with general semimartingale asset prices were

Received April 2021; revised September 2021.

MSC2020 subject classifications. Primary 91G10, 49J55; secondary 49K45.

Key words and phrases. Duality, utility from consumption, supermartingale deflator.

then developed for the terminal wealth problem in a masterly contribution by Kramkov and Schachermayer [19, 20].

The finite horizon consumption problem in a semimartingale market, under the no-arbitrage condition of No Free Lunch with Vanishing Risk (NFLVR), was given a dual treatment by Karatzas and Žitković [17] (who also incorporated a random endowment), building on earlier work by Žitković [34]. The infinite horizon consumption problem remained an open problem to treat via duality methods until fairly recently, when a significant advance was made by Mostovyi [24]. Working under NFLVR, Mostovyi [24] was able to show that most of the tenets of duality theory for utility maximisation, as espoused by Kramkov and Schachermayer [19, 20], do hold true for the infinite horizon consumption problem. This was extended by Chau et al. [3] to cover the case where the no-arbitrage condition was weakened to the No Unbounded Profit with Bounded Risk (NUPBR) condition, so that one need not insist on the existence of equivalent local martingale measures (ELMMs). This is a general observation, first made in explicit terms by Karatzas and Kardaras [12], that all one needs for a well-posed utility maximisation problem is the existence of a suitable class of dual variables, or deflators, which need not be densities of ELMMs, and which multiplicatively deflate primal variables to create local martingales or supermartingales. This fact was implicit in Karatzas et al. [15], which did not use ELMMs at all, and to some extent was an underlying theme in the work of Kramkov and Schachermayer [19, 20] who, despite working under NFLVR (so ELMMs were definitively assumed to exist), expanded the dual domain to a class of supermartingale deflators and found counterexamples where the dual minimiser was not the density of an ELMM. We note that in both Mostovyi [24] and Chau et al. [3] the formulation could encompass other problems, by varying the measure (a stochastic clock) that was used to aggregate utility from consumption over time. (These papers also incorporated the stochastic clock into the wealth dynamics, which amounts to a change of variable from a traditional consumption rate, and we shall say more on this below.) By varying this clock the approach in [3, 24] can treat the finite horizon utility from consumption problem, the terminal wealth problem, as well as the finite horizon problem of utility from both consumption and terminal wealth.

Given the above history, it is as well to point out where there is still work to do and, as this is the focus of this paper, let us now turn to this and describe the contribution.

First, we obtain a stronger duality statement than in Mostovyi [24] and Chau et al. [3], in the following sense. In [24] and [3], the initial dual domain was based either on martingale deflators (in [24], working under NFLVR) or on local martingale deflators (in [3], working under NUPBR). The dual domain was then defined as the closure (in an appropriate topology) of processes dominated by some element of the set of deflators in question. The authors of [3, 24] were forced into taking the aforementioned closure in order to obtain a closed dual domain, which could then be shown to be the bipolar of the original domain of deflators, and thus also the polar of the primal domain. Contrast this with the result of Kramkov and Schachermayer [19], Lemma 4.1, in the terminal wealth problem. There, one begins with a dual domain of supermartingales (such that deflated admissible wealth is a supermartingale for all strategies), then enlarges this domain to consider random variables dominated by the terminal value of some deflator. No closure is taken, but it is nevertheless shown that the enlarged dual domain is naturally closed, so one reaches the bipolar of the set of deflators, and perfect bipolarity between the primal and dual domains is achieved. Herein lies our first contribution: we are able to extend the prescription of Kramkov and Schachermayer [19]. First, we base our dual domain on a set of supermartingales, this time such that *deflated wealth plus cumulative deflated consumption is a supermartingale* for all admissible consumption plans. Then, again in the spirit of [19], we enlarge the dual domain to encompass processes dominated by the deflators. Crucially, no closure needs to be taken. We show that the enlarged

dual domain is closed in the appropriate topology, so that we reach the bipolar of the original domain of supermartingales and obtain the duality between the primal and dual optimisation problems without having to take a closure in defining the enlarged dual domain. Finally, we show that our enlarged dual domain coincides with the closure of processes dominated by local martingale deflators, that is, the dual domain used in Chau et al. [3]. Thus, the set of processes dominated by local martingale deflators is dense in our dual domain. This result (Proposition 5.1) is confirmation that we have chosen the dual domain in just the right way to achieve a strong duality statement. The underlying bipolarity results are obtained by exploiting the Stricker and Yan [31] version of the Optional Decomposition Theorem (ODT), which uses deflators rather than ELMs, so we do not use any constructions whatsoever involving equivalent measures. We shall say more on this aspect very shortly.

The second strengthening of the results in Mostovyi [24] and Chau et al. [3] is fundamental. In addition to the optimal consumption, we characterise the associated optimal wealth process (and by extension the optimal strategy). Somewhat surprisingly, neither of [24] or [3] (or the earlier works [17, 34]) made any statement whatsoever regarding the optimal wealth. This turns out to be a satisfying analysis which shows that, at the optimum, deflated wealth is a supermartingale and also a potential, decaying to zero, while deflated wealth plus cumulative deflated consumption at the optimum is a uniformly integrable martingale. This is natural, though to the best of our knowledge has not been shown before in a general semimartingale infinite horizon consumption problem. It is the natural generalisation of the Kramkov and Schachermayer [19, 20] terminal wealth result that, at the optimum, deflated wealth is transformed from a supermartingale to a uniformly integrable martingale.

The next aspect of our work concerns the use of, or more accurately the avoidance of, any constructions involving ELMs. We are working on an infinite horizon, and it is well known that in this case hardly any models will admit ELMs, because the candidate change of measure density is not a uniformly integrable martingale over the infinite timescale. While this can be dealt with, by (e.g.) eliminating the tail σ -algebra in some way when wishing to use equivalent measures restricted to a finite horizon σ -field, we bypass any such pitfalls by exploiting the Stricker and Yan [31] version of the ODT and so avoiding ELMs. As we are working under NUPBR, where ELMs might not exist at all (a case in point is a stock driven by a three-dimensional Bessel process, which we use in an example of a utility maximisation problem in our framework in Section 8), it is natural to construct proofs which avoid any use of ELMs if possible, and this is what we do.

Finally, the proof of the main duality theorem in our approach reveals an interesting structure of the consumption problem compared with the terminal wealth problem. In contrast to [3, 24], we do not incorporate a stochastic clock into the wealth dynamics, so our consumption rate is with respect to calendar time. The change of variable used in [3, 24] was convenient in those papers, as it allowed the authors to assume that a constant “consumption” stream was allowed. This amounts to, in essence, a decaying real consumption rate. (It is manifestly the case that with a true consumption rate, one cannot guarantee being able to consume at a constant rate for ever.) By choosing to work with the real consumption rate, two aspects of the problem’s underlying structure emerge. First, it naturally leads to the correct supermartingale constraint that one should apply at the outset: that deflated wealth plus cumulative deflated consumption is a supermartingale. This leads to the correct choice of dual domain. Second, it reveals a role reversal for the primal and dual domains compared with the terminal wealth problem of Kramkov and Schachermayer [19, 20]. In [19, 20], because the constant wealth $X^0 \equiv 1$ is admissible, the dual domain is bounded in $L^1(\mathbb{P})$. But in the consumption problem it is the *primal* domain that is bounded in L^1 (with respect to an appropriate measure). This role reversal of the primal and dual domains then manifests itself in the proofs. In numerous steps of the program, a method that works for the primal domain in

[19, 20] is diverted to the dual domain here, and vice versa. A prime example is the proof of conjugacy of the value functions: in the terminal wealth problem one creates a compact subset of the primal domain so as to apply the minimax theorem, and proves that the dual value function is the convex conjugate of the primal value function. Here, instead, one creates the compact subset in the dual domain, and applies a transformed minimax theorem (replacing maximisation with minimisation, and a concave function with a convex one, and so on) and proves that the primal value function is the concave conjugate of the dual value function. There are many other instances of this role reversal, which will be pointed out in the course of the proof of the duality theorems in Section 7. In view of these facets, we choose to give a complete and self-contained treatment of the duality proofs in their entirety.

The rest of the paper is structured as follows. In Section 2 we describe the financial market, the admissible consumption plans, and the class of dual variables (consumption deflators), alongside the alternatives such as local martingale deflators. In Section 3 we formulate the primal and dual problems. The main duality theorem (Theorem 4.1) is given in Section 4. In Section 5 we give an abstract version of the bipolarity relations (Proposition 5.5) between suitably defined primal and dual domains, an associated abstract version of the duality theorem (Theorem 5.6), and state Proposition 5.1, that the set of processes dominated by local martingale deflators is dense in the set of processes dominated by consumption deflators. The bipolarity relations are proven in Section 6 by considering the infinite horizon budget constraint for consumption, and showing that it is both a necessary and sufficient condition for admissibility. Here, we complete the discussion on ramifications of using an alternative choice of dual domain based on local martingale deflators, and prove Proposition 5.1. In Section 7, we prove the abstract duality, then establish Proposition 7.14 characterising the optimal wealth process, followed by the concrete duality theorem. In Section 8, we give an example with power utility and a stock driven by a three-dimensional Bessel process, with stochastic volatility and correlation, for which the dual minimiser is a strict local martingale, fitting well into our earlier program.

2. The market. We have an infinite horizon financial market containing d stocks and a cash asset, on a complete stochastic basis $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, with the filtration \mathbb{F} satisfying the usual conditions of right-continuity and augmentation with the \mathbb{P} -null sets of \mathcal{F} . We shall use the cash asset as numéraire, so work with discounted quantities. The (discounted) stock price vector is given by a positive d -dimensional càdlàg semimartingale $S = (S^1, \dots, S^d)$.

An agent with initial capital $x > 0$ can trade the stocks and cash and may consume at a nonnegative càdlàg adapted rate $c = (c_t)_{t \geq 0}$, assumed to satisfy the minimal condition $\int_0^t c_s ds < \infty$, almost surely, $\forall t \geq 0$. The associated wealth process is X , given by

$$(2.1) \quad X_t = x + (H \cdot S)_t - \int_0^t c_s ds, \quad t \geq 0, x > 0.$$

In (2.1), $(H \cdot S)$ denotes the stochastic integral and the trading strategy H is a predictable S -integrable vector process for the number of units of each stock held. Write

$$C_t := \int_0^t c_s ds, \quad t \geq 0,$$

for the nondecreasing cumulative consumption process. Then, with

$$(2.2) \quad X^0 := x + (H \cdot S)$$

denoting the wealth process of a self-financing portfolio corresponding to strategy H , we have the decomposition

$$(2.3) \quad X = X^0 - C.$$

2.1. *Admissible consumption plans.* We will assume solvency at all times, so $X \geq 0$ almost surely in (2.1). In this case, for a given $x > 0$, we call the pair (H, c) (or (X, c)) an x -admissible investment-consumption strategy. If, for a consumption process c we can find a predictable S -integrable process H such that (H, c) is an x -admissible investment-consumption strategy, then we say that c is an x -admissible consumption process or, briefly, an admissible consumption plan. Denote the set of x -admissible consumption plans by $\mathcal{A}(x)$:

$$(2.4) \quad \mathcal{A}(x) := \left\{ c \geq 0 : \exists H \text{ such that } X := x + (H \cdot S) - \int_0^\cdot c_s \, ds \geq 0, \text{ a.s.} \right\}, \quad x > 0.$$

For $x = 1$ we write $\mathcal{A} \equiv \mathcal{A}(1)$, and we note that $\mathcal{A}(x) = x\mathcal{A}$ for $x > 0$. We observe that \mathcal{A} is a convex set.

For $c \equiv 0$, the wealth process is that of a self-financing portfolio, with wealth process X^0 as in (2.2). Define $\mathcal{X}(x)$ as the set of almost surely nonnegative self-financing wealth processes with initial value $x > 0$:

$$\mathcal{X}(x) := \{ X^0 : X^0 = x + (H \cdot S) \geq 0, \text{ a.s.} \}, \quad x > 0.$$

As for the admissible consumption plans, we write $\mathcal{X} \equiv \mathcal{X}(1)$, with $\mathcal{X}(x) = x\mathcal{X}$ for $x > 0$, and we note that \mathcal{X} is a convex set.

Given the wealth decomposition in (2.3), an equivalent characterisation of the admissible consumption plans is that there exists a self-financing wealth process which dominates cumulative consumption (such a wealth process will necessarily be nonnegative, so will lie in $\mathcal{X}(x)$).

2.2. *Deflators for consumption plans.* The dual domain for our infinite horizon utility maximisation problem from inter-temporal consumption will be a specialisation of the one used by Kramkov and Schachermayer [19, 20] for the terminal wealth problem. We shall refer to the processes in the dual domain as *deflators* (or, sometimes, as *consumption deflators*, if we need to distinguish them from the corresponding deflators in the absence of consumption).

Define the set of positive càdlàg processes such that deflated wealth plus cumulative deflated consumption is a supermartingale for every admissible consumption plan:

$$(2.5) \quad \mathcal{Y}(y) := \left\{ Y > 0, \text{ càdlàg}, Y_0 = y : XY + \int_0^\cdot c_s Y_s \, ds \text{ is a supermartingale, } \forall c \in \mathcal{A} \right\}.$$

Using \mathcal{A} rather than $\mathcal{A}(x)$ in (2.5) is without loss of generality, given $\mathcal{A}(x) = x\mathcal{A}$, $x > 0$. As usual, we write $\mathcal{Y} \equiv \mathcal{Y}(1)$ and we have $\mathcal{Y}(y) = y\mathcal{Y}$ for $y > 0$. In (2.5), the wealth process X is the one on the left-hand side of (2.1) or (2.3) with $x = 1$, so incorporating consumption. We note that, since $(X, c) \equiv (1, 0)$ is an admissible consumption-investment pair, each $Y \in \mathcal{Y}(y)$ is a supermartingale. The set \mathcal{Y} is easily seen to be convex.

In the case $c \equiv 0$ (which is admissible) we have that deflated self-financing wealth is a supermartingale for any choice of consumption deflator. Thus, the set $\mathcal{Y}(y)$ is included in the set of *wealth deflators* that were used by Kramkov and Schachermayer [19, 20]. We shall write Y^0 to denote such deflators, and the set of wealth deflators will be denoted by $\mathcal{Y}^0(y)$:

$$\mathcal{Y}^0(y) := \{ Y^0 > 0, \text{ càdlàg}, Y_0^0 = y : X^0 Y^0 \text{ is a supermartingale, for all } X^0 \in \mathcal{X} \}.$$

As before, we write $\mathcal{Y}^0 \equiv \mathcal{Y}^0(1)$ and we have $\mathcal{Y}^0(y) = y\mathcal{Y}^0$ for $y > 0$. Since $X^0 \equiv 1$ lies in \mathcal{X} , each $Y^0 \in \mathcal{Y}^0(y)$ is a supermartingale. The wealth deflators are also known as *super-martingale deflators*. Clearly, the set \mathcal{Y}^0 is convex.

The set \mathcal{Z} of *local martingale deflators* (LMDs) is composed of positive càdlàg local martingales Z with unit initial value such that deflated self-financing wealth $X^0 Z$, for all $X^0 \in \mathcal{X}$, is a local martingale:

$$(2.6) \quad \mathcal{Z} := \{ Z > 0, \text{ càdlàg}, Z_0 = 1 : X^0 Z \text{ is a local martingale, for all } X^0 \in \mathcal{X} \}.$$

Since the local martingale $X^0 Z \geq 0$ for all $X^0 \in \mathcal{X}$, it is also a supermartingale and, since $X^0 \equiv 1$ lies in \mathcal{X} , each $Z \in \mathcal{Z}$ is also a supermartingale. The set \mathcal{Z} contains the density processes of equivalent local martingale measures (ELMMs) in situations where those would exist. We shall not, however, be using any constructions involving ELMMs, even restricted to a finite horizon. We shall say more on this in Section 2.2.1.

We observe that we have the inclusions

$$\mathcal{Z} \subseteq \mathcal{Y} \subseteq \mathcal{Y}^0.$$

(To see the first inclusion, recall the wealth decomposition in (2.3). Applying the Itô product rule to the process CZ gives $XZ + \int_0^t c_s Z_s ds = X^0 Z - \int_0^t C_{s-} dZ_s$, the left-hand side of which is nonnegative, with the right-hand-side a local martingale, so the left-hand side is a nonnegative local martingale and thus a supermartingale. Thus, any LMD $Z \in \mathcal{Z}$ also lies in \mathcal{Y} .)

The standing no-arbitrage assumption we shall make is that the set of supermartingale deflators is nonempty:

$$(2.7) \quad \mathcal{Y}^0 \neq \emptyset.$$

It is well known that (2.7) is equivalent to the no unbounded profit with bounded risk (NUPBR) condition (also known as no arbitrage of the first kind, or NA_1), weaker than the no free lunch with vanishing risk (NFLVR) condition, the latter requiring the existence of ELMMs, which is often problematic over the infinite horizon, as we discuss in Section 2.2.1. There are a number of equivalent characterisations of NUPBR, including that the set \mathcal{Z} of LMDs is nonempty: see Karatzas and Kardaras [12], Kardaras [18], Takaoka and Schweizer [32] and Chau et al. [3], as well as the recent overview by Kabanov, Kardaras and Song [11].

2.2.1. *Completion of the stochastic basis and equivalent measures.* As indicated earlier, we shall avoid completely any constructions which invoke equivalent local martingale measures (ELMMs), even restricted to a finite horizon. This is partly for aesthetic reasons: since we work under NUPBR and assume only the existence of various classes of deflators, which is the minimal requirement for well-posed utility maximisation problems, it seems natural to seek proofs which use only deflators. This is what we do.

There is some mathematical rationale for avoiding ELMMs. We are working on an infinite horizon and have assumed the usual conditions. Thus, each element of the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ includes all the \mathbb{P} -null sets of $\mathcal{F} := \sigma(\bigcup_{t \geq 0} \mathcal{F}_t) =: \mathcal{F}_\infty$, the tail σ -algebra. So, ultimate events (as time $t \uparrow \infty$) of measure zero are included in any finite time σ -field $\mathcal{F}_T, T < \infty$.

It is well known that in such a scenario many financial models will not admit an equivalent martingale measure over the infinite horizon, because the candidate change of measure density is not a uniformly integrable martingale. (This is true of the Black-Scholes model; see Karatzas and Shreve [16], Section 1.7.) One then has to proceed with caution when invoking arguments which utilise equivalent measures, by finding a consistent way to eliminate the tail σ -algebra from the picture when restricting to a finite horizon $T < \infty$.

One route forward is to not complete the space, as in Huang and Pagès [10], in an infinite horizon consumption model in a complete Brownian market. This is sound, though care is needed to ensure that no results are used which require the usual hypotheses to hold.

Another way to proceed, if one wishes to consider equivalent measures restricted to a finite horizon $T < \infty$, is to augment the space with null events of a σ -field generated over a finite horizon at least as big as T , that is by $\sigma(\bigcup_{0 \leq t \leq T'} \mathcal{F}_t)$, for some $0 \leq T \leq T' < \infty$. This can be done in a consistent way, and relies on an application of Carathéodory’s extension theorem (Rogers and Williams [28], Theorem II.5.1). One can then obtain equivalent measures in an

infinite horizon model when restricting such measures to any finite horizon. This procedure is carried out in a Brownian filtration in Karatzas and Shreve ([16], Section 1.7), with a cautionary example ([16], Example 1.7.6) showing that augmenting the σ -field generated by Brownian motion over any finite horizon with null sets of the corresponding tail σ -algebra would render invalid the construction of equivalent measures, even over a finite horizon.

The message is that one has to be careful in using any constructions involving equivalent measures, even restricted to a finite horizon, when working in infinite horizon financial model.

We avoid having to invoke such fixes, since we avoid all constructions involving ELMs. In particular, in Section 6 we establish bipolarity results between the primal and dual domains using only the Stricker and Yan [31] version of the optional decomposition theorem, relying on deflators rather than equivalent measures.

3. The consumption problem and its dual. Let $U : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a utility function, strictly concave, strictly increasing, continuously differentiable on \mathbb{R}_+ and satisfying the Inada conditions

$$(3.1) \quad \lim_{x \downarrow 0} U'(x) = +\infty, \quad \lim_{x \rightarrow \infty} U'(x) = 0.$$

To guarantee a well-posed consumption problem, one could also impose here the reasonable asymptotic elasticity condition of Kramkov and Schachermayer [19]:

$$(3.2) \quad \text{AE}(U) := \limsup_{x \rightarrow \infty} \frac{xU'(x)}{U(x)} < 1.$$

The condition in (3.2) was shown in [19] to be a minimal condition, in an arbitrary market model, to guarantee that the terminal wealth utility maximisation problem satisfied all the tenets of a general duality theory. It was later shown, again by Kramkov and Schachermayer [20], that if one instead assumes a market model such that the weak condition of a finite dual value function holds, then this alternative set-up gives a consistent duality theory. Furthermore, finiteness of the dual problem, along with a minimal condition on the primal value function (to be finitely valued for at least one value of initial capital) so as to exclude a trivial problem, implies the reasonable asymptotic elasticity condition. For this reason, we shall follow the spirit of [20] and just impose weak finiteness conditions on the primal and dual value functions so as to exclude trivial problems, and then later make the (standard) remark in the style of [20], Note 2, on how these are consistent with (3.2) (see Remark 5.7).

Let $\kappa : (0, \infty) \rightarrow \mathbb{R}_+$ be a positive finite measure which will determine how utility of consumption is discounted through time, assumed to be almost surely absolutely continuous with respect to Lebesgue measure and satisfying

$$(3.3) \quad \frac{d\kappa_t}{dt} \leq 1, \quad \text{almost surely, } t \geq 0, \quad \mathbb{E}\left[\int_0^\infty c_t d\kappa_t\right] \leq K < \infty, \quad \forall c \in \mathcal{A},$$

for some constant $K > 0$. For later use, define the positive process $\gamma = (\gamma_t)_{t \geq 0}$ as the reciprocal of $(d\kappa_t/dt)_{t \geq 0}$:

$$(3.4) \quad \gamma_t := \left(\frac{d\kappa_t}{dt}\right)^{-1}, \quad t \geq 0.$$

Define the primal value function from optimal consumption by

$$(3.5) \quad u(x) := \sup_{c \in \mathcal{A}(x)} \mathbb{E}\left[\int_0^\infty U(c_t) d\kappa_t\right], \quad x > 0.$$

To exclude a trivial problem, we shall assume throughout that $u(x) > -\infty$ for all $x > 0$. This is guaranteed by the weak condition that $\mathbb{E}[\int_0^\infty \min[0, U(c_t)] d\kappa_t] > -\infty$.

The supremum in (3.5) is written as one over consumption processes. This should not obscure the fact that an optimal consumption process must also determine an associated optimal wealth process (equivalently an optimal trading strategy). This is clear from the definition in (2.4), where the consumption process is defined with reference to the associated investment strategy. Indeed, in traditional formulations of the problem, this is acknowledged in the notation by writing the value function as a supremum over a pair of controls involving either (X, c) or (H, c) . Our goal is to find an optimal consumption process \hat{c} , but to also characterise the associated optimal wealth process \hat{X} . Note that no such characterisation of the optimal wealth process was given in either of Mostovyi [24] or Chau et al. [3]. This turns out to be an interesting feature of the analysis, with a nice result (Proposition 7.14) incorporated into the main duality theorem: at the optimum, the deflated wealth process is a supermartingale and a potential, while the deflated wealth plus cumulative deflated consumption is a uniformly integrable martingale. These results are the natural extensions of the result for the terminal wealth problem in Kramkov and Schachermayer [19, 20], in which optimal deflated wealth is a uniformly integrable martingale.

EXAMPLE 3.1 (Infinite horizon discounted utility from consumption). The example we are primarily interested in is the case where $dk_t = e^{-\alpha t} dt$, for some positive impatience parameter $\alpha > 0$ (which could also be made stochastic). In this case we have $\gamma_t = e^{\alpha t}$, $t \geq 0$, which is the factor which inflates the natural deflators in the dual problem, as we shall see.

The problem in (3.5) is then $\mathbb{E}[\int_0^\infty e^{-\alpha t} U(c_t) dt] \rightarrow \max!$ We shall illustrate the solution of such a problem with a stock driven by a three-dimensional Bessel process, and with stochastic volatility and correlation, in Example 8.1.

3.1. *On stochastic clocks.* We discuss briefly some variations of the problem (3.5) which can be incorporated into our framework (but which are not the main focus of our analysis).

In Mostovyi [24] and Chau et al. [3] the measure κ is taken to be a *stochastic clock*, that is, a nondecreasing, càdlàg adapted process satisfying

$$\kappa_0 = 0, \kappa_\infty \leq K < \infty, \text{ a.s., } \mathbb{P}[\kappa_\infty > 0] > 0,$$

for some finite positive constant K . As shown by Mostovyi [24], Examples 2.5–2.9, by appropriate choice of the stochastic clock a number of different problems can be included within the framework of (3.5), such as the terminal wealth problem, the finite horizon consumption problem, the finite horizon consumption and terminal wealth problem, as well as the infinite horizon problem in Example 3.1. The same observation applies to our problem, provided we choose the measure κ to be a stochastic clock of the appropriate type. Our primary focus, however, is to give a definitive treatment of the traditional infinite horizon discounted utility of consumption problem.

Note also that in [3, 24], the stochastic clock was incorporated into the wealth dynamics: for some process \bar{c} , (2.1) was replaced by

$$X_t = x + (H \cdot S)_t - \int_0^t \bar{c}_s d\kappa_s, \quad t \geq 0, x > 0.$$

Thus, the process \bar{c} (let us call it a pseudo-consumption rate, to distinguish it from our variable) of those papers involves a change of variable from our consumption rate. The approach in [3, 24] allows for a constant positive pseudo-consumption rate, which can sometimes be mathematically convenient. With a true consumption rate and an infinite horizon, a constant consumption plan is not possible. Each approach can be converted to the other, as we now illustrate.

For concreteness, suppose the measure κ is as in (3.3). The pseudo-consumption rate \bar{c} is then related to the real consumption rate by $\bar{c}_t = \gamma_t c_t, t \geq 0$. The problems considered in [3, 24] are of the form

$$(3.6) \quad \mathbb{E} \left[\int_0^\infty \bar{U}(t, \gamma_t c_t) d\kappa_t \right] \rightarrow \max!$$

for some time dependent utility function $\bar{U}(\cdot, \cdot)$. (This utility was also stochastic in [3, 24], but this makes no difference to the argument here.) To make the problem in (3.6) equivalent to our problem in (3.5) requires $\bar{U}(t, \gamma_t c_t) = U(c_t)$ almost surely for all $t \geq 0$, and this is easy to satisfy. For example, if $\gamma_t = e^{\alpha t}, t \geq 0$ and $U(\cdot) = \log(\cdot)$ is logarithmic utility, we choose $\bar{U}(t, \bar{c}) = \log(\bar{c}) - \alpha t$. If $U(c) = c^p/p, p < 1, p \neq 0$ is power utility, then we choose $\bar{U}(t, \bar{c}) = e^{-\alpha p t} \bar{c}^p/p$. Hence, we can always restore a problem of the form in (3.5) (equivalent to the problems in [3, 24] up to an additive or multiplicative constant, typically).

We choose in this work to adopt the classical definition of consumption. Part of our reason for doing so is to make very transparent the underlying supermartingale constraint on deflated wealth plus cumulative deflated consumption that one must apply, if one is to show how the program of Kramkov and Schachermayer [19, 20], suitably modified and extended, creates a natural procedure for characterising the classical consumption duality. As will be seen, this reveals an interesting role reversal of the primal and dual domains in many steps of the proofs, compared with the terminal wealth problem, because it turns out that the primal domain in the consumption problem is L^1 -bounded (with respect to a suitable measure), but it is the dual domain that has this property in the terminal wealth case.

REMARK 3.2 (Discounted units). There is no loss of generality in working with discounted quantities (so in effect a zero interest rate). To see this, suppose instead that we have a positive interest rate process $r = (r_t)_{t \geq 0}$, so the cash asset with initial value 1 has positive price process $A_t = e^{\int_0^t r_s ds}, t \geq 0$. If \tilde{c} is the un-discounted consumption process, then the problem in (3.5) is $\mathbb{E}[\int_0^\infty U(\tilde{c}_t/A_t) d\kappa_t] \rightarrow \max!$ We can define another utility function $\tilde{U} : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ such that $\tilde{U}(A_t, \tilde{c}_t) = U(\tilde{c}_t/A_t), t \geq 0$, and the problem in (3.5) can then be transported to one in terms of the raw (un-discounted) consumption rate. For example, if $\gamma_t = e^{\alpha t}, t \geq 0$ and $U(\cdot) = \log(\cdot)$ is logarithmic utility, we choose $\tilde{U}(A, \tilde{c}) = \log(\tilde{c}) - \log(A)$. If $U(c) = c^p/p, p < 1, p \neq 0$ is power utility, then we choose $\tilde{U}(A, \tilde{c}) = A^{-p} \tilde{c}^p/p$.

REMARK 3.3 (Stochastic utility). In the problem (3.5) we can allow $U(\cdot)$ to be stochastic, so to also depend on $\omega \in \Omega$ in an optional way, as done by Mostovyi [24]. The analysis is unaffected, as the reader can easily verify, so one can read the proofs with a stochastic utility in mind and with dependence on $\omega \in \Omega$ suppressed throughout.

3.2. *The dual problem.* Let $V : \mathbb{R}_+ \rightarrow \mathbb{R}$ denote the convex conjugate of $U(\cdot)$, defined by

$$V(y) := \sup_{x > 0} [U(x) - xy], \quad y > 0.$$

The map $y \mapsto V(y), y > 0$, is strictly convex, strictly decreasing, continuously differentiable on $\mathbb{R}_+, -V(\cdot)$ satisfies the Inada conditions, and we have the bi-dual relation

$$U(x) := \inf_{y > 0} [V(y) + xy], \quad x > 0,$$

as well as $V'(\cdot) = -I(\cdot) = -(U')^{-1}(\cdot)$, where $I(\cdot)$ denotes the inverse of marginal utility. In particular, we have the inequality

$$(3.7) \quad V(y) \geq U(x) - xy, \quad \forall x, y > 0, \text{ with equality iff } U'(x) = y.$$

For each consumption deflator $Y \in \mathcal{Y}(y)$ defined in (2.5), define a process Y^γ by

$$(3.8) \quad Y_t^\gamma := \gamma_t Y_t, \quad t \geq 0,$$

where γ was defined in (3.4). For later use, denote the set of such processes by $\tilde{\mathcal{Y}}(y)$:

$$(3.9) \quad \tilde{\mathcal{Y}}(y) := \{Y^\gamma : Y^\gamma \text{ is given by (3.8), with } Y \in \mathcal{Y}(y)\}, \quad y > 0,$$

so the set $\tilde{\mathcal{Y}}(y)$ is in one-to-one correspondence with the set $\mathcal{Y}(y)$ of consumption deflators. As usual, we write $\tilde{\mathcal{Y}} \equiv \tilde{\mathcal{Y}}(1)$, and we have $\tilde{\mathcal{Y}}(y) = y\tilde{\mathcal{Y}}$ for $y > 0$.

The dual problem to (3.5) has value function $v : \mathbb{R}_+ \rightarrow \mathbb{R}$ defined by

$$(3.10) \quad v(y) := \inf_{Y \in \mathcal{Y}(y)} \mathbb{E} \left[\int_0^\infty V(\gamma_t Y_t) d\kappa_t \right], \quad y > 0.$$

We shall assume throughout that $v(y) < \infty$ for all $y > 0$.

4. The duality theorem. Here is the main result, the perpetual consumption duality. It is somewhat stronger and mathematically more robust than previous results. We describe how the theorem differs from, and in which senses it strengthens, existing results, after presenting the theorem.

THEOREM 4.1 (Perpetual consumption duality under NUPBR). *Define the primal consumption problem by (3.5) and the corresponding dual problem by (3.10). Assume (2.7), (3.1) and that*

$$u(x) > -\infty, \quad \forall x > 0, \quad v(y) < \infty, \quad \forall y > 0.$$

Then:

(i) $u(\cdot)$ and $v(\cdot)$ are conjugate:

$$v(y) = \sup_{x>0} [u(x) - xy], \quad u(x) = \inf_{y>0} [v(y) + xy], \quad x, y > 0.$$

(ii) The primal and dual optimisers $\hat{c}(x) \in \mathcal{A}(x)$ and $\hat{Y}(y) \in \mathcal{Y}(y)$ exist and are unique, so that

$$u(x) = \mathbb{E} \left[\int_0^\infty U(\hat{c}_t(x)) d\kappa_t \right], \quad v(y) = \mathbb{E} \left[\int_0^\infty V(\gamma_t \hat{Y}_t(y)) d\kappa_t \right], \quad x, y > 0.$$

(iii) With $y = u'(x)$ (equivalently, $x = -v'(y)$), the primal and dual optimisers are related by

$$(4.1) \quad U'(\hat{c}_t(x)) = \gamma_t \hat{Y}_t(y), \quad \text{equivalently,} \quad \hat{c}_t(x) = -V'(\gamma_t \hat{Y}_t(y)), \quad t \geq 0,$$

and satisfy

$$(4.2) \quad \mathbb{E} \left[\int_0^\infty \hat{c}_t(x) \hat{Y}_t(y) dt \right] = xy.$$

Moreover, the associated optimal wealth process $\hat{X}(x)$ is given by

$$(4.3) \quad \hat{X}_t(x) \hat{Y}_t(y) = \mathbb{E} \left[\int_t^\infty \hat{c}_s(x) \hat{Y}_s(y) ds \mid \mathcal{F}_t \right], \quad t \geq 0,$$

and the process $\hat{X}(x) \hat{Y}(y) + \int_0^\cdot \hat{c}_s(x) \hat{Y}_s(y) ds$ is a uniformly integrable martingale.

(iv) The functions $u(\cdot)$ and $-v(\cdot)$ are strictly increasing, strictly concave, satisfy the Inada conditions, and for all $x, y > 0$ their derivatives satisfy

$$xu'(x) = \mathbb{E} \left[\int_0^\infty U'(\hat{c}_t(x)) \hat{c}_t(x) d\kappa_t \right], \quad yv'(y) = \mathbb{E} \left[\int_0^\infty V'(\gamma_t \hat{Y}_t(y)) \hat{Y}_t(y) dt \right].$$

The proof of Theorem 4.1 will be given in Section 7, and will rely on bipolarity results and an abstract version of the duality stated in Section 5, with the bipolarity results proven in Section 6. A duality result of this form was established by Mostovyi [24] under NFLVR. This was strengthened to a result under NUPBR by Chau et al [3]. Compared to these papers, Theorem 4.1 makes a stronger statement in other ways.

First, we characterise the optimal wealth process, a statement that was missing from [3, 24]. This turns out to be a nice result to prove (see Proposition 7.14), showing that the optimal process $\widehat{X}\widehat{Y}$ is a supermartingale and a potential, while $\widehat{X}\widehat{Y} + \int_0^\cdot \widehat{c}_s \widehat{Y}_s ds$ is a uniformly integrable martingale. This is the natural extension of the result in the terminal wealth problem that, at the optimum, deflated wealth is a uniformly integrable martingale (see Kramkov and Schachermayer [19, 20]), and confirms that the supermartingale condition we placed on the process $XY + \int_0^\cdot c_s Y_s ds$ for admissibility is the right criterion to start from.

Further, as we shall see in the course of proving Theorem 4.1, the dual domain $\mathcal{Y}(y)$ will need to be enlarged, in a spirit akin to Kramkov and Schachermayer [19, 20], to consider processes which are dominated by some element of the original dual domain. This enlargement, as is known from the terminal wealth scenario of [19, 20], is needed in order to reach the bipolar of the original dual domain, so that the (enlarged) dual domain is closed in an appropriate topology. This in turn guarantees that a unique dual optimiser will exist. This is one of the key contributions made in [19, 20]. One does not assume a priori that either the primal or dual domains are closed.

Here, for the consumption problem, we shall see that we do not need to enlarge the primal domain, only the dual domain. Mostovyi [24] and Chau et al. [3] found a similar phenomenon, but with the important caveat that they took the enlarged dual domain to be the closure (in the appropriate topology) of the set of processes dominated by local martingale deflators (in [3]) or martingale deflators (in [24]).

Here, we do not explicitly make the dual domain closed (in the manner of [3, 24]) by construction, so we obtain a stronger result. We merely enlarge the dual domain in a manner analogous to Kramkov and Schachermayer [19, 20], by considering processes dominated by consumption deflators, and then show that the enlarged domain is closed using supermartingale convergence results which exploit so-called Fatou convergence of processes. We also prove that our enlarged domain coincides with the closure of processes dominated by local martingale deflators (see Proposition 5.1), so coincides with that used in [3]. In other words, the domain used in [3] is *dense* in our domain. The proof of Proposition 5.1 will also reveal why the supermartingale convergence results, used to show that our enlarged domain is closed, cannot provide the same result for the (preclosure) domains used in [3, 24]. Basically, the limiting supermartingale is just that, a supermartingale, and it cannot be shown to be a (local) martingale deflator.

This all reveals that, in a real sense, we have found just the right dual domain for a strong duality statement.

Lastly, regarding some steps underlying the proof of Theorem 4.1, and in particular the arguments in Section 6 used to establish bipolarity relations connecting the primal and dual domains, our proofs make no use at any point of constructions involving equivalent measures, such as ELMs, but use only deflators. Since we are working under NUPBR this is natural, and in some senses even desirable. Moreover, as we have alluded to in Section 2.2.1, there are potential complications in using equivalent measures when working on an infinite horizon, so there are sound reasons for taking the course we follow here.

In our scenario, therefore, we provide an unambiguously robust route through the proofs which avoids any use of ELMs. This, in addition to the features described above, of showing that the naturally enlarged dual domain is closed, without taking its closure to guarantee this, makes Theorem 4.1 a quite distinct infinite horizon consumption duality result from those in [3, 24].

REMARK 4.2 (Incorporating a stochastic clock into the wealth dynamics). As discussed in Section 3.1, one can incorporate a stochastic clock into the wealth dynamics, as done by Mostovyi [24] and Chau et al [3]. Our entire program works with this change, and we point out here how Theorem 4.1 would be altered. We modify the wealth dynamics (2.1) to

$$X_t = x + (H \cdot S)_t - \int_0^t c_s \, d\kappa_s, \quad t \geq 0, x > 0,$$

where κ is a stochastic clock of the form described in Section 3.1. The process c was denoted by \bar{c} in Section 3.1, but for a clean notation we shall not make this adjustment here. The primal value function is still given by (3.5). The consumption deflators are also unchanged, but the key supermartingale constraint in (2.5) is altered to reflect the change of consumption variable, to

$$XY + \int_0^\infty c_s Y_s \, d\kappa_s \quad \text{is a supermartingale, } \forall c \in \mathcal{A},$$

where admissible consumption plans are still those for which the wealth process X is non-negative. In other words, one simply alters the measure used in the cumulative deflated consumption term in the fundamental supermartingale constraint. As a result, the form of the dual problem in (3.10) is altered to

$$v(y) := \inf_{Y \in \mathcal{Y}(y)} \mathbb{E} \left[\int_0^\infty V(Y_t) \, d\kappa_t \right], \quad y > 0,$$

so one loses the extraneous process γ in the argument of $V(\cdot)$ in the definition of the dual value function, and hence in the expression for this function in item (ii) of the theorem. Similar adjustments occur in the remaining results of Theorem 4.1. Thus, item (iii) of the theorem is altered to

With $y = u'(x)$ (equivalently, $x = -v'(y)$), the primal and dual optimisers are related by

$$U'(\hat{c}_t(x)) = \hat{Y}_t(y), \quad \text{equivalently, } \hat{c}_t(x) = -V'(\hat{Y}_t(y)), \quad t \geq 0,$$

and satisfy

$$\mathbb{E} \left[\int_0^\infty \hat{c}_t(x) \hat{Y}_t(y) \, d\kappa_t \right] = xy,$$

with the associated optimal wealth process $\hat{X}(x)$ given by

$$\hat{X}_t(x) \hat{Y}_t(y) = \mathbb{E} \left[\int_t^\infty \hat{c}_s(x) \hat{Y}_s(y) \, d\kappa_s \mid \mathcal{F}_t \right], \quad t \geq 0,$$

and the process $\hat{X}(x) \hat{Y}(y) + \int_0^\cdot \hat{c}_s(x) \hat{Y}_s(y) \, d\kappa_s$ is a uniformly integrable martingale.

Item (iv) of the theorem is altered to:

The functions $u(\cdot)$ and $-v(\cdot)$ are strictly increasing, strictly concave, satisfy the Inada conditions, and for all $x, y > 0$ their derivatives satisfy

$$x u'(x) = \mathbb{E} \left[\int_0^\infty U'(\hat{c}_t(x)) \hat{c}_t(x) \, d\kappa_t \right], \quad y v'(y) = \mathbb{E} \left[\int_0^\infty V'(\hat{Y}_t(y)) \hat{Y}_t(y) \, d\kappa_t \right].$$

5. Abstract bipolarity and duality. In this section, we state a bipolarity result in abstract form, leading to an abstract duality theorem, from which Theorem 4.1 will follow. Proofs of these results will follow in subsequent sections.

Set $\Omega := [0, \infty) \times \Omega$. Let \mathcal{G} denote the optional σ -algebra on Ω , that is, the sub- σ -algebra of $\mathcal{B}([0, \infty)) \otimes \mathcal{F}$ generated by evanescent sets and stochastic intervals of the form $\llbracket T, \infty \rrbracket$ for arbitrary stopping times T . Define the measure $\mu := \kappa \times \mathbb{P}$ on (Ω, \mathcal{G}) . On the resulting

finite measure space $(\Omega, \mathcal{G}, \mu)$, denote by $L_+^0(\mu)$ the space of nonnegative μ -measurable functions, corresponding to nonnegative infinite horizon processes.

The primal and dual domains for our optimisation problems (3.5) and (3.10) are now considered as subsets of $L_+^0(\mu)$. The abstract primal domain $\mathcal{C}(x)$ is identical to the set of admissible consumption plans, now considered as a subset of $L_+^0(\mu)$:

$$(5.1) \quad \mathcal{C}(x) := \{g \in L_+^0(\mu) : g = c, \mu\text{-a.e.}, \text{ for some } c \in \mathcal{A}(x)\}, \quad x > 0.$$

As always we write $\mathcal{C} \equiv \mathcal{C}(1)$, with $\mathcal{C}(x) = x\mathcal{C}$ for $x > 0$, and the set \mathcal{C} is convex. (Since $\mathcal{C} = \mathcal{A}$ we do not really need to introduce the new notation, and do so only for some notational symmetry in the abstract formulation.) In the abstract notation, the primal value function (3.5) is written as

$$(5.2) \quad u(x) := \sup_{g \in \mathcal{C}(x)} \int_{\Omega} U(g) \, d\mu, \quad x > 0.$$

For the dual problem, the abstract dual domain is an enlargement of the original domain to accommodate processes dominated by the original dual variables. To this end, define the set

$$(5.3) \quad \mathcal{D}(y) := \{h \in L_+^0(\mu) : h \leq \gamma Y, \mu\text{-a.e.}, \text{ for some } Y \in \mathcal{Y}(y)\}, \quad y > 0.$$

As usual, we write $\mathcal{D} \equiv \mathcal{D}(1)$, we have $\mathcal{D}(y) = y\mathcal{D}$ for $y > 0$, and the set \mathcal{D} is convex. With this notation, and since $V(\cdot)$ is decreasing, the dual problem (3.10) takes the form

$$(5.4) \quad v(y) := \inf_{h \in \mathcal{D}(y)} \int_{\Omega} V(h) \, d\mu, \quad y > 0.$$

The enlargement of the dual domain from \mathcal{Y} (equivalently, $\tilde{\mathcal{Y}}$ in (3.9)) to \mathcal{D} is needed for the same reason as in Kramkov and Schachermayer [19, 20] in the context of the terminal wealth problem (where one enlarged the dual domain from supermartingale deflators to elements of $L_+^0(\mathbb{P})$ that were dominated by terminal values of supermartingale deflators). The enlargement will ensure that \mathcal{D} is closed with respect to convergence in measure μ (proven in Lemma 6.7). This in turn ensures that we reach a perfect bipolarity between the primal and dual domains (as given in Proposition 5.5), which is a key ingredient in establishing full duality between the primal and dual problems. Contrast this enlargement with the approach taken in Chau et al. [3] and Mostovyi [24] as described immediately below.

5.1. Alternative dual domains. In Chau et al. [3] (respectively, Mostovyi [24]) the dual domain was not based on the deflators $Y \in \mathcal{Y}$ but instead on the local martingale deflators $Z \in \mathcal{Z}$ (respectively, equivalent martingale deflators). Thus, translated into our formulation (so using a true rather than a pseudo-consumption rate), Chau et al. [3] use, in place of $\mathcal{D}(y)$, a domain defined as the closure, with respect to the topology of convergence in measure μ , of a set $D(y)$, where $D(y)$ is defined analogously to $\mathcal{D}(y)$ but with local martingale deflators replacing the consumption deflators. Thus, with $\bar{A} \equiv \text{cl}(A)$ denoting the closure of any set $A \subseteq L_+^0(\mu)$, we have

$$(5.5) \quad \bar{\mathcal{D}}(y) \equiv \text{cl}(D) := \text{cl}\{h \in L_+^0(\mu) : h \leq y\gamma Z, \text{ for some } Z \in \mathcal{Z}\}, \quad y > 0.$$

As usual we write $D \equiv D(1)$, and $D(y) = yD$ for $y > 0$, with the same convention for \bar{D} . In this formulation, therefore, the dual value function is represented as in (5.4) but with $\bar{D}(y)$ in place of $\mathcal{D}(y)$.

The salient point here is the fact that the *closure* of $D(y)$ has been taken in (5.5). The reason for this will become transparent in the proofs of Section 6, but we outline the issue here, and state a nice result (Proposition 5.1) which connects the domains D , D and \bar{D} .

In the approach of [3] (and also of [24], with martingale deflators in place of local martingale deflators), if one does not take the aforementioned closure, it becomes impossible (as far as we can see) to prove that the dual domain is closed. It thus becomes impossible to obtain a perfect bipolarity between the primal and dual domains, on which the duality proofs ultimately rest. The technical reason for this is that the closed property of \mathcal{D} is established (see Lemma 6.7) using a supermartingale convergence result based on Fatou convergence of processes. The limiting supermartingale in this procedure is known only to be a supermartingale in \mathcal{Y} , so is not guaranteed to be a local martingale deflator. This is the driving force behind our choice of dual domain based on a supermartingale criterion. The approach in [3, 24] is simply not amenable to this procedure, which is why those papers had to invoke the closure in (5.5).

In this way, we strengthen the duality theorems in [3, 24], by not forcing the dual domain to be closed by construction. This point is well made by Rogers [27], who observes that having to take the closure of the dual domain in its definition “makes the statement of the main result somewhat weaker”. We do denigrate in any way, however, the advances made in [3, 24].

What is more, we have the proposition below, which reaffirms in some sense that our choice of dual domain is the correct one: we have chosen it in just the right way to reach the bipolar of the original dual domain and hence the polar of the primal domain.

PROPOSITION 5.1. *With respect to the topology of convergence in measure μ , the set*

$$D := \{h \in L_+^0(\mu) : h \leq \gamma Z, \text{ for some } Z \in \mathcal{Z}\},$$

is dense in the set $\mathcal{D} \equiv \mathcal{D}(1)$ of (5.3). That is, we have

$$\mathcal{D} = \overline{D} \equiv \text{cl}(D).$$

The proof of Proposition 5.1 will be given in Section 6, alongside the proof of the bipolarity result in Proposition 5.5 that is the subject of the next subsection.

5.2. *Abstract bipolarity.* The abstract duality theorem relies on the abstract bipolarity result in Proposition 5.5 below which connects the sets \mathcal{C} and \mathcal{D} . The result is of course in the spirit of Kramkov and Schachermayer [19], Proposition 3.1.

We shall sometimes employ the notation

$$\langle g, h \rangle := \int_{\Omega} gh \, d\mu, \quad g, h \in L_+^0(\mu).$$

Let us recall some definitions, particularly the concepts of set solidity and the polar of a set.

DEFINITION 5.2 (Solid set, closed set). A subset $A \subseteq L_+^0(\mu)$ is called *solid* if $f \in A$ and $0 \leq g \leq f, \mu$ -a.e. implies that $g \in A$.

A set is *closed in μ -measure*, or simply *closed*, if it is closed with respect to the topology of convergence in measure μ .

DEFINITION 5.3 (Polar of a set). The *polar*, A° , of a set $A \subseteq L_+^0(\mu)$, is defined by

$$A^\circ := \{h \in L_+^0(\mu) : \langle g, h \rangle \leq 1, \text{ for each } g \in A\}.$$

For clarity and for later use, we state here the bipolar theorem of Brannath and Schachermayer [2], Theorem 1.3, originally proven in a probability space, and adapted here to the measure space $(\Omega, \mathcal{G}, \mu)$.

THEOREM 5.4 (Bipolar theorem, Brannath and Schachermayer [2], Theorem 1.3). *On the finite measure space $(\Omega, \mathcal{G}, \mu)$:*

- (i) *For a set $A \subseteq L^0_+(\mu)$, its polar A° is a closed, convex, solid subset of $L^0_+(\mu)$.*
- (ii) *The bipolar $A^{\circ\circ}$, defined by*

$$A^{\circ\circ} := \{g \in L^0_+(\mu) : \langle g, h \rangle \leq 1, \text{ for each } h \in A^\circ\},$$

is the smallest closed, convex, solid set in $L^0_+(\mu)$ containing A .

PROPOSITION 5.5 (Abstract bipolarity). *Under the condition (2.7), the abstract primal and dual sets \mathcal{C} and \mathcal{D} satisfy the following properties:*

- (i) *\mathcal{C} and \mathcal{D} are both closed with respect to convergence in measure μ , convex and solid;*
- (ii) *\mathcal{C} and \mathcal{D} satisfy the bipolarity relations*

$$(5.6) \quad g \in \mathcal{C} \iff \langle g, h \rangle \leq 1, \quad \forall h \in \mathcal{D}, \text{ that is, } \mathcal{C} = \mathcal{D}^\circ,$$

$$(5.7) \quad h \in \mathcal{D} \iff \langle g, h \rangle \leq 1, \quad \forall g \in \mathcal{C}, \text{ that is, } \mathcal{D} = \mathcal{C}^\circ;$$

- (iii) *\mathcal{C} and \mathcal{D} are bounded in $L^0(\mu)$, and \mathcal{C} is also bounded in $L^1(\mu)$.*

The proof of Proposition 5.5 will be given in Section 6, where we shall establish the infinite horizon budget constraint, giving a necessary condition for admissible consumption plans, and a reverse implication, leading to a sufficient condition for admissibility, culminating in the full bipolarity relations once we enlarge the dual domain. The derivations in Section 6 are quite distinct from previous approaches, and are the bedrock of the mathematical results. As indicated earlier, we shall establish the bipolarity results without any recourse whatsoever to constructions involving ELMMs, by exploiting ramifications of the Stricker and Yan [31] version of the optional decomposition theorem.

5.3. Abstract duality. Armed with the abstract bipolarity in Proposition 5.5, we have the following abstract version of the convex duality relations between the primal problem (5.2) and its dual (5.4). The theorem shows that all the natural tenets of utility maximisation theory, as established by Kramkov and Schachermayer [19] in the terminal wealth problem under NFLVR, extend to infinite horizon inter-temporal problems under NUPBR, with weak underlying assumptions on the primal and dual domains.

THEOREM 5.6 (Abstract duality theorem). *Define the primal value function $u(\cdot)$ by (5.2) and the dual value function by (5.4). Assume that the utility function satisfies the Inada conditions (3.1) and that*

$$(5.8) \quad u(x) > -\infty, \quad \forall x > 0, \quad v(y) < \infty, \quad \forall y > 0.$$

Then, with Proposition 5.5 in place, we have:

- (i) *$u(\cdot)$ and $v(\cdot)$ are conjugate:*

$$(5.9) \quad v(y) = \sup_{x>0} [u(x) - xy], \quad u(x) = \inf_{y>0} [v(y) + xy], \quad x, y > 0.$$

- (ii) *The primal and dual optimisers $\widehat{g}(x) \in \mathcal{C}(x)$ and $\widehat{h}(y) \in \mathcal{D}(y)$ exist and are unique, so that*

$$u(x) = \int_{\Omega} U(\widehat{g}(x)) \, d\mu, \quad v(y) = \int_{\Omega} V(\widehat{h}(y)) \, d\mu, \quad x, y > 0.$$

(iii) With $y = u'(x)$ (equivalently, $x = -v'(y)$), the primal and dual optimisers are related by

$$U'(\widehat{g}(x)) = \widehat{h}(y), \quad \text{equivalently,} \quad \widehat{g}(x) = -V'(\widehat{h}(y)),$$

and satisfy

$$\langle \widehat{g}(x), \widehat{h}(y) \rangle = xy.$$

(iv) $u(\cdot)$ and $-v(\cdot)$ are strictly increasing, strictly concave, satisfy the Inada conditions, and their derivatives satisfy

$$xu'(x) = \int_{\Omega} U'(\widehat{g}(x))\widehat{g}(x) \, d\mu, \quad yv'(y) = \int_{\Omega} V'(\widehat{h}(y))\widehat{h}(y) \, d\mu, \quad x, y > 0.$$

The proof of Theorem 5.6 will be given in Section 7, and uses as its starting point the bipolarity result in Proposition 5.5.

The duality proof itself follows some of the classical steps (with adaptations) of Kramkov and Schachermayer [19, 20], but there is an interesting role reversal for the primal and dual sets. In the terminal wealth problem, the dual domain is bounded in $L^1(\mathbb{P})$, because the constant wealth process $\mathbb{1} : \Omega \mapsto 1$ lies in the primal domain. In the infinite horizon consumption problem, by contrast, the constant consumption stream $c \equiv 1$ is not admissible, so the dual domain is not bounded in $L^1(\mu)$. Instead, it turns out that $L^1(\mu)$ -boundedness is satisfied by the primal domain. The upshot is that, in a number of places, the method of proof used in [19, 20] for a property of the primal domain is applied in our case to a corresponding property in the dual domain (and vice versa). Examples include the proofs of uniform integrability of the families $(U^+(g))_{g \in \mathcal{C}(x)}$ and $(V^-(h))_{h \in \mathcal{D}(y)}$, a reversed application of the minimax theorem (replacing a maximisation with a minimisation and so forth) in proving conjugacy of the value functions, and some characterisations of the derivatives of the value functions at zero and infinity. We shall point out these features when proving the results. This is one of the reasons for our choosing to give a complete, self-contained treatment with full proofs.

We conclude this section with a small remark (that is by now standard, but does need stating) on reasonable asymptotic elasticity as an alternative to assuming finiteness of the dual value function.

REMARK 5.7 (Reasonable asymptotic elasticity). In Theorem 5.6 we have assumed only the minimal conditions in (5.8) to guarantee nontrivial primal and dual problems. It is well known that, in place of the second condition in (5.8) of a finitely-valued dual problem, we could have imposed the reasonable asymptotic elasticity condition of Kramkov and Schachermayer [19] as given in (3.2), along with the assumption that $u(x) < \infty$ for some $x > 0$. Then, as in Kramkov and Schachermayer ([20], Note 2), these conditions would have implied that $v(y) < \infty$ for all $y > 0$.

6. Budget constraint and bipolarity relations.

6.1. *The budget constraint.* The first step in the proof of the duality theorem is to establish bipolarity relations between the primal and dual domains. We shall do this in stages, first deriving the infinite horizon budget constraint. This yields the form of the dual problem as a byproduct. The derivation also lends itself to a discussion of the rationale for choosing the dual domain to be the set $\mathcal{Y}(y)$ of consumption deflators, and what would have been the ramifications of instead choosing the wealth deflators or the local martingale deflators as the dual variables.

LEMMA 6.1 (Infinite horizon budget constraint). *Let $c \in \mathcal{A}(x)$ be any admissible consumption plan and let $Y \in \mathcal{Y}(y)$ be any consumption deflator. We then have the infinite horizon budget constraint:*

$$(6.1) \quad \mathbb{E} \left[\int_0^\infty c_t Y_t dt \right] \leq xy, \quad \forall c \in \mathcal{A}(x), Y \in \mathcal{Y}(y).$$

PROOF. Recall the wealth process X incorporating consumption in (2.1). Since $XY + \int_0^\cdot c_s Y_s ds$ is a supermartingale and $XY \geq 0$, we have

$$\mathbb{E} \left[\int_0^t c_s Y_s ds \right] \leq xy, \quad t \geq 0.$$

Letting $t \uparrow \infty$ and using monotone convergence we obtain (6.1). \square

REMARK 6.2 (On alternative choices of dual domain). The derivation of Lemma 6.1 allows us to give some of the rationale for choosing the dual domain as we did.

Suppose instead that we chose the dual domain to be the set $\mathcal{Y}^0(y)$ of supermartingale deflators. Recall the decomposition in (2.3) of a wealth process X incorporating consumption into a self-financing wealth process X^0 minus cumulative consumption $C = \int_0^\cdot c_s ds$. Now, for any wealth deflator $Y^0 \in \mathcal{Y}^0(y)$ and $c \in \mathcal{A}(x)$ we have, on using the Itô product rule on the process CY^0 and rearranging,

$$(6.2) \quad XY^0 + \int_0^\cdot c_s Y_s^0 ds = X^0 Y^0 - \int_0^\cdot C_{s-} dY_s^0.$$

The right-hand side of (6.2) is a difference of supermartingales, so not necessarily a supermartingale, and we would fail to achieve the infinite horizon budget constraint.

Suppose, on the other hand, that we chose the dual domain to be constructed from the set \mathcal{Z} of local martingale deflators. This is the route taken by Chau et al. [3] and by Mostovyi [24] (except that the deflators were martingales in [24], in tandem with the NFLVR scenario in that paper.) We would then reach the analogue of (6.2) in the form

$$XZ + \int_0^\cdot c_s Z_s ds = X^0 Z - \int_0^\cdot C_{s-} dZ_s,$$

for any $Z \in \mathcal{Z}$. Now, $X^0 Z$ is a nonnegative local martingale and thus a supermartingale, so using this and that $XZ \geq 0$, we would obtain

$$(6.3) \quad \mathbb{E} \left[\int_0^t c_s Z_s ds \right] \leq x - \mathbb{E} \left[\int_0^t C_{s-} dZ_s \right], \quad t \geq 0.$$

The process $M := \int_0^\cdot C_{s-} dZ_s$ is a local martingale. With $(T_n)_{n \in \mathbb{N}}$ a localising sequence for M , so that $\mathbb{E}[\int_0^{T_n} C_{s-} dZ_s] = 0$, $n \in \mathbb{N}$, (6.3) would convert to $\mathbb{E}[\int_0^{T_n} c_s Z_s ds] \leq x$, $n \in \mathbb{N}$. Letting $n \uparrow \infty$ and using monotone convergence we would obtain a budget constraint $\mathbb{E}[\int_0^\infty c_t Z_t dt] \leq x$. So far so good. The difficulty in taking this route would arise later, when enlarging the dual domain to try to reach its bipolar. One seeks to enlarge the dual domain to processes which are dominated by some process in the original dual domain, and then to show that the enlarged domain is closed with respect to convergence in measure μ . The closedness proof relies on exploiting Fatou convergence of supermartingales. The limit in this procedure is known to be a supermartingale, but there is no guarantee that it is a local martingale deflator. So duality would ultimately fail, unless the enlarged dual domain was made closed by explicit construction. This is why Mostovyi [24] (respectively, Chau et al [3]) used a construction of the form in (5.5), invoking the closure. Ultimately, as stated in Proposition 5.1, all avenues reach the same goal, but the difference is that in our approach we did not have to invoke a closure. We shall return to this discussion of dual domains and their relations in Remark 6.11, once we have established full bipolarity between our abstract primal and dual domains.

From Lemma 6.1, we obtain the form of the dual problem to (3.5) by bounding the achievable utility in the familiar way. For any $c \in \mathcal{A}(x)$ and $Y \in \mathcal{Y}(y)$, we have

$$\begin{aligned} \mathbb{E}\left[\int_0^\infty U(c_t) d\kappa_t\right] &\leq \mathbb{E}\left[\int_0^\infty U(c_t) d\kappa_t\right] + xy - \mathbb{E}\left[\int_0^\infty c_t Y_t dt\right] \\ &= \mathbb{E}\left[\int_0^\infty (U(c_t) - c_t \gamma_t Y_t) d\kappa_t\right] + xy \\ &\leq \mathbb{E}\left[\int_0^\infty V(\gamma_t Y_t) d\kappa_t\right] + xy, \quad x, y > 0, \end{aligned}$$

the last inequality a consequence of (3.7). This motivates the definition of the dual problem associated with the primal problem (3.5), with dual value function $v(\cdot)$ defined by (3.10).

6.2. *Bipolar relations.* In economic terms, the budget constraint (6.1) says that initial capital can finance future consumption, and constitutes a necessary condition for admissible consumption processes. Indeed, another way of defining admissible consumption plans is to insist that, at any time $t \geq 0$, current wealth (suitably deflated) must finance future deflated consumption. We would thus require

$$X_t Y_t \geq \mathbb{E}\left[\int_t^\infty c_s Y_s ds \mid \mathcal{F}_t\right], \quad t \geq 0,$$

for all deflators $Y \in \mathcal{Y}(y)$. Rearranging the above inequality, we have

$$\mathbb{E}\left[\int_0^\infty c_s Y_s ds \mid \mathcal{F}_t\right] \leq X_t Y_t + \int_0^t c_s Y_s ds, \quad t \geq 0.$$

Taking expectations, one recovers the infinite horizon budget constraint provided that the supermartingale condition in (2.5) holds. This is another justification for the choice of dual domain as we have presented it.

Setting $x = y = 1$ in (6.1), the budget constraint gives us that, for $c \in \mathcal{A}$ and $Y \in \mathcal{Y}$, we have $\mathbb{E}[\int_0^\infty c_t Y_t dt] \leq 1$. We thus have the implications

$$(6.4) \quad c \in \mathcal{A} \implies \mathbb{E}\left[\int_0^\infty c_t Y_t dt\right] \leq 1, \quad \forall Y \in \mathcal{Y},$$

and

$$(6.5) \quad Y \in \mathcal{Y} \implies \mathbb{E}\left[\int_0^\infty c_t Y_t dt\right] \leq 1, \quad \forall c \in \mathcal{A}.$$

We wish to establish the reverse implications in some form, if need be by enlarging the domains. First, we establish the reverse implication to (6.4) in Lemma 6.4 below. This requires some version of the Optional Decomposition Theorem (ODT), whose original form is due to El Karoui and Quenez [7] in a Brownian setting. This was generalised to the locally bounded semimartingale case by Kramkov [21], extended to the nonlocally bounded case by Föllmer and Kabanov [8], and to models with constraints by Föllmer and Kramkov [9].

The relevant version of the ODT for us is the one due to Stricker and Yan [31], which uses *deflators* (and in particular LMDs) rather than ELMs. In the proof of Lemma 6.4 we shall apply a part of the Stricker and Yan ODT which applies to the super-hedging of American claims, so is designed to construct a process which can super-replicate a payoff at an arbitrary time. The salient observation is that this result can also be used to dominate a consumption stream, which is how we shall employ it. For clarity and convenience of the reader, we state here the ODT results we need, and afterwards specify precisely which results from [31] we have taken.

For $t \geq 0$, let $\mathcal{T}(t)$ denote the set of \mathbb{F} -stopping times with values in $[t, \infty)$. For $t = 0$, write $\mathcal{T} \equiv \mathcal{T}(0)$, and recall the set \mathcal{Z} of local martingale deflators in (2.6).

THEOREM 6.3 (Stricker and Yan [31] ODT).

(i) Let W be an adapted nonnegative process. The process ZW is a supermartingale for each $Z \in \mathcal{Z}$ if and only if W admits a decomposition of the form

$$W = W_0 + (\phi \cdot S) - A,$$

where ϕ is a predictable S -integrable process such that $Z(\phi \cdot S)$ is a local martingale for each $Z \in \mathcal{Z}$, A is an adapted increasing process with $A_0 = 0$, and for all $Z \in \mathcal{Z}$ and $T \in \mathcal{T}$, $\mathbb{E}[Z_T A_T] < \infty$. In this case, moreover, we have $\sup_{Z \in \mathcal{Z}, T \in \mathcal{T}} \mathbb{E}[Z_T A_T] \leq W_0$.

(ii) Let $b = (b_t)_{t \geq 0}$ be a nonnegative càdlàg process such that $\sup_{Z \in \mathcal{Z}, T \in \mathcal{T}} \mathbb{E}[Z_T b_T] < \infty$. Then there exists an adapted càdlàg process W that dominates b : $W_t \geq b_t$ almost surely for all $t \geq 0$, ZW is a supermartingale for each $Z \in \mathcal{Z}$, and the smallest such process W is given by

$$(6.6) \quad W_t = \operatorname{ess\,sup}_{Z \in \mathcal{Z}, T \in \mathcal{T}(t)} \frac{1}{Z_t} \mathbb{E}[Z_T b_T | \mathcal{F}_t], \quad t \geq 0.$$

Part (i) of Theorem 6.3 is taken from [31], Theorem 2.1. Part (ii) is a combination of [31], Lemma 2.4 and Remark 2.

The following lemma establishes the reverse implication to (6.4).

LEMMA 6.4. Suppose c is a nonnegative adapted càdlàg process that satisfies, for all $Y \in \mathcal{Y}$,

$$(6.7) \quad \mathbb{E} \left[\int_0^\infty c_t Y_t \, dt \right] \leq 1.$$

Then $c \in \mathcal{A}$.

PROOF. Since c is assumed to satisfy (6.7) for all deflators $Y \in \mathcal{Y}$, and since $\mathcal{Z} \subseteq \mathcal{Y}$, (6.7) is satisfied for any $Z \in \mathcal{Z}$. For such a local martingale deflator, and for any stopping time $T \in \mathcal{T}$, the integration by parts formula gives

$$(6.8) \quad C_T Z_T = \int_0^T C_{s-} \, dZ_s + \int_0^T c_s Z_s \, ds, \quad T \in \mathcal{T},$$

where $C := \int_0^\cdot c_s \, ds$ is the nondecreasing candidate cumulative consumption process. The process $M := \int_0^\cdot C_{s-} \, dZ_s$ is a local martingale. Let $(T_n)_{n \in \mathbb{N}}$ be a localising sequence for M , an almost surely increasing sequence of stopping times with $\lim_{n \rightarrow \infty} T_n = \infty$ a.s. such that the stopped process $M_t^{T_n} := M_{t \wedge T_n}$, $t \geq 0$ is a uniformly integrable martingale for each $n \in \mathbb{N}$. Therefore, $\mathbb{E}[\int_0^{T \wedge T_n} C_{s-} \, dZ_s] = 0$ for each $n \in \mathbb{N}$. Using this along with the finiteness of $T \in \mathcal{T}$ and the uniform integrability of M^{T_n} , we have

$$\mathbb{E} \left[\int_0^T C_{s-} \, dZ_s \right] = \mathbb{E} \left[\lim_{n \rightarrow \infty} \int_0^{T \wedge T_n} C_{s-} \, dZ_s \right] = \lim_{n \rightarrow \infty} \mathbb{E} \left[\int_0^{T \wedge T_n} C_{s-} \, dZ_s \right] = 0.$$

Using this in (6.8), we obtain

$$\mathbb{E}[Z_T C_T] = \mathbb{E} \left[\int_0^T Z_s c_s \, ds \right] \leq 1,$$

the last inequality a consequence of the assumption (6.7) and $\mathcal{Z} \subseteq \mathcal{Y}$. Since $Z \in \mathcal{Z}$ and $T \in \mathcal{T}$ were arbitrary, we have

$$\sup_{Z \in \mathcal{Z}, T \in \mathcal{T}} \mathbb{E}[Z_T C_T] \leq 1 < \infty.$$

Thus, from part (ii) of Theorem 6.3, there exists a càdlàg process W that dominates C , so $W_t \geq C_t$, a.s., $\forall t \geq 0$, and ZW is a super-martingale for each $Z \in \mathcal{Z}$. From (6.6), the smallest such W given by

$$W_t = \operatorname{ess\,sup}_{Z \in \mathcal{Z}, T \in \mathcal{T}(t)} \frac{1}{Z_t} \mathbb{E}[Z_T C_T | \mathcal{F}_t], \quad t \geq 0,$$

so that $W_0 \leq 1$. Further, by part (i) of Theorem (6.3), there exists a predictable S -integrable process H and an adapted increasing process A , with $A_0 = 0$, such that W has decomposition $W = W_0 + (H \cdot S) - A$, with $Z(H \cdot S)$ a local martingale for each $Z \in \mathcal{Z}$, and $\mathbb{E}[Z_T A_T] < \infty$ for all $Z \in \mathcal{Z}$ and $T \in \mathcal{T}$.

Since W dominates C , we can define a process X^0 by

$$X_t^0 := 1 + (H \cdot S)_t, \quad t \geq 0,$$

which also dominates C , since its initial value is no smaller than W_0 and we have dispensed with the increasing process A . We observe that X^0 corresponds to the value of a self-financing wealth process with initial capital 1 which dominates C , so that $c \in \mathcal{A}$. \square

We can now assemble consequences of the budget constraint and of Lemma 6.4 which, combined with the bipolar theorem, gives the following polarity properties of the set \mathcal{A} .

LEMMA 6.5 (Polarity properties of \mathcal{A}). *The set $\mathcal{A} \equiv \mathcal{A}(1)$ of admissible consumption plans with initial capital $x = 1$ is a closed, convex and solid subset of $L_+^0(\mu)$. It is equal to the polar of the set $\tilde{\mathcal{Y}} \equiv \tilde{\mathcal{Y}}(1)$ of (3.9) with respect to measure μ :*

$$(6.9) \quad \mathcal{A} = \tilde{\mathcal{Y}}^\circ,$$

so that

$$(6.10) \quad \mathcal{A}^\circ = \tilde{\mathcal{Y}}^{\circ\circ},$$

and \mathcal{A} is equal to its bipolar:

$$(6.11) \quad \mathcal{A}^{\circ\circ} = \mathcal{A}.$$

PROOF. Lemma 6.4, combined with the implication in (6.4), gives the equivalence

$$c \in \mathcal{A} \iff \mathbb{E} \left[\int_0^\infty c_t Y_t dt \right] \leq 1, \quad \forall Y \in \mathcal{Y}.$$

In terms of the measure κ of (3.3) and the set $\tilde{\mathcal{Y}}$ in (3.9) of processes $\gamma Y, Y \in \mathcal{Y}$, we have

$$c \in \mathcal{A} \iff \mathbb{E} \left[\int_0^\infty c_t Y_t^\gamma d\kappa_t \right] \leq 1, \quad \forall Y^\gamma \in \tilde{\mathcal{Y}}.$$

Equivalently, in terms of the measure μ , we have

$$(6.12) \quad c \in \mathcal{A} \iff \int_\Omega c Y^\gamma d\mu \leq 1, \quad \forall Y^\gamma \in \tilde{\mathcal{Y}}.$$

The characterisation (6.12) is the dual representation of \mathcal{A} :

$$\mathcal{A} = \{c \in L_+^0(\mu) : \langle c, Y^\gamma \rangle \leq 1, \text{ for each } Y^\gamma \in \tilde{\mathcal{Y}}\}.$$

This says that \mathcal{A} is the polar of $\tilde{\mathcal{Y}}$, establishing (6.9) and thus (6.10).

Part (i) of the bipolar theorem, Theorem 5.4, along with (6.9), imply that \mathcal{A} is a closed, convex and solid subset of $L_+^0(\mu)$ (since it is equal to the polar of a set) as claimed. Part (ii) of Theorem 5.4 gives $\mathcal{A}^{\circ\circ} \supseteq \mathcal{A}$ with $\mathcal{A}^{\circ\circ}$ the smallest closed, convex, solid set containing \mathcal{A} . But since \mathcal{A} is itself closed, convex and solid, we have (6.11). \square

REMARK 6.6. There are other ways to obtain the closed, convex and solid properties of \mathcal{A} . First, the equivalence (6.12) along with Fatou’s lemma yields that the set \mathcal{A} is closed with respect to the topology of convergence in measure μ . To see this, let $(c^n)_{n \in \mathbb{N}}$ be a sequence in \mathcal{A} which converges μ -a.e. to an element $c \in L^0_+(\mu)$. For arbitrary $Y^\gamma \in \tilde{\mathcal{Y}}$ we obtain, via Fatou’s lemma and the fact that $c^n \in \mathcal{A}$ for each $n \in \mathbb{N}$,

$$\int_{\Omega} cY^\gamma \, d\mu \leq \liminf_{n \rightarrow \infty} \int_{\Omega} c^n Y^\gamma \, d\mu \leq 1,$$

so by (6.12), $c \in \mathcal{A}$, and thus \mathcal{A} is closed. Further, it is straightforward to establish the convexity of \mathcal{A} from its definition. Finally, solidity of \mathcal{A} is also clear: if one can dominate a consumption plan $c \in \mathcal{A}$ with a self-financing wealth process, then one can also dominate any smaller consumption plan with the same portfolio.

The next step is to attempt to reach some form of reverse polarity result to (6.9). It is here that the enlargement of the dual domain from $\tilde{\mathcal{Y}}$ to the set \mathcal{D} of (5.3) comes into play.

To see why this enlargement is needed, we first observe from (6.5) that we have

$$(6.13) \quad Y^\gamma \in \tilde{\mathcal{Y}} \implies \langle c, Y^\gamma \rangle \leq 1, \quad \forall c \in \mathcal{A},$$

which implies that

$$(6.14) \quad \tilde{\mathcal{Y}} \subseteq \mathcal{A}^\circ.$$

We do not have the reverse inclusion, because we do not have the reverse implication to (6.13), so cannot write a full bipolarity relation between sets \mathcal{A} and $\tilde{\mathcal{Y}}$. The enlargement from $\tilde{\mathcal{Y}}$ to the set \mathcal{D} resolves the issue, yielding the consumption bipolarity of Lemma 6.8 below. This procedure, in the spirit of Kramkov and Schachermayer [19], requires us to establish that the enlarged domain is closed in an appropriate topology. Here is the relevant result.

LEMMA 6.7. *The enlarged dual domain $\mathcal{D} \equiv \mathcal{D}(1)$ of (5.3) is closed with respect to the topology of convergence in measure μ .*

The proof of Lemma 6.7 will be given further below. First, we use the result of the lemma to establish the consumption bipolarity result below.

LEMMA 6.8 (Consumption bipolarity). *Given Lemma 6.7, the set \mathcal{D} is a closed, convex and solid subset of $L^0_+(\mu)$, and the sets \mathcal{A} and \mathcal{D} satisfy the bipolarity relations*

$$(6.15) \quad \mathcal{A} = \mathcal{D}^\circ, \quad \mathcal{D} = \mathcal{A}^\circ.$$

PROOF. For any $h \in \mathcal{D}$ there will exist an element $Y^\gamma \in \tilde{\mathcal{Y}}$ such that $h \leq Y^\gamma$, μ -almost everywhere. Hence, the implication (6.13) holds true with \mathcal{D} in place of $\tilde{\mathcal{Y}}$:

$$h \in \mathcal{D} \implies \langle c, h \rangle \leq 1, \quad \forall c \in \mathcal{A},$$

which yields the analogue of (6.14):

$$(6.16) \quad \mathcal{D} \subseteq \mathcal{A}^\circ.$$

Combining (6.10) and (6.16) we have

$$(6.17) \quad \mathcal{D} \subseteq \tilde{\mathcal{Y}}^{\circ\circ}.$$

Part (ii) of the bipolar theorem, Theorem 5.4, says that $\tilde{\mathcal{Y}}^{\circ\circ} \supseteq \tilde{\mathcal{Y}}$ and that $\tilde{\mathcal{Y}}^{\circ\circ}$ is the smallest closed, convex, solid set which contains $\tilde{\mathcal{Y}}$. But \mathcal{D} is also closed, convex and solid (closed due

to Lemma 6.7, convexity following easily from the convexity of $\tilde{\mathcal{Y}}$, and solidity is obvious), and by definition $\mathcal{D} \supseteq \tilde{\mathcal{Y}}$, so we also have

$$(6.18) \quad \mathcal{D} \supseteq \tilde{\mathcal{Y}}^{\circ\circ}.$$

Thus, (6.17) and (6.18) give

$$(6.19) \quad \mathcal{D} = \tilde{\mathcal{Y}}^{\circ\circ}.$$

In other words, in enlarging from $\tilde{\mathcal{Y}}$ to \mathcal{D} we have succeeded in reaching the bipolar of the former.

Combining (6.19) and (6.10) we see that \mathcal{D} is the polar of \mathcal{A} ,

$$(6.20) \quad \mathcal{D} = \mathcal{A}^\circ,$$

so we have the second equality in (6.15). From (6.20) we get $\mathcal{D}^\circ = \mathcal{A}^{\circ\circ}$ which, combined with (6.11), yields the first equality in (6.15), and the proof is complete. \square

It remains to prove Lemma 6.7, which we used above. We recall the concept of Fatou convergence of stochastic processes from Föllmer and Kramkov [9], that will be needed.

DEFINITION 6.9 (Fatou convergence). Let $(Y^n)_{n \in \mathbb{N}}$ be a sequence of processes on a stochastic basis $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, uniformly bounded from below, and let τ be a dense subset of \mathbb{R}_+ . The sequence $(Y^n)_{n \in \mathbb{N}}$ is said to be *Fatou convergent on τ* to a process Y if

$$Y_t = \limsup_{s \downarrow t, s \in \tau} \limsup_{n \rightarrow \infty} Y_s^n = \liminf_{s \downarrow t, s \in \tau} \liminf_{n \rightarrow \infty} Y_s^n, \quad \text{a.s. } \forall t \geq 0.$$

If $\tau = \mathbb{R}_+$, the sequence is simply called *Fatou convergent*.

The relevant consequence for our purposes is Föllmer and Kramkov [9], Lemma 5.2, giving a Fatou convergence result for supermartingales, on a *countable* dense subset of \mathbb{R}_+ . For the convenience of the reader, we state the result here.

LEMMA 6.10 (Fatou convergence of supermartingales, Föllmer and Kramkov [9], Lemma 5.2). *Let $(S^n)_{n \in \mathbb{N}}$ be a sequence of supermartingales, uniformly bounded from below, with $S_0^n = 0, n \in \mathbb{N}$. Let τ be a dense countable subset of \mathbb{R}_+ . Then there is a sequence $(Y^n)_{n \in \mathbb{N}}$ of supermartingales, with $Y^n \in \text{conv}(S^n, S^{n+1}, \dots)$, and a supermartingale Y with $Y_0 \leq 0$, such that $(Y^n)_{n \in \mathbb{N}}$ is Fatou convergent on τ to Y .*

In Lemma 6.10, $\text{conv}(S^n, S^{n+1}, \dots)$ denotes a convex combination $\sum_{k=n}^{N(n)} \lambda_k S^k$ for $\lambda_k \in [0, 1]$ with $\sum_{k=n}^{N(n)} \lambda_k = 1$. The requirement that $S_0^n = 0$ is of course no restriction, since for a supermartingale with (say) $S_0^n = 1$ (as we shall have when we apply these results below for supermartingales in \mathcal{Y}), we can always subtract the initial value 1 to reach a process which starts at zero.

With this preparation, we can now prove Lemma 6.7.

PROOF OF LEMMA 6.7. Let $(h^n)_{n \in \mathbb{N}}$ be a sequence in \mathcal{D} , converging μ -a.e. to some $h \in L^0_+(\mu)$. We want to show that $h \in \mathcal{D}$.

Since $h^n \in \mathcal{D}$, for each $n \in \mathbb{N}$ we have $h^n \leq \gamma \widehat{Y}^n$, μ -a.e for some supermartingale $\widehat{Y}^n \in \mathcal{Y}$. With τ a dense countable subset of \mathbb{R}_+ , Lemma 6.10 implies that there exists a sequence $(Y^n)_{n \in \mathbb{N}}$ of supermartingales, with each $Y^n \in \text{conv}(\widehat{Y}^n, \widehat{Y}^{n+1}, \dots)$, and a supermartingale Y , such that $(Y^n)_{n \in \mathbb{N}}$ is Fatou convergent on τ to Y .

Note that, because \mathcal{Y} is a convex set, for each $n \in \mathbb{N}$ we have $Y^n \in \mathcal{Y}$. Furthermore, by Žitković [34], Lemma 8, (proven there for finite horizon processes, but it is straightforward to verify that the proof goes through without alteration for infinite horizon processes), there is a countable set $K \subset \mathbb{R}_+$ such that for $t \in \mathbb{R}_+ \setminus K$, we have $Y_t = \liminf_{n \rightarrow \infty} Y_t^n$ almost surely, and hence also $Y = \liminf_{n \rightarrow \infty} Y^n$, μ -almost everywhere (since these differ only on a set of measure zero), and indeed $Y = \liminf_{n \rightarrow \infty} Y^n$, $\text{Leb} \times \mathbb{P}$ -almost everywhere, where Leb denotes Lebesgue measure on \mathbb{R}_+ . We shall use these latter properties shortly.

With $c \in \mathcal{A}$ an admissible consumption plan and X the associated wealth process, define a supermartingale sequence $(\widehat{V}^n)_{n \in \mathbb{N}}$ by $\widehat{V}^n := X\widehat{Y}^n + \int_0^\cdot c_s \widehat{Y}_s^n ds$. Then, with $Y^n = \sum_{k=n}^{N(n)} \lambda_k \widehat{Y}^k$ denoting the convex combination which constructs $(Y^n)_{n \in \mathbb{N}}$ from $(\widehat{Y}^n)_{n \in \mathbb{N}}$, define a corresponding sequence $(V^n)_{n \in \mathbb{N}}$ by

$$V^n := \sum_{k=n}^{N(n)} \lambda_k \widehat{V}^k = \sum_{k=n}^{N(n)} \lambda_k \left(X\widehat{Y}^k + \int_0^\cdot c_s \widehat{Y}_s^k ds \right) = XY^n + \int_0^\cdot c_s Y_s^n ds.$$

Because $Y^n \in \mathcal{Y}$ and $c \in \mathcal{A}$ is an admissible consumption plan (equivalently, (X, c) is an admissible investment-consumption strategy), $XY^n + \int_0^\cdot c_s Y_s^n ds$ is a supermartingale for each $n \in \mathbb{N}$, so that

$$\mathbb{E} \left[X_t Y_t^n + \int_s^t c_u Y_u^n du \mid \mathcal{F}_s \right] \leq X_s Y_s^n, \quad 0 \leq s \leq t < \infty, n \in \mathbb{N}.$$

Using this, along with the property that $Y = \liminf_{n \rightarrow \infty} Y^n$, $\text{Leb} \times \mathbb{P}$ -almost everywhere and Fatou’s lemma, we have

$$\begin{aligned} \mathbb{E} \left[X_t Y_t + \int_s^t c_u Y_u du \mid \mathcal{F}_s \right] &= \mathbb{E} \left[\liminf_{n \rightarrow \infty} X_t Y_t^n + \int_s^t \liminf_{n \rightarrow \infty} c_u Y_u^n du \mid \mathcal{F}_s \right] \\ &\leq \liminf_{n \rightarrow \infty} \mathbb{E} \left[X_t Y_t^n + \int_s^t c_u Y_u^n du \mid \mathcal{F}_s \right] \\ &\leq \liminf_{n \rightarrow \infty} X_s Y_s^n \\ &= X_s Y_s, \quad 0 \leq s \leq t < \infty, \end{aligned}$$

which yields the supermartingale property for the process $XY + \int_0^\cdot c_u Y_u du$, and hence that $Y \in \mathcal{Y}$ (since (X, c) is an admissible investment-consumption strategy).

Because $h^n \leq \gamma \widehat{Y}^n$, μ -a.e. for each $n \in \mathbb{N}$, and since $Y^n = \sum_{k=n}^{N(n)} \lambda_k \widehat{Y}^k$, we have

$$(6.21) \quad \gamma Y^n = \sum_{k=n}^{N(n)} \lambda_k \gamma \widehat{Y}^k \geq \sum_{k=n}^{N(n)} \lambda_k h^k, \quad \mu\text{-a.e.}$$

Now using that $Y = \liminf_{n \rightarrow \infty} Y^n$, μ -almost everywhere, taking the limit inferior in (6.21) and recalling that $(h^n)_{n \in \mathbb{N}}$ converges μ -a.e. to h , we obtain

$$\gamma Y \geq \liminf_{n \rightarrow \infty} \sum_{k=n}^{N(n)} \lambda_k h^k = h, \quad \mu\text{-a.e.}$$

That is, $h \leq \gamma Y$ μ -a.e., for $Y \in \mathcal{Y}$, so $h \in \mathcal{D}$, and thus \mathcal{D} is closed. \square

With the consumption bipolarity of Lemma 6.8, we have in fact established Proposition 5.5, so let us confirm this.

PROOF OF PROPOSITION 5.5. With the identification $\mathcal{C} = \mathcal{A}$ (from the definition (5.1)), and the properties of \mathcal{A} established in Lemma 6.5, we have all the claimed properties of \mathcal{C} in items (i) and (ii). The corresponding assertions for \mathcal{D} follow from Lemma 6.8.

The property that \mathcal{C} is bounded in $L^1(\mu)$ follows from the second property in (3.3), that the integral of cumulative consumption with respect to $\kappa \times \mathbb{P} \equiv \mu$, is bounded, and thus \mathcal{C} is bounded in $L^1(\mu)$ and hence in $L^0(\mu)$.

For the L^0 -boundedness of \mathcal{D} , we shall find a positive element $\bar{g} \in \mathcal{C}$ and show that \mathcal{D} is bounded in $L^1(\bar{g} d\mu)$, and hence bounded in $L^0(\mu)$. Choose $\mathcal{A} \ni c_t \equiv \bar{c}_t := e^{-\delta t}$, $t > 0$, for some $\delta > 1$. It is easy to verify that with $x = 1$ and $H \equiv 0$ in (2.1), we have $X \geq 0$, μ -a.e., so $\bar{c} \in \mathcal{A}$. We observe that $\mathcal{C} \ni \bar{g} \equiv \bar{c}$ is strictly positive except on a set of μ -measure zero. We then have, for any $h \in \mathcal{D}$, so $h \leq \gamma Y$ for some $Y \in \mathcal{Y}$ (satisfying $\mathbb{E}[Y_t] \leq 1$, $t \geq 0$),

$$\int_{\Omega} \bar{g} h d\mu \leq \mathbb{E} \left[\int_0^{\infty} e^{-\delta t} Y_t dt \right] = \int_0^{\infty} e^{-\delta t} \mathbb{E}[Y_t] dt \leq \frac{1}{\delta}.$$

Thus, \mathcal{D} is bounded in $L^1(\bar{g} d\mu)$ and hence bounded in $L^0(\mu)$. \square

The $L^1(\mu)$ -boundedness of the primal domain \mathcal{C} is to be contrasted with the terminal wealth problem of Kramkov and Schachermayer [19, 20], in which the dual domain is bounded in $L^1(\mathbb{P})$. This is the source of a switching of roles of the primal and dual domains in the consumption problem compared with the terminal wealth problem, and will manifest itself on numerous occasions in the course of proving the duality theorem in the next section.

6.3. *Local martingale deflators versus consumption deflators.* We can now return to the discussion of Section 5.1, in which we made comparisons with the approaches to bipolarity in Chau et al [3] and Mostovyi [24]. This will lead us to the proof of Proposition 5.1. The proof will demonstrate that the approach in [3, 24] can get a fair way towards establishing bipolarity between \mathcal{A} and the set $\tilde{\mathcal{Z}}$, defined analogously to $\tilde{\mathcal{Y}}$ in (3.9), by

$$\tilde{\mathcal{Z}} := \{Z^\gamma : Z^\gamma := \gamma Z, Z \in \mathcal{Z}\}.$$

One can get a little further by enlarging to D , but it is then necessary to invoke the closure \overline{D} to reach full bipolarity. This establishes the result of the proposition, and shows how the dual domain we chose is not too big, and not too small, to establish bipolarity. A byproduct of the proof is that it shows how results analogous to Mostovyi [24], Lemma 4.2, and Chau et al [3], Lemma 1, which give an equivalence between an admissible consumption plan and an appropriate budget constraint involving either local martingale deflators (in [3]) or martingale deflators (in [24]) can be established without recourse to constructions involving equivalent measures, by judicious use of the Stricker and Yan [31] ODT, rather like the proof of Lemma 6.4. As we pointed out in Section 2.2.1, this is both an aesthetic and mathematically desirable feature.

PROOF OF PROPOSITION 5.1. Consider a consumption plan with initial capital $x = 1$. Using the same arguments as in Remark 6.2 we establish the analogue of (6.3) for $x = 1$:

$$(6.22) \quad \mathbb{E} \left[\int_0^t c_s Z_s ds \right] \leq 1 - \mathbb{E} \left[\int_0^t C_{s-} dZ_s \right], \quad t \geq 0.$$

The process $M := \int_0^\cdot C_{s-} dZ_s$ is a local martingale. With $(T_n)_{n \in \mathbb{N}}$ a localising sequence for M , so that $\mathbb{E}[\int_0^{T_n} C_{s-} dZ_s] = 0$, $n \in \mathbb{N}$, (6.22) converts to $\mathbb{E}[\int_0^{T_n} c_s Z_s ds] \leq 1$, $n \in \mathbb{N}$. Letting $n \uparrow \infty$ and using monotone convergence we obtain a budget constraint in the form $\mathbb{E}[\int_0^\infty c_t Z_t dt] \leq 1$. We thus have the implications analogous to (6.4) and (6.5):

$$(6.23) \quad c \in \mathcal{A} \implies \mathbb{E} \left[\int_0^\infty c_t Z_t dt \right] \leq 1, \quad \forall Z \in \mathcal{Z},$$

and

$$(6.24) \quad Z \in \mathcal{Z} \implies \mathbb{E} \left[\int_0^\infty c_t Z_t dt \right] \leq 1, \quad \forall c \in \mathcal{A}.$$

We can then establish the reverse implication to (6.23) in exactly the same manner as in the proof of Lemma 6.4. In other words, if c is a nonnegative process satisfying the budget constraint, then it is an admissible consumption plan. That is, we have

$$(6.25) \quad \mathbb{E} \left[\int_0^\infty c_t Z_t dt \right] \leq 1, \quad \forall Z \in \mathcal{Z} \implies c \in \mathcal{A}.$$

Thus, following the same arguments as in the proof of Lemma 6.5, we have, from (6.23) and (6.25),

$$\mathcal{A} = \{c \in L^0_+(\mu) : \langle c, \gamma Z \rangle \leq 1, \text{ for each } Z^\gamma = \gamma Z \in \tilde{\mathcal{Z}}\},$$

so that \mathcal{A} is the polar of $\tilde{\mathcal{Z}}$:

$$(6.26) \quad \mathcal{A} = \tilde{\mathcal{Z}}^\circ,$$

implying

$$(6.27) \quad \mathcal{A}^\circ = \tilde{\mathcal{Z}}^{\circ\circ},$$

and that \mathcal{A} is equal to its bipolar:

$$\mathcal{A}^{\circ\circ} = \mathcal{A},$$

by the same arguments as in the proof of Lemma 6.5.

Now, (6.24) gives us that

$$\tilde{\mathcal{Z}} \subseteq \mathcal{A}^\circ,$$

by the same arguments that led to (6.14). We do not have the reverse inclusion, because we do not have the reverse implication to (6.24), so cannot write a full bipolarity relation between sets \mathcal{A} and $\tilde{\mathcal{Z}}$. To this end, one can try enlarging the dual domain from $\tilde{\mathcal{Z}}$ to D , in the same manner that we enlarged from $\tilde{\mathcal{Y}}$ to the set \mathcal{D} when using consumption deflators as dual variables. This yields, in the same manner as we established (6.16),

$$(6.28) \quad D \subseteq \mathcal{A}^\circ.$$

Combining (6.27) and (6.28) we have

$$(6.29) \quad D \subseteq \tilde{\mathcal{Z}}^{\circ\circ}.$$

Here is the crucial point: to establish the reverse inclusion to (6.29) would require that the set D is closed with respect to the topology of convergence in measure μ . But the arguments we used for the proof of Lemma 6.7 to establish this property for the domain \mathcal{D} , break down when applied to the set D , because the limiting supermartingale in the Fatou convergence argument is known only to be a supermartingale, and cannot be shown to be a local martingale deflator. So we are forced to enlarge D itself to its closure \bar{D} .

With this enlargement to \bar{D} , we first show that (6.28), and hence (6.29), extend from D to \bar{D} . Suppose $(h^n)_{n \in \mathbb{N}}$ is a sequence in $D \subseteq \bar{D}$ that converges μ -a.e. to some element $h \in L^0_+(\mu)$. Then $h \in \bar{D}$, since \bar{D} is closed in μ -measure (and a μ -a.e. convergent sequence must also converge in measure μ). Using Fatou's lemma and that $h^n \in D$ we have, for any $c \in \mathcal{A}$

$$\langle c, h \rangle = \left\langle c, \lim_{n \rightarrow \infty} h^n \right\rangle \leq \lim_{n \rightarrow \infty} \langle c, h^n \rangle \leq 1.$$

Thus, we get the implication $h \in \overline{D} \implies \langle c, h \rangle \leq 1, \forall c \in \mathcal{A}$, so we extend (6.28) and, in particular, (6.29) from D to \overline{D} :

$$\overline{D} \subseteq \tilde{\mathcal{Z}}^{\circ\circ},$$

which is the analogue of (6.17). Finally, using the bipolar theorem in the same manner as the last part of the proof of Lemma 6.8, we establish bipolarity between \overline{D} and \mathcal{A} :

$$(6.30) \quad \mathcal{A} = \overline{D}^\circ, \quad \overline{D} = \mathcal{A}^\circ.$$

Comparing (6.30) with (6.15) shows that we have $\mathcal{D} = \overline{D}$, so D is dense in \mathcal{D} , and the proof is complete. \square

REMARK 6.11 (Relations between the bipolars of dual domains). We can now round off the discussion initiated in Remark 6.2, regarding relations between the various dual domains that one might use in establishing a consumption duality. In particular, we examine how these relations transform when passing to the bipolar in the product space.

For brevity, in this remark we shall use the product space $\nu := \text{Leb} \times \mathbb{P}$. Similar remarks pertain with respect to $\mu = \kappa \times \mathbb{P}$, bringing in the extraneous factor γ in defining the solid hulls of the dual domains. Denote by $\text{polar}(A)$ the polar of any set $A \subset L^0_+(\nu)$, with $\text{bipolar}(\cdot)$ and $\text{solid}(\cdot)$ denoting the bipolar and solid hull, respectively.

In the original dual space, we have the inclusions

$$\mathcal{Z} \subseteq \mathcal{Y} \subseteq \mathcal{Y}^0,$$

as noted in Section 2.2. The budget constraint of Lemma 6.1 combined with Lemma 6.4 yielded the properties in Lemma 6.5, and in particular the property (6.9) that the primal domain was the polar of the original dual domain of consumption deflators. The proof of Proposition 5.1 showed that the same property held with local martingale deflators as dual variables (see (6.26)). Thus, in the notation of this remark, we have

$$\text{polar}(\mathcal{Z}) = \text{polar}(\mathcal{Y}),$$

with both sets equal to the primal domain \mathcal{A} .

Passing to the bipolar, we established perfect bipolarity between the primal and dual domains by enlarging the dual space to its solid hull and showing (in Lemma 6.7) that the resulting domain was closed. In the notation of this remark, we have $\text{bipolar}(\mathcal{Y}) = \text{solid}(\mathcal{Y})$, and thus

$$(6.31) \quad \text{bipolar}(\mathcal{Z}) = \text{solid}(\mathcal{Y}).$$

The content of Proposition 5.1 is that we cannot replace $\text{bipolar}(\mathcal{Z})$ with $\text{solid}(\mathcal{Z})$ in (6.31) unless we take the closure, $\text{cl}(\text{solid}(\mathcal{Z}))$, because the Fatou supermartingale convergence method could not guarantee a local martingale deflator as the limiting supermartingale.

The final step in this chain of results is to incorporate the set \mathcal{Y}^0 of wealth deflators. Clearly $\text{solid}(\mathcal{Y}^0) \supseteq \text{solid}(\mathcal{Y})$. Moreover, by a similar (and easier) proof as for Lemma 6.7, $\text{solid}(\mathcal{Y}^0)$ is closed (in essence, a convex combination of wealth deflators Fatou converges to a supermartingale, while the corresponding deflated self-financing wealth also converges to a deflated wealth supermartingale, so the limiting supermartingale is a wealth deflator, and the rest of the proof rests on the solidity of $\text{solid}(\mathcal{Y}^0)$). The remaining arguments are as for $\text{solid}(\mathcal{Y})$, except for the crucial proviso that, as indicated in Remark 6.2, we are not able to establish the infinite horizon budget constraint with wealth deflators, so we appear to have $\mathcal{A} \not\subseteq \text{polar}(\mathcal{Y}^0)$, indicating that the inclusion $\text{solid}(\mathcal{Y}^0) \supset \text{solid}(\mathcal{Y})$ is strict. We thus have

$$\text{bipolar}(\mathcal{Z}) = \text{solid}(\mathcal{Y}) \subset \text{solid}(\mathcal{Y}^0).$$

This remark thus indicates some open questions. If one could show that the budget constraint holds with wealth deflators, then one would arrive at the equality $\mathcal{Y} = \mathcal{Y}^0$. (By the same token, if one were able to show that $\text{solid}(\mathcal{Z})$ is closed, one would arrive at $\mathcal{Z} = \mathcal{Y}$.) One can envisage concrete models where the local martingale deflators and the wealth deflators coincide with the consumption deflators (think of the Brownian models that form the building block of the monograph of Karatzas and Shreve [16]). The message here is that, at the level of abstraction of this paper, utilising Fatou convergence techniques for supermartingales, it does not appear possible to show that the dual domains coalesce. It would appear that in order to establish (for instance) $\mathcal{Y} = \mathcal{Y}^0$ in full generality, would require new techniques.

7. Proofs of the duality theorems. In this section we prove the abstract duality of Theorem 5.6, from which the concrete duality of Theorem 4.1 is then deduced. Throughout this section, we have in place the result of Proposition 5.5, as this bipolarity is the starting point of the duality proof. The proof of Theorem 5.6 proceeds via a series of lemmas. Some of them have a similar flavour to the steps in the celebrated Kramkov and Schachermayer [19, 20] abstract duality proof, but in many places the roles of the primal and dual domains are reversed compared to [19, 20]. This is because in [19, 20] the dual domain is $L^1(\mathbb{P})$ -bounded, but here it is the primal domain that is $L^1(\mu)$ -bounded.

Let us state the basic properties that are taken as given throughout this section.

FACT 7.1. Throughout this section, assume that the utility function satisfies the Inada conditions (3.1), that the sets \mathcal{C} and \mathcal{D} satisfy all the properties in Proposition 5.5, and that the abstract primal and dual value functions in (5.2) and (5.4) satisfy the minimal conditions in (5.8).

All subsequent lemmata and propositions in this section implicitly take Fact 7.1 as given. The first step is to establish weak duality.

LEMMA 7.2 (Weak duality). *The primal and dual value functions $u(\cdot)$ and $v(\cdot)$ of (5.2) and (5.4) satisfy the weak duality bounds*

$$(7.1) \quad v(y) \geq \sup_{x>0} [u(x) - xy], \quad y > 0, \quad \text{equivalently} \quad u(x) \leq \inf_{y>0} [v(y) + xy], \quad x > 0.$$

As a result, $u(x)$ is finitely valued for all $x > 0$. Moreover, we have the limiting relations

$$(7.2) \quad \limsup_{x \rightarrow \infty} \frac{u(x)}{x} \leq 0, \quad \liminf_{y \rightarrow \infty} \frac{v(y)}{y} \geq 0.$$

PROOF. For any $g \in \mathcal{C}(x)$ and $h \in \mathcal{D}(y)$, using the polarity relations in (5.6) and (5.7) we may bound the achievable utility according to

$$(7.3) \quad \begin{aligned} \int_{\Omega} U(g) \, d\mu &\leq \int_{\Omega} U(g) \, d\mu + xy - \int_{\Omega} gh \, d\mu \\ &= \int_{\Omega} (U(g) - gh) \, d\mu + xy \\ &\leq \int_{\Omega} V(h) \, d\mu + xy, \quad x, y > 0, \end{aligned}$$

the last inequality a consequence of (3.7). Maximising the left-hand-side of (7.3) over $g \in \mathcal{C}(x)$ and minimising the right-hand-side over $h \in \mathcal{D}(y)$ gives $u(x) \leq v(y) + xy$ for all $x, y > 0$, and (7.1) follows.

The assumption that $v(y) < \infty$ for all $y > 0$ immediately yields that $u(x)$ is finitely valued for some $x > 0$. Since $U(\cdot)$ is strictly increasing and strictly concave, and given the convexity of \mathcal{C} , these properties are inherited by $u(\cdot)$, which is therefore finitely valued for all $x > 0$. Finally, the relations in (7.1) easily lead to those in (7.2). \square

Above, we obtained concavity and monotonicity of $u(\cdot)$ by using convexity of \mathcal{C} and the properties of $U(\cdot)$. Similar arguments show that $v(\cdot)$ is strictly decreasing and strictly convex. We shall see these properties reproduced in proofs of existence and uniqueness of the optimisers for $u(\cdot)$, $v(\cdot)$.

The next step is to give a compactness lemma for the primal domain.

LEMMA 7.3 (Compactness lemma for \mathcal{C}). *Let $(\tilde{g}^n)_{n \in \mathbb{N}}$ be a sequence in \mathcal{C} . Then there exists a sequence $(g^n)_{n \in \mathbb{N}}$ with $g^n \in \text{conv}(\tilde{g}^n, \tilde{g}^{n+1}, \dots)$, which converges μ -a.e. to an element $g \in \mathcal{C}$ that is μ -a.e. finite.*

PROOF. Delbaen and Schachermayer [6], Lemma A1.1, (adapted from a probability space to the finite measure space $(\Omega, \mathcal{G}, \mu)$) implies the existence of a sequence $(g^n)_{n \in \mathbb{N}}$, with $g^n \in \text{conv}(\tilde{g}^n, \tilde{g}^{n+1}, \dots)$, which converges μ -a.e. to an element g that is μ -a.e. finite because \mathcal{C} is bounded in $L^0(\mu)$ (the finiteness also following from [6], Lemma A1.1). By convexity of \mathcal{C} , each $g^n, n \in \mathbb{N}$ lies in \mathcal{C} . Finally, by Fatou’s lemma, for every $h \in \mathcal{D}$ we have

$$\int_{\Omega} gh \, d\mu = \int_{\Omega} \liminf_{n \rightarrow \infty} g^n h \, d\mu \leq \liminf_{n \rightarrow \infty} \int_{\Omega} g^n h \, d\mu \leq 1,$$

so that $g \in \mathcal{C}$. \square

Results in the style of Lemma 7.3 are standard in these duality proofs. We will see a similar result for the dual domain \mathcal{D} shortly. Typically, the program is to first prove such a result in the dual domain and to follow this with a uniform integrability result for the family $(V^-(h))_{h \in \mathcal{D}(y)}$. This facilitates a proof of existence and uniqueness of the dual minimiser, and of the conjugacy for the value functions by establishing the first relation in (5.9).

Here, as we have alluded to earlier, the natural course of events is switched on its head: one works instead first in the primal domain, with the next step to prove a uniform integrability result for the family $(U^+(g))_{g \in \mathcal{C}(x)}$. This leads to existence and uniqueness of the primal maximiser, and to conjugacy in the form of the second (bi-conjugate) relation in (5.9). The style of proof in the dual domain for the classical program transfers to the primal domain here. This switching of the roles of the primal and dual domains will be an almost continual feature of the analysis of this section, and we shall point out further instances of it in due course. All this stems from the $L^1(\mu)$ -boundedness of the primal (as opposed to the dual) domain in the consumption problem, as pointed out in the first paragraph of this section.

Here is the next step in this chain of results.

LEMMA 7.4 (Uniform integrability of $(U^+(g))_{g \in \mathcal{C}(x)}$). *The family $(U^+(g))_{g \in \mathcal{C}(x)}$ is uniformly integrable, for any $x > 0$.*

The style of the proof is along identical lines to Kramkov and Schachermayer [19], Lemma 3.2, but there it was applied to the concave function $-V(\cdot)$ and in the dual domain to prove the uniform integrability of $(V^-(h))_{h \in \mathcal{D}(y)}$. We are witnessing the switching of the roles of \mathcal{C} and \mathcal{D} .

PROOF OF LEMMA 7.4. Since $U(\cdot)$ is increasing, we need only consider the case where $U(\infty) := \lim_{x \rightarrow \infty} U(x) = +\infty$ (otherwise there is nothing to prove). Let $\varphi :$

$(U(0), U(\infty)) \mapsto (0, \infty)$ denote the inverse of $U(\cdot)$. Then $\varphi(\cdot)$ is strictly increasing. For any $g \in \mathcal{C}(x)$ (so $\int_{\Omega} g \, d\mu \leq Kx$, for some $K < \infty$) we have, for all $x > 0$,

$$\int_{\Omega} \varphi(U^+(g)) \, d\mu \leq \varphi(0) + \int_{\Omega} \varphi(U(g)) \, d\mu = \varphi(0) + \int_{\Omega} g \, d\mu \leq \varphi(0) + Kx.$$

Then, using l'Hôpital's rule and the change of variable $\varphi(x) = y \iff x = U(y)$, we have

$$(7.4) \quad \lim_{x \rightarrow U(\infty)} \frac{\varphi(x)}{x} = \lim_{x \rightarrow \infty} \frac{\varphi(x)}{x} = \lim_{y \rightarrow \infty} \frac{y}{U(y)} = \lim_{y \rightarrow \infty} \frac{1}{U'(y)} = +\infty,$$

on using the Inada conditions (3.1). The $L^1(\mu)$ -boundedness of $\mathcal{C}(x)$ means we can apply the de la Vallée–Poussin theorem (Pham [25], Theorem A.1.2) which, combined with (7.4), implies the uniform integrability of the family $(U^+(g))_{g \in \mathcal{C}(x)}$. \square

REMARK 7.5. There is another way to establish Lemma 7.4 which matches more closely the style of proof in Kramkov and Schachermayer ([20], Lemma 1), and which we shall see applied to the dual domain in Lemma 7.10 to establish uniform integrability of $(V^-(h))_{h \in \mathcal{D}(y)}$. We mention this method here, because at first glance the method of [20], Lemma 1, will not work to establish Lemma 7.4, due to the fact that \mathcal{D} is not bounded in $L^1(\mu)$. However, as we show here, a slight adjustment to the proof can rectify matters. Here is the argument.

Let $(g^n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{C}(x)$, for any fixed $x > 0$. We want to show that the sequence $(U^+(g^n))_{n \in \mathbb{N}}$ is uniformly integrable.

Fix $x > 0$. If $U(\infty) \leq 0$ there is nothing to prove, so assume $U(\infty) > 0$.

If the sequence $(U^+(g^n))_{n \in \mathbb{N}}$ is not uniformly integrable, then, passing if need be to a subsequence still denoted by $(g^n)_{n \in \mathbb{N}}$, we can find a constant $\alpha > 0$ and a disjoint sequence $(A_n)_{n \in \mathbb{N}}$ of sets of (Ω, \mathcal{G}) (so $A_n \in \mathcal{G}$, $n \in \mathbb{N}$ and $A_i \cap A_j = \emptyset$ if $i \neq j$) such that

$$\int_{\Omega} U^+(g^n) \mathbb{1}_{A_n} \, d\mu \geq \alpha, \quad n \in \mathbb{N}.$$

(See, e.g., Pham [25], Corollary A.1.1.) Define, for some $g^0 \in \mathcal{C}$, a sequence $(f^n)_{n \in \mathbb{N}}$ of elements in $L^0_+(\mu)$ by

$$(7.5) \quad f^n := x_0 g^0 + \sum_{k=1}^n g^k \mathbb{1}_{A_k},$$

where $x_0 := \inf\{x > 0 : U(x) \geq 0\}$. (It is here where we are amending the arguments in Kramkov and Schachermayer ([20], Lemma 1): there, one defines the sequence $(f^n)_{n \in \mathbb{N}}$ by $f^n := x_0 + \sum_{k=1}^n g^k \mathbb{1}_{A_k}$, but an examination of the rest of the argument we now give shows that this will require $\int_{\Omega} h \, d\mu \leq 1, \forall h \in \mathcal{D}$, which we do not have, because the constant consumption stream $c \equiv 1$ is not admissible. But we do have instead $\int_{\Omega} gh \, d\mu \leq 1, \forall g \in \mathcal{C}, h \in \mathcal{D}$, which allows the alternative definition of the sequence $(f^n)_{n \in \mathbb{N}}$ in (7.5) to make things work.)

For any $h \in \mathcal{D}$, we have

$$\int_{\Omega} f^n h \, d\mu = \int_{\Omega} \left(x_0 g^0 + \sum_{k=1}^n g^k \mathbb{1}_{A_k} \right) h \, d\mu \leq x_0 + \sum_{k=1}^n \int_{\Omega} g^k h \mathbb{1}_{A_k} \, d\mu \leq x_0 + nx.$$

Thus, $f^n \in \mathcal{C}(x_0 + nx), n \in \mathbb{N}$.

On the other hand, since $U^+(\cdot)$ is nonnegative and nondecreasing,

$$\begin{aligned} \int_{\Omega} U(f^n) \, d\mu &= \int_{\Omega} U^+(f^n) \, d\mu \\ &= \int_{\Omega} U^+\left(x_0 g^0 + \sum_{k=1}^n g^k \mathbb{1}_{A_k}\right) \, d\mu \\ &\geq \int_{\Omega} U^+\left(\sum_{k=1}^n g^k \mathbb{1}_{A_k}\right) \, d\mu \\ &= \sum_{k=1}^n \int_{\Omega} U^+(g^k \mathbb{1}_{A_k}) \, d\mu \geq \alpha n. \end{aligned}$$

Therefore,

$$\begin{aligned} \limsup_{z \rightarrow \infty} \frac{u(z)}{z} &= \limsup_{n \rightarrow \infty} \frac{u(x_0 + nx)}{x_0 + nx} \\ &\geq \limsup_{n \rightarrow \infty} \frac{\int_{\Omega} U(f^n) \, d\mu}{x_0 + nx} \\ &\geq \limsup_{n \rightarrow \infty} \left(\frac{\alpha n}{x_0 + nx}\right) = \frac{\alpha}{x} > 0, \end{aligned}$$

which contradicts the limiting weak duality bound in (7.2). This contradiction establishes the result.

One can now proceed to prove either existence of a unique optimiser in the primal problem, or conjugacy of the value functions. We proceed first the former, followed by conjugacy.

LEMMA 7.6 (Primal existence). *The optimal solution $\widehat{g}(x) \in \mathcal{C}(x)$ to the primal problem (5.2) exists and is unique, so that $u(\cdot)$ is strictly concave.*

PROOF. Fix $x > 0$. Let $(g^n)_{n \in \mathbb{N}}$ be a maximising sequence in $\mathcal{C}(x)$ for $u(x) < \infty$ (the finiteness proven in Lemma 7.2). That is

$$(7.6) \quad \lim_{n \rightarrow \infty} \int_{\Omega} U(g^n) \, d\mu = u(x) < \infty.$$

By the compactness lemma for \mathcal{C} (and thus also for $\mathcal{C}(x) = x\mathcal{C}$), Lemma 7.3, we can find a sequence $(\widehat{g}^n)_{n \in \mathbb{N}}$ of convex combinations, so $\mathcal{C}(x) \ni \widehat{g}^n \in \text{conv}(g^n, g^{n+1}, \dots)$, $n \in \mathbb{N}$, which converges μ -a.e. to some element $\widehat{g}(x) \in \mathcal{C}(x)$. We claim that $\widehat{g}(x)$ is the primal optimiser. That is, that we have

$$(7.7) \quad \int_{\Omega} U(\widehat{g}(x)) \, d\mu = u(x).$$

By concavity of $U(\cdot)$ and (7.6), we have

$$\lim_{n \rightarrow \infty} \int_{\Omega} U(\widehat{g}^n) \, d\mu \geq \lim_{n \rightarrow \infty} \int_{\Omega} U(g^n) \, d\mu = u(x),$$

which, combined with the obvious inequality $u(x) \geq \lim_{n \rightarrow \infty} \int_{\Omega} U(\widehat{g}^n) \, d\mu$ means that we also have, further to (7.6),

$$\lim_{n \rightarrow \infty} \int_{\Omega} U(\widehat{g}^n) \, d\mu = u(x).$$

In other words

$$(7.8) \quad \lim_{n \rightarrow \infty} \int_{\Omega} U^+(\widehat{g}^n) \, d\mu - \lim_{n \rightarrow \infty} \int_{\Omega} U^-(\widehat{g}^n) \, d\mu = u(x) < \infty,$$

and note therefore that both integrals in (7.8) are finite.

From Fatou’s lemma, we have

$$(7.9) \quad \lim_{n \rightarrow \infty} \int_{\Omega} U^-(\widehat{g}^n) \, d\mu \geq \int_{\Omega} U^-(\widehat{g}(x)) \, d\mu.$$

From Lemma 7.4 we have uniform integrability of $(U^+(\widehat{g}^n))_{n \in \mathbb{N}}$, so that

$$(7.10) \quad \lim_{n \rightarrow \infty} \int_{\Omega} U^+(\widehat{g}^n) \, d\mu = \int_{\Omega} U^+(\widehat{g}(x)) \, d\mu.$$

Thus, using (7.9) and (7.10) in (7.8), we obtain

$$u(x) \leq \int_{\Omega} U(\widehat{g}(x)) \, d\mu,$$

which, combined with the obvious inequality $u(x) \geq \int_{\Omega} U(\widehat{g}(x)) \, d\mu$, yields (7.7). The uniqueness of the primal optimiser follows from the strict concavity of $U(\cdot)$, as does the strict concavity of $u(\cdot)$. For this last claim, fix $x_1 < x_2$ and $\lambda \in (0, 1)$, note that $\lambda \widehat{g}(x_1) + (1 - \lambda)\widehat{g}(x_2) \in \mathcal{C}(\lambda x_1 + (1 - \lambda)x_2)$ (yet must be sub-optimal for $u(\lambda x_1 + (1 - \lambda)x_2)$) as it is not guaranteed to equal $\widehat{g}(\lambda x_1 + (1 - \lambda)x_2)$) and therefore, using the strict concavity of $U(\cdot)$,

$$\begin{aligned} u(\lambda x_1 + (1 - \lambda)x_2) &\geq \int_{\Omega} U(\lambda \widehat{g}(x_1) + (1 - \lambda)\widehat{g}(x_2)) \, d\mu \\ &> \lambda u(x_1) + (1 - \lambda)u(x_2). \end{aligned} \quad \square$$

We now establish conjugacy of the value functions. Compared with the classical method of proof in Kramkov and Schachermayer [19], Lemma 3.4, our method is similar, but instead of bounding the elements in the primal domain to create a compact set for the weak* topology $\sigma(L^\infty, L^1)$ on $L^\infty(\mu)$, we bound the elements in the dual domain.¹ Accordingly, we apply a version of the minimax theorem with a minimisation over a compact set and a maximisation over a subset of a vector space (see, e.g., Aubin and Ekeland [1], Theorem 7, Page 319), as opposed to the maximisation over a compact set and a minimisation over a subset of a vector space (as in Strasser [30], Theorem 45.8). (See also Sion [29], Theorem 3.2 and Corollary 3.3, in which either one of the convex spaces involved can be taken to be compact.) This reversal is appropriate because the primal domain is a subset of $L^1(\mu)$, whereas in the terminal wealth problem the dual domain is a subset of $L^1(\mathbb{P})$. The consequence is that we prove the second (bi-conjugate) relation in (5.9), as opposed to the first. Here is the minimax theorem as we shall use it.

THEOREM 7.7 (Minimax). *Let \mathcal{X} be a convex subset of a normed vector space E and let \mathcal{Y} be a $\sigma(E', E)$ -compact convex, subset of the topological dual E' of E . Assume that $f : \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R}$ satisfies the following conditions:*

1. $x \mapsto f(x, y)$ is concave on \mathcal{X} for every $y \in \mathcal{Y}$;
2. $y \mapsto f(x, y)$ is lower semicontinuous and convex on \mathcal{Y} for every $x \in \mathcal{X}$.

¹Recall that a sequence $(h^n)_{n \in \mathbb{N}}$ in $L^\infty(\mu)$ converges to $h \in L^\infty(\mu)$ with respect to the weak* topology $\sigma(L^\infty, L^1)$ if and only if $\langle g, h^n \rangle$ converges to $\langle g, h \rangle$ for each $g \in L^1(\mu)$.

Then:

$$\inf_{y \in \mathcal{Y}} \sup_{x \in \mathcal{X}} f(x, y) = \sup_{x \in \mathcal{X}} \inf_{y \in \mathcal{Y}} f(x, y).$$

Here is the conjugacy result for the primal and dual value functions.

LEMMA 7.8 (Conjugacy). *The primal value function in (5.2) satisfies the bi-conjugacy relation*

$$u(x) = \inf_{y > 0} [v(y) + xy], \quad \text{for each } x > 0,$$

where $v(\cdot)$ is the dual value function in (5.4).

PROOF. For $n \in \mathbb{N}$ denote by \mathcal{B}_n the set of elements in $L^0_+(\mu)$ lying in a ball of radius n :

$$\mathcal{B}_n := \{h \in L^0_+(\mu) : h \leq n, \mu\text{-a.e.}\}.$$

The sets $(\mathcal{B}_n)_{n \in \mathbb{N}}$ are $\sigma(L^\infty, L^1)$ -compact. Because each $g \in \mathcal{C}(x)$ is μ -integrable, $\mathcal{C}(x)$ is a closed, convex subset of the vector space $L^1(\mu)$, so we apply the minimax theorem as given in Theorem 7.7 to the compact set \mathcal{B}_n (n fixed) and the set $\mathcal{C}(x)$, with the function $f(g, h) := \int_\Omega (V(h) + gh) d\mu$, for $g \in \mathcal{C}(x)$, $h \in \mathcal{B}_n$, to give

$$(7.11) \quad \inf_{h \in \mathcal{B}_n} \sup_{g \in \mathcal{C}(x)} \int_\Omega (V(h) + gh) d\mu = \sup_{g \in \mathcal{C}(x)} \inf_{h \in \mathcal{B}_n} \int_\Omega (V(h) + gh) d\mu.$$

By the bipolarity relation $\mathcal{D} = \mathcal{C}^\circ$ in (5.7), an element $h \in L^0_+(\mu)$ lies in $\mathcal{D}(y)$ if and only if $\sup_{g \in \mathcal{C}(x)} \int_\Omega gh d\mu \leq xy$. Thus, the limit as $n \rightarrow \infty$ on the left-hand-side of (7.11) is given as

$$(7.12) \quad \begin{aligned} \lim_{n \rightarrow \infty} \inf_{h \in \mathcal{B}_n} \sup_{g \in \mathcal{C}(x)} \int_\Omega (V(h) + gh) d\mu &= \inf_{y > 0} \inf_{h \in \mathcal{D}(y)} \left(\int_\Omega V(h) d\mu + xy \right) \\ &= \inf_{y > 0} [v(y) + xy]. \end{aligned}$$

Now consider the right-hand side of (7.11). Define

$$U_n(x) := \inf_{0 < y \leq n} [V(y) + xy], \quad x > 0, n \in \mathbb{N}.$$

The right-hand side of (7.11) is then given as

$$\sup_{g \in \mathcal{C}(x)} \inf_{h \in \mathcal{B}_n} \int_\Omega (V(h) + gh) d\mu = \sup_{g \in \mathcal{C}(x)} \int_\Omega U_n(g) d\mu =: u_n(x),$$

so that taking the limit as $n \rightarrow \infty$ and equating this with the limit obtained in (7.12), we have

$$(7.13) \quad \lim_{n \rightarrow \infty} u_n(x) = \inf_{y > 0} [v(y) + xy] \geq u(x),$$

with the inequality due to the weak duality bound in (7.1). Consequently, we will be done if we can now show that we also have

$$\lim_{n \rightarrow \infty} u_n(x) \leq u(x).$$

Evidently, $(u_n(x))_{n \in \mathbb{N}}$ is a decreasing sequence satisfying the limiting inequality in (7.13). Let $(\tilde{g}^n)_{n \in \mathbb{N}}$ be a maximising sequence in $\mathcal{C}(x)$ for $\lim_{n \rightarrow \infty} u_n(x)$, so such that

$$\lim_{n \rightarrow \infty} \int_\Omega U_n(\tilde{g}^n) d\mu = \lim_{n \rightarrow \infty} u_n(x).$$

The compactness lemma for \mathcal{C} , Lemma 7.3, implies the existence of a sequence $(g^n)_{n \in \mathbb{N}}$ in $\mathcal{C}(x)$, with $g^n \in \text{conv}(\tilde{g}^n, \tilde{g}^{n+1}, \dots)$, which converges μ -a.e. to an element $g \in \mathcal{C}(x)$. Now, $U_n(x) = U(x)$ for $x \geq I(n)$, where $I(\cdot) = -V'(\cdot)$ is the inverse of $U'(\cdot)$ (and $U_n(\cdot) \rightarrow U(\cdot)$ as $n \rightarrow \infty$). So we deduce from Lemma 7.4 that the sequence $(U_n^+(g^n))_{n \in \mathbb{N}}$ is uniformly integrable, and hence that

$$(7.14) \quad \lim_{n \rightarrow \infty} \int_{\Omega} U_n^+(g^n) \, d\mu = \int_{\Omega} U^+(g) \, d\mu.$$

On the other hand, from Fatou’s lemma, we have

$$(7.15) \quad \lim_{n \rightarrow \infty} \int_{\Omega} U_n^-(g^n) \, d\mu \geq \int_{\Omega} U^-(g) \, d\mu,$$

so (7.14) and (7.15) give

$$(7.16) \quad \lim_{n \rightarrow \infty} \int_{\Omega} U_n(g^n) \, d\mu \leq \int_{\Omega} U(g) \, d\mu.$$

Finally, using concavity of $U_n(\cdot)$ and (7.16), we obtain

$$\lim_{n \rightarrow \infty} u_n(x) = \lim_{n \rightarrow \infty} \int_{\Omega} U_n(\tilde{g}^n) \, d\mu \leq \lim_{n \rightarrow \infty} \int_{\Omega} U_n(g^n) \, d\mu \leq \int_{\Omega} U(g) \, d\mu \leq u(x),$$

and the proof is complete. \square

We now move on to the dual side of the analysis. We begin with a similar compactness lemma to Lemma 7.3, but now for the dual domain. The proof is identical to the proof of Lemma 7.3 so is omitted.

LEMMA 7.9 (Compactness lemma for \mathcal{D}). *Let $(\tilde{h}^n)_{n \in \mathbb{N}}$ be a sequence in \mathcal{D} . Then there exists a sequence $(h^n)_{n \in \mathbb{N}}$ with $h^n \in \text{conv}(\tilde{h}^n, \tilde{h}^{n+1}, \dots)$, which converges μ -a.e. to an element $h \in \mathcal{D}$ that is μ -a.e. finite.*

Next, we have an analogous result to Lemma 7.4, but for the dual variables, concerning the uniform integrability of a sequence $(V^-(h^n))_{n \in \mathbb{N}}$ for $h^n \in \mathcal{D}(y)$ (which will subsequently lead to a lemma on existence and uniqueness of the dual optimiser). The proof is in the style of Kramkov and Schachermayer ([20], Lemma 1), but there the technique was applied to a corresponding primal result akin to Lemma 7.4.

LEMMA 7.10 (Uniform integrability of $(V^-(h^n))_{n \in \mathbb{N}}$, $h^n \in \mathcal{D}(y)$). *Let $(h^n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{D}(y)$, for any fixed $y > 0$. The sequence $(V^-(h^n))_{n \in \mathbb{N}}$ is uniformly integrable.*

PROOF. Fix $y > 0$. If $V(\infty) \geq 0$ there is nothing to prove, so assume $V(\infty) < 0$.

If the sequence $(V^-(h^n))_{n \in \mathbb{N}}$ is not uniformly integrable, then, passing if need be to a subsequence still denoted by $(h^n)_{n \in \mathbb{N}}$, we can find a constant $\alpha > 0$ and a disjoint sequence $(A_n)_{n \in \mathbb{N}}$ of sets of (Ω, \mathcal{G}) (so $A_n \in \mathcal{G}$, $n \in \mathbb{N}$ and $A_i \cap A_j = \emptyset$ if $i \neq j$) such that

$$\int_{\Omega} V^-(h^n) \mathbb{1}_{A_n} \, d\mu \geq \alpha, \quad n \in \mathbb{N}.$$

(See, e.g., Pham [25], Corollary A.1.1.) Define, for some $h^0 \in \mathcal{D}$, a sequence $(f^n)_{n \in \mathbb{N}}$ of elements in $L^0_+(\mu)$ by

$$f^n := y_0 h^0 + \sum_{k=1}^n h^k \mathbb{1}_{A_k},$$

where $y_0 := \inf\{y > 0 : V(y) \leq 0\}$. For any $g \in \mathcal{C}$ we have

$$\int_{\Omega} g f^n \, d\mu = \int_{\Omega} g \left(y_0 h^0 + \sum_{k=1}^n h^k \mathbb{1}_{A_k} \right) \, d\mu \leq y_0 + \sum_{k=1}^n \int_{\Omega} g h^k \mathbb{1}_{A_k} \, d\mu \leq y_0 + n y.$$

Thus, $f^n \in \mathcal{D}(y_0 + n y)$, $n \in \mathbb{N}$.

On the other hand, since $V^-(\cdot)$ is nonnegative and nondecreasing,

$$\begin{aligned} \int_{\Omega} V(f^n) \, d\mu &= - \int_{\Omega} V^-(f^n) \, d\mu \\ &= - \int_{\Omega} V^-\left(y_0 + \sum_{k=1}^n h^k \mathbb{1}_{A_k}\right) \, d\mu \\ &\leq - \int_{\Omega} V^-\left(\sum_{k=1}^n h^k \mathbb{1}_{A_k}\right) \, d\mu \\ &= - \sum_{k=1}^n \int_{\Omega} V^-(h^k \mathbb{1}_{A_k}) \, d\mu \leq -\alpha n. \end{aligned}$$

Therefore,

$$\begin{aligned} \liminf_{z \rightarrow \infty} \frac{v(z)}{z} &= \liminf_{n \rightarrow \infty} \frac{v(y_0 + n y)}{y_0 + n y} \\ &\leq \liminf_{n \rightarrow \infty} \frac{\int_{\Omega} V(f^n) \, d\mu}{y_0 + n y} \leq \liminf_{n \rightarrow \infty} \left(\frac{-\alpha n}{y_0 + n y} \right) = -\frac{\alpha}{y} < 0, \end{aligned}$$

which contradicts the limiting weak duality bound in (7.2). This contradiction establishes the result. \square

One can now proceed to prove existence of a unique optimiser in the dual problem. The proof is on similar lines to the proof of primal existence (Lemma 7.6), with adjustments for minimisation as opposed to maximisation, convexity of $V(\cdot)$ replacing concavity of $U(\cdot)$ and Lemma 7.10 replacing Lemma 7.4. For brevity, therefore, the proof is omitted.

LEMMA 7.11 (Dual existence). *The optimal solution $\widehat{h}(y) \in \mathcal{D}(y)$ to the dual problem (5.4) exists and is unique, so that $v(\cdot)$ is strictly convex.*

We now move on to further characterise the derivatives of the value functions, as well as the primal and dual optimisers. The first result is on the derivative of the primal value function $u(\cdot)$ at infinity (equivalently, the derivative of the dual value function $v(\cdot)$ at zero). Once again, because of the switching of the roles of the primal and dual sets in our proofs compared with those of the terminal wealth problem, the proof of the following lemma matches closely the proof in Kramkov and Schachermayer [19], Lemma 3.5, of the derivative of $v(\cdot)$ at infinity (giving the derivative of $u(\cdot)$ at zero).

LEMMA 7.12. *The derivatives of the primal value function in (5.2) at infinity and of the dual value function in (5.4) at zero are given by*

$$(7.17) \quad u'(\infty) := \lim_{x \rightarrow \infty} u'(x) = 0, \quad -v'(0) := \lim_{y \downarrow 0} (-v'(y)) = +\infty.$$

PROOF. By the conjugacy result in Lemma 7.8 between the value functions, the assertions in (7.17) are equivalent. We shall prove the first assertion.

The primal value function $u(\cdot)$ is strictly concave and strictly increasing, so there is a finite nonnegative limit $u'(\infty) := \lim_{x \rightarrow \infty} u'(x)$. Because $U(\cdot)$ is increasing with $\lim_{x \rightarrow \infty} U'(x) = 0$, for any $\epsilon > 0$ there exists a number C_ϵ such that $U(x) \leq C_\epsilon + \epsilon x, \forall x > 0$. Using this, the $L^1(\mu)$ -boundedness of \mathcal{C} (so that $\int_{\Omega} g \, d\mu \leq Kx, \forall g \in \mathcal{C}(x)$, for some $K < \infty$) and l'Hôpital's rule, we have, with $\int_{\Omega} d\mu =: \delta > 0$,

$$\begin{aligned} 0 &\leq \lim_{x \rightarrow \infty} u'(x) = \lim_{x \rightarrow \infty} \frac{u(x)}{x} \\ &= \lim_{x \rightarrow \infty} \sup_{g \in \mathcal{C}(x)} \int_{\Omega} \frac{U(g)}{x} \, d\mu \\ &\leq \lim_{x \rightarrow \infty} \sup_{g \in \mathcal{C}(x)} \int_{\Omega} \frac{C_\epsilon + \epsilon g}{x} \, d\mu \\ &\leq \lim_{x \rightarrow \infty} \left(\frac{C_\epsilon \delta}{x} + \epsilon K \right) = \epsilon K, \end{aligned}$$

and taking the limit as $\epsilon \downarrow 0$ gives the result. \square

The final step in the series of lemmas that will furnish us with the proof of Theorem 5.6 is to characterise the derivative of the primal value function $u(\cdot)$ at zero (equivalently, the derivative of the dual value function $v(\cdot)$ at infinity) along with a duality characterisation of the primal and dual optimisers.

LEMMA 7.13.

(i) *The derivatives of the primal value function in (5.2) at zero and of the dual value function in (5.4) at infinity are given by*

$$(7.18) \quad u'(0) := \lim_{x \downarrow 0} u'(x) = +\infty, \quad -v'(\infty) := \lim_{y \rightarrow \infty} (-v'(y)) = 0.$$

(ii) *For any fixed $x > 0$, with $y = u'(x)$ (equivalently $x = -v'(y)$), the primal and dual optimisers $\widehat{g}(x), \widehat{h}(y)$ are related by*

$$(7.19) \quad U'(\widehat{g}(x)) = \widehat{h}(y) = \widehat{h}(u'(x)), \quad \mu\text{-a.e.},$$

and satisfy

$$(7.20) \quad \int_{\Omega} \widehat{g}(x) \widehat{h}(y) \, d\mu = xy = xu'(x).$$

(iii) *The derivatives of the value functions satisfy the relations*

$$(7.21) \quad xu'(x) = \int_{\Omega} U'(\widehat{g}(x)) \widehat{g}(x) \, d\mu, \quad yv'(y) = \int_{\Omega} V'(\widehat{h}(y)) \widehat{h}(y) \, d\mu, \quad x, y > 0.$$

PROOF. Recall the inequality (3.7), which also applies to the value functions because they are also conjugate by Lemma 7.8. We thus have, in addition to (3.7),

$$(7.22) \quad v(y) \geq u(x) - xy, \quad \forall x, y > 0, \text{ with equality iff } y = u'(x).$$

With $\widehat{g}(x) \in \mathcal{C}(x), x > 0$ and $\widehat{h}(y) \in \mathcal{D}(y), y > 0$ denoting the primal and dual optimisers, the bipolarity relations (5.6) and (5.7) imply that we have

$$\int_{\Omega} \widehat{g}(x) \widehat{h}(y) \, d\mu \leq xy, \quad x, y > 0.$$

Using this as well as (3.7) and (7.22) we have

$$(7.23) \quad 0 \leq \int_{\Omega} (V(\widehat{h}(y)) - U(\widehat{g}(x)) + \widehat{g}(x)\widehat{h}(y)) \, d\mu \leq v(y) - u(x) + xy, \quad x, y > 0,$$

The right-hand-side of (7.23) is zero if and only if $y = u'(x)$, due to (7.22), and the nonnegative integrand must then be μ -a.e. zero, which by (3.7) can only happen if (7.19) holds, which establishes that primal-dual relation.

Thus, for any fixed $x > 0$ and with $y = u'(x)$, and hence equality in (7.23), we have

$$\begin{aligned} 0 &= \int_{\Omega} (V(\widehat{h}(y)) - U(\widehat{g}(x)) + \widehat{g}(x)\widehat{h}(y)) \, d\mu \\ &= v(y) - u(x) + \int_{\Omega} \widehat{g}(x)\widehat{h}(y) \, d\mu \\ &= v(y) - u(x) + xy, \quad y = u'(x), \end{aligned}$$

which implies that (7.20) must hold. Inserting the explicit form of $\widehat{h}(y) = U'(\widehat{g}(x))$ into (7.20) yields the first relation in (7.21). Similarly, setting $\widehat{g}(x) = I(\widehat{h}(y)) = -V'(\widehat{h}(y))$ into (7.20), with $x = -v'(y)$ (equivalent to $y = u'(x)$), yields the second relation in (7.21).

It remains to establish the relations in (7.18), which are equivalent assertions. We shall prove the first one. This will use the fact that \mathcal{C} is a subset of $L^1(\mu)$. In the terminal wealth case, one typically proves the second assertion using the property that the dual domain lies within $L^1(\mathbb{P})$. This is the switching of the roles of the primal and dual domains in the consumption problem, that we have witnessed throughout this section.

From the first relation in (7.21) and the fact that

$$(7.24) \quad \int_{\Omega} gh \, d\mu \leq xy, \quad \forall g \in \mathcal{C}(x), h \in \mathcal{D}(y), x, y > 0,$$

we see that, for any $x > 0$, we have $U'(\widehat{g}(x)) \in \mathcal{D}(u'(x))$. Thus, for any $g \in \mathcal{C}$, (7.24) implies that

$$(7.25) \quad u'(x) \geq \int_{\Omega} U'(\widehat{g}(x))g \, d\mu, \quad \forall g \in \mathcal{C},$$

which we shall make use of shortly.

Since $\mathcal{C}(x)$ is a subset of $L^1(\mu)$, we have $\int_{\Omega} \widehat{g}(x) \, d\mu \leq Kx$, for some $K < \infty$, and hence

$$(7.26) \quad \int_{\Omega} \frac{\widehat{g}(x)}{x} \, d\mu \leq K, \quad \forall x > 0.$$

Using Fatou's lemma in (7.26) we have

$$K \geq \liminf_{x \downarrow 0} \int_{\Omega} \frac{\widehat{g}(x)}{x} \, d\mu \geq \int_{\Omega} \liminf_{x \downarrow 0} \left(\frac{\widehat{g}(x)}{x} \right) \, d\mu,$$

which, given that $\widehat{g}(x)/x$ is nonnegative, gives that $\liminf_{x \downarrow 0} (\widehat{g}(x)/x) < \infty$, μ -a.e. Therefore, writing $\widehat{g}(x) =: x\widehat{g}^x$, which defines a unique element $\widehat{g}^x \in \mathcal{C}$, we have

$$\widehat{g}^0 := \liminf_{x \downarrow 0} \widehat{g}^x = \liminf_{x \downarrow 0} \frac{\widehat{g}(x)}{x} < \infty, \quad \mu\text{-a.e.}$$

Using this property and applying Fatou's lemma to (7.25) we obtain, on using $U'(0) = +\infty$,

$$+\infty \geq \liminf_{x \downarrow 0} u'(x) \geq \liminf_{x \downarrow 0} \int_{\Omega} U'(x\widehat{g}^x)g \, d\mu \geq \int_{\Omega} \liminf_{x \downarrow 0} U'(x\widehat{g}^x)g \, d\mu = +\infty,$$

which gives us the first relation in (7.18). \square

We have now established all results that give the duality in Theorem 5.6, so let us confirm this.

PROOF OF THEOREM 5.6. Lemma 7.8 implies the relations (5.9) of item (i). The statements in item (ii) are implied by Lemma 7.6 and Lemma 7.11. Items (iii) and (iv) follow from Lemma 7.12 and Lemma 7.13. \square

We are almost ready to prove the concrete duality in Theorem 4.1, because Theorem 5.6 readily implies nearly all of the assertions of Theorem 4.1. The outstanding assertion is the characterisation of the optimal wealth process in (4.3) and the associated uniformly integrable martingale property of the deflated wealth plus cumulative deflated consumption process $\widehat{X}(x)\widehat{Y}(y) + \int_0^\cdot \widehat{c}_s(x)\widehat{Y}_s(y) ds$. So we proceed to establish these assertions in the proposition below, which turns out to be interesting in its own right. We take as given the other assertions of Theorem 4.1, and in particular the optimal budget constraint in (4.2). We shall confirm the proof of Theorem 4.1 in its entirety after the proof of the next result.

PROPOSITION 7.14 (Optimal wealth process). *Given the saturated budget constraint equality in (4.2), the optimal wealth process is characterised by (4.3). The process*

$$\widehat{M}_t := \widehat{X}_t(x)\widehat{Y}_t(y) + \int_0^t \widehat{c}_s(x)\widehat{Y}_s(y) ds, \quad 0 \leq t < \infty,$$

is a uniformly integrable martingale, converging to an integrable random variable \widehat{M}_∞ , so the martingale extends to $[0, \infty]$. The process $\widehat{X}(x)\widehat{Y}(y)$ is a potential, that is, a nonnegative supermartingale satisfying $\lim_{t \rightarrow \infty} \mathbb{E}[\widehat{X}_t(x)\widehat{Y}_t(y)] = 0$. Moreover, $\widehat{X}_\infty(x)\widehat{Y}_\infty(y) = 0$, almost surely.

PROOF. It simplifies notation if we take $x = y = 1$, and is without loss of generality: although $y = u'(x)$ in (4.2), one can always multiply the utility function by an arbitrary constant so as to ensure that $u'(1) = 1$. We thus have the optimal budget constraint

$$(7.27) \quad \mathbb{E} \left[\int_0^\infty \widehat{c}_t \widehat{Y}_t dt \right] = 1,$$

for $\widehat{c} \equiv \widehat{c}(1) \in \mathcal{A}$ and $\widehat{Y} \equiv \widehat{Y}(1) \in \mathcal{Y}$. Since $\widehat{c} \in \mathcal{A}$, we know there exists an optimal wealth process $\widehat{X} \equiv \widehat{X}(1)$ and an associated optimal trading strategy \widehat{H} , such that

$$\widehat{X} = 1 + (\widehat{H} \cdot S) - \int_0^\cdot \widehat{c}_s ds \geq 0,$$

and such that $\widehat{M} := \widehat{X}\widehat{Y} + \int_0^\cdot \widehat{c}_s \widehat{Y}_s ds$ is a supermartingale over $[0, \infty)$. The supermartingale condition, by the same arguments that led to the derivation of the budget constraint in Lemma 6.1, leads to the inequality $\mathbb{E}[\int_0^\infty \widehat{c}_t \widehat{Y}_t dt] \leq 1$ instead of the equality (7.27). Similarly, if the supermartingale is strict, we get a strict inequality in place of (7.27). We thus deduce that \widehat{M} must be a martingale over $[0, \infty)$. We shall show that this extends to $[0, \infty]$, along with the other claims in the proposition.

Since \widehat{M} is a martingale, the (nonnegative càdlàg) deflated wealth process $\widehat{X}\widehat{Y}$ is a martingale minus a nondecreasing process, so is a nonnegative càdlàg supermartingale, and thus (by Cohen and Elliott [4], Corollary 5.2.2, e.g.) converges to an integrable limiting random variable $\widehat{X}_\infty\widehat{Y}_\infty := \lim_{t \rightarrow \infty} \widehat{X}_t\widehat{Y}_t$ (and moreover $\widehat{X}_t\widehat{Y}_t \geq \mathbb{E}[\widehat{X}_\infty\widehat{Y}_\infty]$, $t \geq 0$). The nondecreasing integral in \widehat{M} clearly also converges to an integrable random variable, by virtue of the budget constraint. Thus, \widehat{M} also converges to an integrable random variable $\widehat{M}_\infty := \widehat{X}_\infty\widehat{Y}_\infty + \int_0^\infty \widehat{c}_t \widehat{Y}_t dt$. By Protter [26], Theorem I.13, the extended martingale over $[0, \infty]$, $(\widehat{M}_t)_{t \in [0, \infty]}$ is then uniformly integrable, as claimed.

The martingale condition gives

$$\mathbb{E}\left[\widehat{X}_t \widehat{Y}_t + \int_0^t \widehat{c}_s \widehat{Y}_s \, ds\right] = 1, \quad 0 \leq t < \infty.$$

Taking the limit as $t \rightarrow \infty$, using monotone convergence in the second term within the expectation and utilising (7.27) yields

$$\lim_{t \rightarrow \infty} \mathbb{E}[\widehat{X}_t \widehat{Y}_t] = 0,$$

so that $\widehat{X}\widehat{Y}$ is a potential, as claimed.

Using the uniform integrability of \widehat{M} and taking the limit as $t \rightarrow \infty$ in $\mathbb{E}[\widehat{M}_t] = 1, t \geq 0$, we have

$$1 = \lim_{t \rightarrow \infty} \mathbb{E}[\widehat{M}_t] = \mathbb{E}\left[\lim_{t \rightarrow \infty} \widehat{M}_t\right] = \mathbb{E}[\widehat{X}_\infty \widehat{Y}_\infty] + 1,$$

on using (7.27). Hence, we get $\mathbb{E}[\widehat{X}_\infty \widehat{Y}_\infty] = 0$ and, since $\widehat{X}_\infty \widehat{Y}_\infty$ is nonnegative, we deduce that $\widehat{X}_\infty \widehat{Y}_\infty = 0$, almost surely as claimed.

We can now assemble these ingredients to arrive at the optimal wealth process formula (4.3). Applying the martingale condition again, this time over $[t, u]$ for some $t \geq 0$, we have

$$\mathbb{E}\left[\widehat{X}_u \widehat{Y}_u + \int_0^u \widehat{c}_s \widehat{Y}_s \, ds \mid \mathcal{F}_t\right] = \widehat{X}_t \widehat{Y}_t + \int_0^t \widehat{c}_s \widehat{Y}_s \, ds, \quad 0 \leq t \leq u < \infty.$$

Taking the limit as $u \rightarrow \infty$ and using the uniform integrability of \widehat{M} we obtain

$$\mathbb{E}\left[\lim_{u \rightarrow \infty} \left(\widehat{X}_u \widehat{Y}_u + \int_0^u \widehat{c}_s \widehat{Y}_s \, ds\right) \mid \mathcal{F}_t\right] = \widehat{X}_t \widehat{Y}_t + \int_0^t \widehat{c}_s \widehat{Y}_s \, ds, \quad t \geq 0,$$

which, on using $\widehat{X}_\infty \widehat{Y}_\infty = 0$, rearranges to

$$\widehat{X}_t \widehat{Y}_t = \mathbb{E}\left[\int_t^\infty \widehat{c}_s \widehat{Y}_s \, ds \mid \mathcal{F}_t\right], \quad t \geq 0,$$

which establishes (4.3), and the proof is complete. \square

PROOF OF THEOREM 4.1. Given the definitions of the sets $\mathcal{C}(x)$ and $\mathcal{D}(y)$ in (5.1) and (5.3), respectively, and the identification of the abstract value functions in (5.2) and (5.4) with their concrete counterparts in (3.5) and (3.10), Theorem 5.6 implies all the assertions of Theorem 4.1, with the exception of the optimal wealth process formula (4.3) and the uniform integrability of $\widehat{X}(x)\widehat{Y}(y) + \int_0^\cdot \widehat{c}_s(x)\widehat{Y}_s(y) \, ds$, which are established by Proposition 7.14. \square

8. An example: Bessel process with stochastic volatility and correlation. We end with an example of an infinite horizon consumption problem in an incomplete market model with strict local martingale deflators, which is covered in our framework.

EXAMPLE 8.1 (Three-dimensional Bessel process with stochastic volatility and correlation, CRRA utility). Take an infinite horizon complete stochastic basis $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, with \mathbb{F} satisfying the usual hypotheses. Let (W, W^\perp) be a two-dimensional Brownian motion. We take \mathbb{F} to be the augmented filtration generated by (W, W^\perp) .

Let B denote the process which solves the stochastic differential equation

$$dB_t = \frac{1}{B_t} dt + dW_t =: \lambda_t dt + dW_t, \quad B_0 = 1.$$

The process B is the well-known three-dimensional Bessel process. The process $\lambda := 1/B$ will be the market price of risk of a stock with price process S and stochastic volatility

process $Y > 0$, driven by the correlated Brownian motion $\widetilde{W} := \rho W + \sqrt{1 - \rho^2} W^\perp$, and with $\rho \in [-1, 1]$ some \mathbb{F} -adapted stochastic correlation. We need not specify the dynamics of Y or ρ any further for the purposes of the example. The stock price dynamics are given by

$$dS_t = Y_t S_t dB_t = Y_t S_t (\lambda_t dt + dW_t).$$

Take a constant relative risk aversion (CRRA) utility function: $U(x) := x^p/p$, $p < 1$, $p \neq 0$, $x > 0$. The results for logarithmic utility $U(\cdot) = \log(\cdot)$ can be recovered by setting $p = 0$ in the final formulae, and this can be verified by carrying out the analysis directly for that case. Take the measure κ to be given by $d\kappa_t = e^{-\alpha t} dt$, for a positive discount rate α , so that $\gamma_t = e^{\alpha t}$, $t \geq 0$. The primal value function is

$$u(x) := \sup_{c \in \mathcal{A}(x)} \mathbb{E} \left[\int_0^\infty e^{-\alpha t} U(c_t) dt \right], \quad x > 0.$$

The wealth process incorporating consumption satisfies

$$dX_t = Y_t \pi_t (\lambda_t dt + dW_t) - c_t dt, \quad X_0 = x,$$

where $\pi = HS$ is the trading strategy expressed in terms of the wealth placed in the stock, with H the process for the number of shares.

With $\mathcal{E}(\cdot)$ denoting the stochastic exponential, the deflators in this model are given by local martingale deflators of the form

$$(8.1) \quad Z := \mathcal{E}(-\lambda \cdot W - \psi \cdot W^\perp),$$

for an arbitrary process ψ satisfying $\int_0^t \psi_s^2 ds < \infty$ almost surely for all $t \geq 0$, with each such ψ leading to a different deflator: this market is of course incomplete. In the case that Y and ρ are deterministic, the market is complete and there is a unique deflator $Z^{(0)} := \mathcal{E}(-\lambda \cdot W)$. It is well-known (see, for instance, Larsen [22], Example 2.2) that $Z^{(0)}$ is a strict local martingale and, what is more, that $Z^{(0)} = \lambda$ and that λ is square integrable. The strict local martingale property is inherited by Z in (8.1), for any choice of integrand ψ .

The deflated wealth plus cumulative deflated consumption process M is then given by

$$(8.2) \quad \begin{aligned} M_t &:= X_t Z_t + \int_0^t c_s Z_s ds \\ &= x + \int_0^t Z_s (Y_s \pi_s - \lambda_s X_s) dW_s - \int_0^t X_s Z_s \psi_s dW_s^\perp, \quad t \geq 0, \end{aligned}$$

which is a nonnegative local martingale and thus a supermartingale.

The convex conjugate of the utility function is $V(y) := -y^q/q$, $y > 0$, where $q < 1$, $q \neq 0$ is the conjugate variable to p , satisfying $1 - q = (1 - p)^{-1}$. The dual value function is given by

$$v(y) := \inf_{Z \in \mathcal{Z}} \mathbb{E} \left[\int_0^\infty e^{-\alpha t} V(y Z_t e^{\alpha t}) dt \right], \quad y > 0.$$

Denote the unique dual minimiser by \widehat{Z} , given by

$$\widehat{Z} := \mathcal{E}(-\lambda \cdot W - \widehat{\psi} \cdot W^\perp),$$

for some optimal integrand $\widehat{\psi}$ in (8.1). For use below, define the nonnegative martingale H by

$$H_t := \mathbb{E} \left[\int_0^\infty e^{-\alpha(1-q)s} \widehat{Z}_s^q ds \mid \mathcal{F}_t \right], \quad t \geq 0.$$

Using Theorem 4.1, and in particular (4.1), the optimal consumption process is given by

$$(8.3) \quad (\widehat{c}_t(x))^{-(1-p)} = u'(x)e^{\alpha t} \widehat{Z}_t, \quad t \geq 0.$$

By (4.2) the optimisers satisfy the saturated budget constraint

$$(8.4) \quad \mathbb{E} \left[\int_0^\infty \widehat{c}_t(x) \widehat{Z}_t dt \right] = x.$$

The relations (8.3) and (8.4) yield

$$\widehat{c}_t(x) = \frac{x}{H_0} e^{-\alpha(1-q)t} \widehat{Z}_t^{-(1-q)}, \quad t \geq 0.$$

Using (4.3), the optimal wealth process is then given by

$$\widehat{X}_t(x) \widehat{Z}_t = \frac{x}{H_0} \mathbb{E} \left[\int_t^\infty e^{-\alpha(1-q)s} \widehat{Z}_s^q ds \mid \mathcal{F}_t \right], \quad t \geq 0.$$

More pertinently, the optimal martingale \widehat{M} , corresponding to the process in (8.2) at the optimum, is computed as

$$\widehat{M}_t := \widehat{X}_t(x) \widehat{Z}_t + \int_0^t \widehat{c}_s(x) \widehat{Z}_s ds = \frac{x}{H_0} H_t, \quad t \geq 0,$$

so is indeed a martingale.

By martingale representation, \widehat{M} will have a stochastic integral representation which, without loss of generality, can be written in the form

$$\widehat{M}_t = x + \int_0^t \widehat{Z}_s \widehat{X}_s(x) (\varphi_s - q\lambda_s) dW_s + \int_0^t \widehat{Z}_s \widehat{X}_s(x) \beta_s dW_s^\perp, \quad t \geq 0,$$

for some integrands φ, β . Comparing with the representation in (8.2) at the optimum yields the optimal trading strategy in terms of the optimal portfolio proportion $\widehat{\theta} := \widehat{\pi} / \widehat{X}(x)$ and the optimal integrand $\widehat{\psi}$ in the form

$$\widehat{\theta}_t := \frac{\widehat{\pi}_t}{\widehat{X}_t(x)} = \frac{\lambda_t}{Y_t(1-p)} + \frac{\varphi_t}{Y_t}, \quad \widehat{\psi}_t = -\beta_t, \quad t \geq 0.$$

In particular, the process φ records the correction to the Merton-type strategy $\lambda/(Y(1-p))$ due to the stochastic volatility and correlation.

This is as far as one can go without computing explicitly the dual minimiser \widehat{Z} , which is typically impossible in closed form for power utility. For the special case of logarithmic utility, one can set $p = 0$ and $q = 0$ in the results for power utility, to show that the process $H = 1/\alpha$ is constant, and $\widehat{M} = x$ is also constant, yielding

$$\widehat{\theta}_t = \frac{\lambda_t}{Y_t}, \quad \widehat{\psi}_t = 0, \quad t \geq 0,$$

giving the classic myopic trading strategy for logarithmic utility and, in particular, that the dual optimiser is the minimal deflator: $\widehat{Z} = Z^{(0)} = \mathcal{E}(-\lambda \cdot W)$. The optimal consumption and wealth processes are given explicitly as

$$\widehat{c}_t(x) = \alpha e^{-\alpha t} \frac{x}{Z_t^{(0)}}, \quad \widehat{X}_t(x) = e^{-\alpha t} \frac{x}{Z_t^{(0)}}, \quad t \geq 0,$$

so that we have the classical relation $\widehat{c}(x) = \alpha \widehat{X}(x)$, as is always the case for infinite horizon logarithmic utility from consumption. The results for logarithmic utility can of course be obtained by going directly through the analysis from scratch in the manner above.

Acknowledgements. The author would like to thank Anastasiya Tanana for helpful comments, as well as two anonymous referees and an Associate Editor and Editor for constructive comments that improved the paper.

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