Utility indifference pricing with market incompleteness

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Abstract. Utility indifference pricing and hedging theory is presented, showing how it leads to linear or to non-linear pricing rules for contingent claims. Convex duality is first used to derive probabilistic representations for exponential utility-based prices, in a general setting with locally bounded semi-martingale price processes. The indifference price for a finite number of claims gives a non-linear pricing rule, which reduces to a linear pricing rule as the number of claims tends to zero, resulting in the so-called marginal utility-based price of the claim. Applications to basis risk models with lognormal price processes, under full and partial information scenarios are then worked out in detail. In the full information case, a claim on a non-traded asset is priced and hedged using a correlated traded asset. The resulting hedge requires knowledge of the drift parameters of the asset price processes, which are very difficult to estimate with any precision. This leads naturally to a further application, a partial information problem, with the drift parameters assumed to be random variables whose values are revealed to the hedger in a Bayesian fashion via a filtering algorithm. The indifference price is given by the solution to a non-linear PDE, reducing to a linear PDE for the marginal price when the number of claims becomes infinitesimally small.

1 Introduction

This chapter presents theory and examples of utility indifference pricing and hedging, a method for managing risk from trading contingent claims in incomplete markets, that has become a classical tool in the valuation of non-hedgeable claims. For a position in n claims, it results in a non-linear pricing rule, which reduces to a linear pricing rule in the limit of a small position in the claim. In this section we qualitatively describe the main idea and highlight some of the literature on the technique. This review is not intended to be exhaustive, but the reader will find ample material and further references in the citations.

Suppose we have a European contingent claim C, a contract which pays the random amount $C \geq 0$ almost surely (a.s.) at some future time $T > 0$. The typical example is where C is dependent on the trajectory of a vector $S = (S_t)_{0 \leq t \leq T}$, which is the price process of a set of $d + 1$ traded securities: $S = (S^0, S^1, \ldots, S^d)$. The zeroth asset S^0 is assumed riskless. For ease of exposition, we shall assume zero interest rates throughout this chapter, so we normalise the riskless asset price to be $S^0_t = 1$ for all $t \in [0, T]$.

As is well-known, in a complete financial market every contingent claim C can be perfectly replicated by a controlled portfolio of the traded securities: a portfolio with wealth process $X = (X_t)_{0 \leq t \leq T}$ exists satisfying $X_T = C$ a.s. In this case, no-arbitrage arguments imply the option price at time 0 is the initial value X_0 of the replicating portfolio, given by $X_0 = E^Q C$, where E^Q denotes expectation under the unique martingale measure Q of the complete market. The hedging strategy for a short position in the claim is simply to hold the replicating portfolio. The classical example of such a replication procedure is the Black-Scholes (BS) [3] option pricing model, where S is the price process of a single stock following a geometric Brownian motion.

By definition, an incomplete financial market is one in which not all claims C can be replicated. In this case writing a claim entails genuine risk, and the pricing and hedging of the claim can only be carried out by specifying the agent's preferences towards such risk. Classically, economists have done this by specifying the agent's utility function U. This was the inspiration for Hodges and Neuberger [12] to introduce the concept of utility indifference pricing, in the context of option pricing under transaction costs in the BS model. In this methodology, the agent seeks to solve for a claim price which, when incorporated into the initial wealth, results in a maximal expected utility when trading claims that is the same as the maximum utility in the absence of claims. The associated hedging strategy is the difference in the agent's optimal stock strategies with and without the random endowment of the claim payoff at the terminal time.

Utility indifference prices usually result in price bounds which mark the bid and ask prices at which a utility maximising investor would be prepared to buy or sell claims. They are nonlinear pricing rules (as we will see in later sections) in that the price for the claim nC ($n \in \mathbb{R}$) is not n times the price for 1 claim. The utility-based price bounds are invariably within the no-arbitrage price bounds, the latter usually being quite wide and therefore of limited practical use.

A marginal version of utility-based pricing, based on classical economic ideas of incremental pricing, was developed by Davis [4]. This gives the utility indifference price for an infinitesimal position in claims, the so-called marginal utility-based price (MUBP), which is (under fairly mild conditions) a unique price within the no-arbitrage interval, and within the bid and ask utility-based prices for a finite position in claims. The marginal price is given by a linear pricing rule, as we shall see. This linear pricing rule will emerge naturally from considering the dual problem to the agent's primal utility maximisation problem.

The utility indifference pricing technique has received much attention in the academic literature. The earliest applications were to transaction cost models. Following Hodges and Neuberger [12], Davis, Panas and Zariphopoulou [6] further developed the application to the BS model with proportional transaction costs. An asymptotic analysis of the Davis, Panas and Zariphopoulou [6] model, valid for small transaction costs, was carried out by Whalley and Wilmott [33]. Monoyios [22, 23] computed the MUBP and associated hedge in a BS model with proportional transaction costs. The marginal price depended on the agent's initial stock holding, and lay within bounds which marked the agent's indifference price for a single claim.

Karatzas and Kou [15] analysed the MUBP in continuous markets with Itô processes for stock prices, and with portfolio constraints. They showed how it nearly always lies within the no-arbitrage bounds. Subsequent developments were to more specific incomplete market scenarios. In particular, a number of papers studied so-called "basis risk" models, in which a claim on a non-traded asset is optimally hedged with a correlated traded asset. These models were studied by [5, 11, 24, 28] among others. In particular, Monoyios [24] showed how exponential utility-based hedging could outperform a "naive" hedge which took the traded asset as a perfect proxy for the non-traded asset. We shall describe basis risk models in some detail in Section 3.

Other applications of utility-based hedging have been to stochastic volatility models [32] and to the pricing of volatility derivatives [10]. The general theory of utility-based pricing, with a particular emphasis on relations with the dual to the primal utility maximisation problem, has been studied by Delbaen et al [7], by Becherer [1], and by Kramkov and co-authors [13, 20, 21].

The rest of the chapter is as follows. In the next section we present the theory of utility-based pricing and hedging in a general semi-martingale setting, and use results from the dual approach to optimal investment to derive probabilistic representations for utility-based prices. In Section 3 we apply the methodology to a basis risk model with full information, in which a claim on a non-traded asset is hedged with a correlated traded asset. The resulting optimal hedge requires knowledge of the asset price drifts, which are virtually impossible to estimate accurately. This motivates Section 4, in which drift parameter uncertainty is acknowledged by modelling the drifts as random variables with a prior distribution, which is updated using a Kalman filter. The resulting model becomes a full information model with random drifts, and we derive representations for utility-based prices and hedges in this case. The resulting hedge is approximated using the MUBP as a pricing rule, and shown to out-perform the naive approach of taking the traded asset as a perfect proxy for the non-traded one. We end with some concluding remarks.

2 Utility-based pricing and hedging: the general set-up

Here we present the theory of utility-based pricing in a fairly general semi-martingale setting, with frictionless markets, meaning no transaction costs on trading of basic securities such as stocks.

We start with a locally bounded vector semi-martingale $S = (S_t)_{0\leq t\leq T}$ on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$. The filtration $\mathbb{F} := (\mathcal{F}_t)_{0\leq t\leq T}$ satisfies the usual conditions, and we take $\mathcal{F}_T = \mathcal{F}$.

There are $d+1$ assets: one savings account and d stocks, so we write $S = (S^0, \ldots, S^d)$, with each S^i a locally bounded semi-martingale, $S^i = (S^i_t)_{0\leq t\leq T}$. The zeroth asset is riskless, with $S^0_t = 1$ for all $t \in [0, T]$, so we are in addition assuming the interest rate is zero, as we shall do throughout this chapter.

A probability measure Q is called an equivalent martingale measure if it is equivalent to P and if S is a local martingale under Q. We denote by $\mathcal{M}$ the family of all such measures and assume that

$$\mathcal{M} \neq \emptyset.$$

This condition is essentially equivalent to the absence of arbitrage opportunities in the market; see Delbaen and Schachermayer [8] for precise statements and more details.

A self-financing portfolio is defined by a pair (x, H) where $x \in \mathbb{R}$ defines the initial capital and $H = (H^i)_{i=1}^d$ is a predictable and S-integrable process specifying the number of shares of each of the stocks held in the portfolio. Hence, for each $i = 1, \ldots, d$, $H^i = (H^i_t)_{0 \leq t \leq T}$. The value process of a self-financing portfolio evolves in time as the stochastic integral of the process H with respect to the stock price:

$$X_t := x + (H \cdot S)_t = x + \int_0^t H_s dS_s, \quad t \in [0, T].$$

(Here, $\int_0^t H_s dS_s \equiv \sum_{i=1}^d \int_0^t H^i_s dS^i_s$.) The market is, in general, incomplete, so not every contingent claim C can be replicated by a self-financing portfolio. Then there is no unique martingale measure, and the possible no-arbitrage prices span an interval given by

$$\left(\inf_{Q \in \mathcal{M}} E^Q C, \sup_{Q \in \mathcal{M}} E^Q C \right). \tag{1}$$

This was shown by Kramkov [18]. When the market is complete, there exists a unique self-financing portfolio satisfying $X_T = C$. In this case we say the strategy H replicates C, and H is the unique hedging strategy for the claim. In this case, there is a unique martingale measure Q, the interval (1) reduces to a single point, and the no-arbitrage price of the claim at time 0 is $p_0^{\mathrm{NA}} := E^Q C$.

In an incomplete market, one is faced with choosing one of the martingale measures $Q \in \mathcal{M}$ as a pricing measure. At first sight, this choice appears to have little to do with optimal investment. But the incompleteness means that selling a claim entails opening oneself up to non-zero terminal risk, as represented by the difference $X_T - C$, where X_T is the terminal wealth of any self-financing portfolio. The question arises as to how one should deal with the residual risk $X_T - C$. This can only be answered by specifying the risk preferences of the financial agent selling the claim. This will be done via a utility function. We shall see that a possible pricing measure then emerges naturally, via the dual to a primal utility maximisation problem.

We consider an economic agent whose preferences over terminal wealth are represented by a utility function $U : \mathbb{R} \to \mathbb{R}$, which is assumed to be strictly increasing, strictly concave, continuously differentiable, and is assumed to satisfy the Inada conditions:

$$U'(-\infty) = \lim_{x \to -\infty} U'(x) = \infty, \quad U'(\infty) = \lim_{x \to \infty} U'(x) = 0.$$

The utility function that we shall employ in this chapter is the exponential utility function:

$$U(x) = -\exp(-\alpha x), \quad \alpha > 0, \tag{2}$$

with constant risk aversion parameter α.

The convex conjugate of the agent's utility function is defined to be the Legendre transform of the convex function $-U(-\cdot)$. That is,

$$V(\eta) := \sup_{x \in \mathbb{R}} [U(x) - x\eta], \quad \eta > 0. \tag{3}$$

The conjugate function is then a continuously differentiable, strictly decreasing and strictly convex function satisfying $-V'(-\infty) = \infty$, $V'(\infty) = 0$.

The supremum in (3) is achieved by $x = x^*$ satisfying

$$U'(x^*) = \eta \Leftrightarrow x^* = I(\eta), \tag{4}$$

where I is the inverse of U'. Then $V(\eta)$ may be written as

$$V(\eta) = U[I(\eta)] - \eta I(\eta). \tag{5}$$

Further, from (3) we have the inequality

$$V(\eta) \geq U(x) - x\eta, \quad \text{with equality iff} \quad x = x^* \quad \text{such that} \quad U'(x^*) = \eta. \tag{6}$$

Also, differentiating (5) gives

$$V'(\eta) = -I(\eta), \tag{7}$$

so that the agent's marginal utility is the inverse of minus the gradient of the convex conjugate:

$$U'(-V'(x)) = x.$$

We note that the defining duality relation (3) is equivalent to the bidual relation

$$U(x) = \inf_{\eta > 0}[V(\eta) + x\eta], \quad x \in \mathbb{R}, \tag{8}$$

since this gives that the value of η achieving the above infimum is η^* satisfying

$$V'(\eta^*) = -x,$$

or, by (7),

$$I(\eta^*) = x \Leftrightarrow U'(x) = \eta^*,$$

which is (4). Note also that the bidual relation (8) implies the inequality (6).

For the exponential utility function, the convex conjugate is given by

$$V(\eta) = \frac{\eta}{\alpha}\left(\log\left(\frac{\eta}{\alpha}\right) - 1\right). \tag{9}$$

For a martingale measure $Q \in \mathcal{M}$, define the relative entropy $\mathcal{H}(Q, P)$ between Q and P by

$$\mathcal{H}(Q, P) := E\left[\frac{dQ}{dP}\log\frac{dQ}{dP}\right].$$

This quantity will play a role in the dual to a primal utility maximisation problem under exponential utility. Denote the set of martingale measures with finite relative entropy by $\mathcal{M}_f$.

For an agent with exponential utility function, we we follow Becherer [1] and define the set $\mathcal{A}$ of admissible strategies H by

$$\mathcal{A} := \{H : (H \cdot S) \text{ is a } Q\text{-martingale for all } Q \in \mathcal{M}_f\}. \tag{10}$$

Other choices for $\mathcal{A}$ are possible. For instance, one may follow Schachermayer [31] and consider strategies with wealth bounded from below, and then maximise over the $L^1(P)$-closure of the set of all random variables $U(\Gamma)$ such that Γ can be super-hedged by some trading strategy. The good news is that these choices lead to the same solution for the dual to the primal utility maximisation problem [7, 14, 31].

For an initial capital x, the agent's primal value function is $u(x)$ defined by

$$u(x) := \sup_{H \in \mathcal{A}} EU(X_T) = \sup_{H \in \mathcal{A}} EU\left(x + \int_0^T H_t dS_t\right). \tag{11}$$

Define, for any $Q \in \mathcal{M}$, the change of measure martingale $Z = (Z_t)_{0 \le t \le T}$ by

$$Z_t := \left.\frac{dQ}{dP}\right|_{\mathcal{F}_t}.$$

Define the dual value function $v(\eta)$ by

$$v(\eta) := \inf_{Q \in \mathcal{M}} EV(\eta Z_T). \tag{12}$$

We shall assume that this function is finite valued. A celebrated body of work has shown that the primal value function may be computed by solving the dual problem. This work culminated in the paper by Kramkov and Schachermayer [19], who considered semimartingale processes S, following earlier work by several authors based on Itô price processes, notably Karatzas et al [16]. The monograph by Karatzas and Shreve [17] contains an authoritative account of this theory for continuous models based on Itô price processes. For an application of duality theory in two-factor incomplete markets, in which explicit representations for the dual minimiser are obtained, see Monoyios [25].

We assume that there is a unique dual optimiser attaining the infimum in (12). This will be the case in the models we tackle in subsequent sections. Denote the dual optimiser by Q^*, the associated change of measure martingale by Z^*, the optimal terminal wealth attaining the supremum in (11) by X_T^*, and the optimal trading strategy by H^*.

The main results of the dual approach to optimal investment are summarised in Theorem 1 below.

Theorem 1 *1. The primal and dual value functions $u(x)$ and $v(\eta)$ in (11) and (12) are conjugate:*

$$v(\eta) = \sup_{x \in \mathbb{R}} [u(x) - x\eta], \qquad u(x) = \inf_{\eta > 0} [v(\eta) + x\eta],$$

so that $u'(x) = \eta$ (equivalently, $v'(\eta) = -x$);

2. *The optimal terminal wealth* X_T^* *and optimal dual minimiser* Q^* *are unique and related by*

$$U'(X_T^*) = \eta \frac{dQ^*}{dP}, \quad equivalently, \quad X_T^* = I\left(\eta \frac{dQ^*}{dP}\right);$$

3. *The following properties for* $u'(x)$ *and* $v'(\eta)$ *hold true:*

$$u'(x) = EU'(X_T^*), \qquad v'(\eta) = E^{Q^*} V'\left(\eta \frac{dQ^*}{dP}\right). \tag{13}$$

For a proof of this theorem, see [19].

To define utility-based prices, we introduce a utility maximisation problem with a random terminal endowment involving a claim C. Define the primal value function with random endowment, $u^{(n)}$, as the maximum expected utility when receiving a terminal payoff of n claims:

$$u^{(n)}(x) := \sup_{H \in \mathcal{A}} EU(X_T + nC) = \sup_{H \in \mathcal{A}} EU\left(x + \int_0^T H_t dS_t + nC\right). \tag{14}$$

Clearly, the value function $u(x)$ of (11) coincides with $u^{(0)}(x)$.

Define the dual problem to (14), with value function $v^{(n)}$, by

$$v^{(n)}(\eta) := \inf_{Q \in \mathcal{M}} E\left[V\left(\eta Z_T\right) + \eta Z_T nC)\right]. \tag{15}$$

Clearly, the value function $v(\eta)$ of (12) coincides with $v^{(0)}(\eta)$.

Denote the dual optimiser in (15) by $Q^{*,n}$, the associated change of measure martingale by $Z^{*,n}$, the optimal terminal wealth attaining the supremum in (14) by $X_T^{*,n}$, and the optimal trading strategy by $H^{*,n}$. Then the corresponding optimal quantities of the problems (11,12) without claims are given by $Q^* \equiv Q^{*,0}$, and similarly for Z^*, X_T^*, H^*.

For the value functions $u^{(n)}, v^{(n)}$ involving a random terminal endowment, and with exponential utility, a similar duality result to Theorem 1 has been obtained by Delbaen et al [7].

Theorem 2 *Given an exponential utility function* $U(x) = -\exp(-\alpha x)$*, we have:*

1. *The value functions* $u^{(n)}(x)$ *and* $v^{(n)}(\eta)$ *of (14) and (15) are conjugate:*

$$v^{(n)}(\eta) = \sup_{x \in \mathbb{R}} [u^{(n)}(x) - x\eta], \qquad u^{(n)}(x) = \inf_{\eta > 0} [v^{(n)}(\eta) + x\eta],$$

so that $u_x^{(n)}(x) = \eta$ *(equivalently,* $v_\eta^{(n)}(\eta) = -x$*);*

2. *The optimal terminal wealth* $X_T^{*,n}$ *and optimal dual minimiser* $Q^{*,n}$ *are unique and related by*

$$U'(X_T^{*,n} + nC) = \eta \frac{dQ^{*,n}}{dP}, \quad equivalently, \quad X_T^{*,n} + nC = I\left(\eta \frac{dQ^{*,n}}{dP}\right); \tag{16}$$

3. *The following properties for* $u_x^{(n)}(x)$ *and* $v_\eta^{(n)}(\eta)$ *hold true:*

$$u_x^{(n)}(x) = EU'(X_T^{*,n} + nC), \qquad v_\eta^{(n)}(\eta) = E^{Q^{*,n}}\left[V'\left(\eta\frac{dQ^*}{dP}\right) + nC\right]. \quad (17)$$

We will use these theorems to derive properties of the indifference prices, the major objects of interest in this chapter, which we now define.

Definition 1 (Utility indifference price) *The* utility indifference price *per claim,* $p^{(n)}$*, for a random endowment of* n *claims, is defined by*

$$u^{(n)}(x - np^{(n)}) = u^{(0)}(x). \quad (18)$$

In other words, when issuing or purchasing claims, the agent ensures that the price per claim results in no loss of utility compared with the alternative strategy of not writing or buying any claims.

The optimal hedging strategy is defined as the difference of the optimal trading strategies with and without the random endowment of n claims.

Definition 2 (Optimal hedging strategy) *The optimal hedging strategy for* n *units of the claim is* $H^{(\mathrm{H})} := (H_t^{(\mathrm{H})})_{0\le t\le T}$ *given by*

$$H_t^{(\mathrm{H})} := H_t^{*,n} - H_t^{*,0}, \quad 0 \le t \le T.$$

The *marginal utility-based price* (MUBP), $\hat{p}$, of the claim, may be defined in a number of ways. One is that it corresponds to a price which solves (18) as $n \to 0$:

Definition 3 (Marginal price) *The marginal utility-based price of the claim at time* 0 *is* $\hat{p}$ *defined by*

$$\hat{p} := \lim_{n\to 0} p^{(n)}. \quad (19)$$

It is well known that with exponential utility the marginal price is also equivalent to the limit of the indifference price as risk aversion goes to zero. Under appropriate conditions (satisfied in this model) it is given by the expectation of the payoff under the optimal measure of the dual to the problem without the claim. We shall derive these results shortly.

Definition 3 is equivalent to the definition below, which was the original definition due to Davis [4]. Suppose one diverts an amount c of the initial capital into the purchase or sale of claims. If the unit price per claim is p then the number of claims traded is c/p. We consider the value function $u^{(c/p)}(x)$ and make the following definition.

Definition 4 (Marginal price [4]) *The* marginal price $\hat{p}$ *of the claim is the price which solves*

$$\left.\frac{\partial u^{(c/p)}}{\partial c}(x - c)\right|_{c=0} = 0. \quad (20)$$

In [4] it is shown that under suitable smoothness conditions $\hat{p}$ is given by the following theorem.

Theorem 3 *The marginal price is given by*

$$\hat{p} = \frac{E\left[U'\left(X_T^*\right)C\right]}{u'(x)}. \tag{21}$$

In the next subsection we shall show the equivalence of (21) and (19), by showing that both representations result in

$$\hat{p} = E^{Q^*}C = E^{Q^E}C,$$

where Q^E is the minimal entropy measure, which is the optimal dual measure Q^* of the dual problem (12) for exponential utility, and is defined by

$$Q^E := \arg\min_{Q\in\mathcal{M}} \mathcal{H}(Q,P).$$

2.1 Dual representations for exponential utility-based prices

Using the fundamental duality results in Theorem 2 we can obtain the following representation for the primal value function $u^{(n)}$, from which dual representations for the indifference prices follow.

Theorem 4 *The value function $u^{(n)}(x)$ in (14) has the representation*

$$u^{(n)}(x) = -\exp\left[-\alpha x - \inf_{Q\in\mathcal{M}}\left(\mathcal{H}(Q,P) + \alpha n E^Q C\right)\right].$$

Proof Using the definition (15) of the dual value function $v^{(n)}$, and the formula (9) for the conjugate function corresponding to exponential utility, we have

$$v^{(n)}(\eta) = V(\eta) + \frac{\eta}{\alpha}\inf_{Q\in\mathcal{M}}\left(\mathcal{H}(Q,P) + \alpha n E^Q C\right). \tag{22}$$

The duality between $u^{(n)}$ and $v^{(n)}$ in Theorem 2 implies $v^{(n)\prime}(\eta) = -x$, so (22) gives that $\eta > 0$ is given by

$$\eta = \alpha\exp\left[-\alpha x - \inf_{Q\in\mathcal{M}}\left(\mathcal{H}(Q,P) + \alpha n E^Q C\right)\right].$$

Using this representation for η in (16) we find that the optimal terminal wealth satisfies

$$\exp(-\alpha X_T^{*,n}) = Z_T^{*,n}\exp\left[-\alpha x - \inf_{Q\in\mathcal{M}}\left(\mathcal{H}(Q,P) + \alpha n E^Q C\right)\right].$$

The result then follows from $u^{(n)}(x) = EU(X_T^{*,n})$.

□

In particular, we have that the primal value function in the absence of claims is given by

$$u^{(0)}(x) = -\exp\left(-\alpha x - \mathcal{H}(Q^E, P)\right),$$

where Q^E is the minimal entropy measure, the optimal dual measure of the problem without claims.

We then have the following immediate corollary, from the definitions of the indifference price.

Corollary 1 *The indifference price $p^{(n)}$ is given by*

$$p^{(n)} = \frac{1}{\alpha n}\left[\inf_{Q\in\mathcal{M}}\left(\mathcal{H}(Q,P) + \alpha n E^Q C\right) - \mathcal{H}(Q^E, P)\right].$$

In particular, for $n > 0$, we have that the utility indifference bid price per claim for n claims is given by

$$p^{(n)}_{\text{bid}} := p^{(n)} = \inf_{Q\in\mathcal{M}}\left[E^Q C + \frac{1}{\alpha n}\left(\mathcal{H}(Q,P) - \mathcal{H}(Q^E,P)\right)\right], \quad n > 0. \tag{23}$$

From this, we see that, in the limit $\alpha \to \infty$, we obtain

$$\lim_{\alpha\to\infty} p^{(n)}_{\text{bid}} = \inf_{Q\in\mathcal{M}} E^Q C,$$

which is the lower bound of the no-arbitrage interval (1).

Similarly, for $n < 0$, say $n = -m$, with $m > 0$, we obtain that the utility indifference ask price per claim for m claims is given by

$$p^{(m)}_{\text{ask}} := p^{(-m)} = \sup_{Q\in\mathcal{M}}\left[E^Q C - \frac{1}{\alpha m}\left(\mathcal{H}(Q,P) - \mathcal{H}(Q^E,P)\right)\right], \quad m > 0. \tag{24}$$

In the limit $\alpha \to \infty$, we obtain

$$\lim_{\alpha\to\infty} p^{(m)}_{\text{ask}} = \sup_{Q\in\mathcal{M}} E^Q C,$$

which is the upper bound of the no-arbitrage interval (1). We conclude that the possible indifference prices for exponential utility span the entire no-arbitrage interval, as we vary the risk aversion parameter.

For the marginal price, we have the following equivalences.

Theorem 5 *The marginal price has the equivalent representations*

$$\hat{p} := \lim_{n\to 0} p^{(n)} = \lim_{\alpha\to 0} p^{(n)} = E^{Q^E} C, \tag{25}$$

and these coincide with the representation (21).

Proof First, (25) follows by letting $\alpha \to 0$ or $n \downarrow 0$ in (23) and letting $\alpha \to 0$ or $m \downarrow 0$ in (24). The equivalence with (21) follows from Theorem 1, which implies that

$$\frac{U'(X_T^*)}{u'(x)} = \frac{dQ^*}{dP},$$

and the fact that for exponential utility, $Q^* = Q^E$, the minimal entropy measure.

□

Theorem 5 confirms that the marginal indifference price is governed by a linear pricing rule, as given by an expectation of the payoff under the optimal dual measure.

3 Basis risk model

Here is a quintessential example of an incomplete market in which the ideas of utility-based pricing given in the previous section can be illustrated with great clarity and explicit solutions. A number of papers ([5,11,24,26,28] to name but a few) have studied such basis risk models.

The setting is a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{0\leq t\leq T}, P)$, where the filtration $\mathbb{F}$ is the P-augmentation of that generated by a two-dimensional Brownian motion $(B, B^{\perp})$. A traded stock price $S := (S_t)_{0\leq t\leq T}$ follows a log-Brownian process given by

$$dS_t = \quad \sigma S_t(\lambda dt + dB_t) \quad =: \sigma S_t d\xi_t, \tag{26}$$

where $\sigma > 0$ and λ are known constants. For simplicity, the interest rate is taken to be 0. The process ξ in (26) defined by $d\xi_t := \lambda dt + dB_t$ will play a role as one component of an "observation process" in a partial information model in the next section, when λ will be treated as a random variable rather than as a known constant.

A non-traded asset price $Y := (Y_t)_{0\leq t\leq T}$ follows the correlated log-Brownian motion

$$dY_t = \quad \beta Y_t(\theta dt + dW_t) \quad =: \beta Y_t d\zeta_t, \tag{27}$$

with $\beta > 0$ and θ known constants. The Brownian motion W is correlated with B according to

$$d[B, W]_t = \rho dt, \quad W = \rho B + \sqrt{1-\rho^2}B^{\perp}, \quad \rho \in [-1, 1],$$

and the process ζ, given by $d\zeta_t := \theta dt + dW_t$, will act as the second component of an observation process in a partial information model in the next section, when θ will be considered a random variable. We shall refer to the Sharpe ratios λ (respectively, θ) as the drift of S (respectively, Y), for brevity.

A European contingent claim pays the non-negative random variable $h(Y_T)$ at time T, where h is a bounded continuous function. If $|\rho| = 1$, the model is complete and a BS-style perfect hedge is possible (as we shall show). But for $|\rho| \neq 1$ the market is incomplete.

Examples of underlying assets that are either not traded (or are difficult to trade) include weather indices or baskets of many stocks. There is no tradeable asset which can be used to perfectly replicate the claim payoff. Traders may resort to using a correlated traded asset to hedge the claim, where the correlation is presumed to be close to 1, in effect taking the traded asset as a perfect proxy for the non-traded one. A typical case is the hedging of a basket option using a futures contract on a stock index, where the composition of the basket and the index are not identical.

The set $\mathcal{M}$ of local martingale measures Q is defined via the density process $Z = (Z_t)_{0\leq t\leq T}$ given by

$$Z_t := \mathcal{E}(-\lambda \cdot B - \psi \cdot B^{\perp})_t, \quad 0 \leq t \leq T,$$

where $\mathcal{E}$ denotes the stochastic exponential, and $\psi = (\psi_t)_{0\leq t\leq T}$ is a process satisfying $\int_0^T \psi_t^2 dt < \infty$ a.s. If, in addition, Z is a martingale, then we may define probability measures Q equivalent to P by

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} = Z_t.$$

The set $\mathcal{M}$ of martingale measures is then in one-to-one correspondence with the set of integrands ψ.

The minimal martingale measure Q^M, which will feature in many of our formulae, corresponds to $\psi = 0$, so has density process with respect to P given by

$$\left.\frac{dQ^M}{dP}\right|_{\mathcal{F}_t} = \mathcal{E}\left(-\lambda \cdot B\right)_t, \quad 0 \leq t \leq T.$$

Under Q^M, (S, Y) follow

$$\begin{aligned} dS_t &= \sigma S_t dB_t^{Q^M}, \\ dY_t &= \beta Y_t \left[(\theta - \rho\lambda)\, dt + dW_t^{Q^M} \right], \end{aligned} \tag{28}$$

where B^{Q^M}, W^{Q^M} are correlated Brownian motions under Q^M. The stock price S is a local Q^M-martingale but this is not the case for the non-traded asset.

3.1 Perfect correlation case

In the perfect correlation case, $\rho = 1, Y$ is effectively a traded asset, so no-arbitrage requires the Q^M-drift of Y to be 0. Therefore, given σ, β, in the $\rho = 1$ case the Sharpe ratios of the assets are related by

$$\theta = \lambda. \tag{29}$$

In fact, with $\rho = 1$, $W = B$, so we have

$$Y_t = Y_0 \left(\frac{S_t}{S_0}\right)^{\beta/\sigma} \exp\left[\frac{1}{2}\beta(\sigma - \beta)\right].$$

We show that in this case the market becomes complete, and perfect hedging is possible. Let the claim price process be $P(t, Y_t), 0 \leq t \leq T$, where $P : [0, T] \times \mathbb{R}^+ \to \mathbb{R}^+$ is smooth enough to apply the Itô formula, so that

$$dP(t, Y_t) = [P_t(t, Y_t) + \mathcal{A}_Y P(t, Y_t)]\, dt + \beta Y_t P_y(t, Y_t) dB_t,$$

where $\mathcal{A}_Y$ is the generator of the process Y in (27), and we have used the fact that $W = B$ if $\rho = 1$.

A self-financing portfolio with initial capital x and $(H_t)_{0 \leq t \leq T}$ shares of stock at time $t \in [0, T]$ has wealth process $X = (X_t)_{0 \leq t \leq T}$ given by

$$X_t = x + \int_0^t H_s dS_s.$$

To hedge a purchase of n claims, one replicates a position in $-n$ claims, so we require $X_T = -nh(Y_T)$ a.s., and in particular

$$X_t = -nP(t, Y_t), \quad 0 \leq t \leq T, \quad dX_t = -ndP(t, Y_t).$$

Then the perfect hedge is $H_t^{(\mathrm{P})}$ (the superscript P denoting "perfect") units of S at $t \in [0, T]$, where

$$H_t^{(\mathrm{P})} = -n \frac{\beta}{\sigma} \frac{Y_t}{S_t} \frac{\partial}{\partial y} P(t, Y_t), \tag{30}$$

and the claim pricing function $P(t, y)$ then satisfies

$$P_t(t, y) + \frac{1}{2} \beta^2 y^2 P_{yy}(t, y) = 0, \quad P(T, y) = h(y),$$

where we have used the no-arbitrage condition (29). This is the BS partial differential equation (PDE), so

$$P(t, Y_t) = \mathrm{BS}(t, Y_t; \beta),$$

where $\mathrm{BS}(t, y; \beta)$ denotes the BS formula at time t for underlying asset price y and volatility β. An important feature of (30) is that the perfect hedge does not require knowledge of the values of the drifts λ, θ.

3.2 Utility-indifference valuation and hedging

Now suppose the correlation is not perfect, so that the market is incomplete. We embed the problem in a utility maximisation framework. Let the agent have risk preferences expressed via the exponential utility function (2). The agent maximises expected utility of terminal wealth at time T, with a random endowment of n units of claim payoff. Define $\pi = (\pi_t)_{0 \leq t \leq T}$ as the process of wealth in the stock, so that $\pi_t := H_t S_t$. Given a starting time $t \in [0, T]$ the objective to be maximised is

$$J^{(n)}(t, x, y; \pi) = E[U(X_T + nh(Y_T)) | X_t = x, Y_t = y].$$

The value function is $u^{(n)}(t,x,y)$, defined by

$$\begin{aligned} u^{(n)}(t,x,y) &:= \sup_{\pi\in\mathcal{A}} J^{(n)}(t,x,y;\pi), && (31)\\ u^{(n)}(T,x,y) &= U(x+nh(y)). && (32) \end{aligned}$$

Denote the optimal trading strategy that achieves the supremum in (31) by $\pi^{*,n}$, and denote the optimal wealth process by $X^{*,n}$.

We assume the random endowment $nh(Y_T)$ is bounded below. This ensures the maximum utility in (31) is well-defined.

The utility-based price at $t\in[0,T]$ is given by the analogue of Definition 1 for a starting time $t\in[0,T]$. Given $X_t=x, Y_t=y$, the price per claim is $p^{(n)}(t,x,y)$, defined by

$$u^{(n)}(t,x-np^{(n)}(t,x,y),y) = u^{(0)}(t,x,y).$$

We allow for possible dependence on t,x,y of $p^{(n)}$ in the above definition, but with exponential preferences it turns out that there is no dependence on x, as we shall see.

The optimal hedging strategy is given by Definition 2. In terms of the variable $\pi := HS$, we have that the optimal hedging strategy for n units of the claim is $\pi^{(\mathrm{H})} := (\pi_t^{(\mathrm{H})})_{0\le t\le T}$ given by

$$\pi_t^{(\mathrm{H})} := \pi_t^{*,n} - \pi_t^{*,0}, \quad 0\le t\le T. \tag{33}$$

The solution to the optimisation problem (31) is well-known, using a so-called distortion transformation (see Zariphopoulou [34]) to linearise the Hamilton-Jacobi-Bellman (HJB) equation for $u^{(n)}$. See [24] for more details of the computation in this model.

The HJB equation for the value function $u^{(n)}$ is

$$u_t^{(n)} + \mathcal{A}_Y u^{(n)} - \frac{\left(\lambda u_x^{(n)} + \rho\beta y u_{xy}^{(n)}\right)^2}{2u_{xx}^{(n)}} = 0. \tag{34}$$

The optimal trading strategy $\pi^{*,n}$ is given by $\pi_t^{*,n} = \Pi^{*,n}(t,X_t^{*,n},Y_t)$, where the function $\Pi^{*,n}:[0,T]\times\mathbb{R}\times\mathbb{R}^+$ is given by

$$\Pi^{*,n}(t,x,y) := -\left(\frac{\lambda u_x^{(n)} + \rho\beta y u_{xy}^{(n)}}{\sigma u_{xx}^{(n)}}\right). \tag{35}$$

We have the following representation for the value function and indifference price.

Proposition 1 *[24] The value function $u\equiv u^{(n)}$ and indifference price $p\equiv p^{(n)}$, given $X_t=x, Y_t=y$ for $t\in[0,T]$, are given by*

$$\begin{aligned} u^{(n)}(t,x,y) &= -e^{-\alpha x - \frac{1}{2}\lambda^2(T-t)}\left[F(t,Y)\right]^{1/(1-\rho^2)}, \\ F(t,y) &= E^{Q^M}\left[\exp\left(-\alpha(1-\rho^2)nh(Y_T)\right)\big| Y_t=y\right], \\ p^{(n)}(t,y) &= -\frac{1}{\alpha(1-\rho^2)n}\log F(t,y). \end{aligned} \tag{36}$$

The function $F(t,y)$ satisfies a linear PDE by virtue of the stochastic representation (36) and the Feynman-Kac theorem. It is easy to verify that the value function as given in the proposition then satisfies the HJB equation (34). The indifference price formula then follows from its definition.

The indifference pricing function $p^{(n)}(t,y)$ satisfies

$$p_t^{(n)} + \beta(\theta - \rho\lambda)yp_y^{(n)} + \frac{1}{2}\beta^2y^2p_{yy}^{(n)} - \frac{1}{2}\beta^2y^2n\alpha(1-\rho^2)(p_y^{(n)})^2 = 0,$$

with $p^{(n)}(T,y) = h(y)$. This is a semi-linear PDE, and in this sense the indifference pricing methodology constitutes a non-linear pricing rule.

For $n = 0$, the indifference pricing PDE becomes linear, and by the Feynman-Kac Theorem we obtain the following representation for the marginal price $\hat{p}(t,y) := \lim_{n\to 0} p^{(n)}(t,y)$:

$$\hat{p}(t,y) = E^{Q^M}[h(Y_T)|Y_t = y].$$

This is a special case of the general representation in Theorem 5, since in the basis risk model the minimal entropy measure Q^E coincides with the minimal martingale measure Q^M. This is because the relative entropy between a martingale measure $Q \in \mathcal{M}$ and P is given by

$$\mathcal{H}(Q,P) = E^Q\left[\frac{1}{2}\left(\lambda^2T + \int_0^T \psi_t^2 dt\right)\right],$$

and this is clearly minimised by $\psi = 0$.

Given the form of the value function, it is easy to show that the expression (35) for the optimal control loses its dependence on x. Then, applying Definition 2 gives the optimal hedging strategy for a position in n claims (see [24] for further details of this derivation).

Proposition 2 *The optimal hedging strategy for a position in n claims is to hold $H_t^{(\mathrm{H})}$ shares at t $\in [0,T]$, given by*

$$H_t^{(\mathrm{H})} = -n\rho\frac{\beta}{\sigma}\frac{Y_t}{S_t}\frac{\partial p^{(n)}}{\partial y}(t,Y_t). \tag{37}$$

We note that for $\rho = 1$ we recover the perfect delta hedge (30), and the claim price then satisfies the BS PDE.

3.3 Residual risk process

Suppose the agent trades n claims at time 0 for price $p^{(n)}(0,Y_0)$ per claim. The agent hedges this position over $[0,T]$ using the strategy $(H_t^{(\mathrm{H})})_{0\le t\le T}$. Her overall position has value process $R := (R_t)_{0\le t\le T}$ given by $R_t = X_t^{(\mathrm{H})} + np^{(n)}(t,Y_t)$, so that

$$dR_t = dX_t^{(\mathrm{H})} + ndp^{(n)}(t,Y_t), \tag{38}$$

where $X^{(\mathrm{H})} = (X_t^{(\mathrm{H})})_{0\le t\le T}$ is the value of the hedging portfolio in S, satisfying

$$\begin{aligned} dX_t^{(\mathrm{H})} &= H_t^{(\mathrm{H})} dS_t, \\ X_0^{(\mathrm{H})} &= -np^{(n)}(0, Y_0). \end{aligned}$$

Using this in (38) along with the Itô formula and the PDE satisfied by $p^{(n)}(t,y)$, we obtain

$$dR_t = \frac{1}{2}\beta^2 n^2 \alpha(1-\rho^2)Y_t^2\left(p_y^{(n)}\right)^2(t,Y_t)dt + \beta n\sqrt{1-\rho^2}Y_t p_y^{(n)}(t,Y_t)dB_t^{\perp}, \qquad (39)$$

with $R_0 = 0$. We call R the *residual risk* (or hedging error) process. The term in $dB_t^{\perp}$, orthogonal to the Brownian increments driving the stock price, is interpreted as the unhedgeable component of risk. If $\rho = 1$ we see that the process R becomes riskless (recall that the interest rate is zero), reflecting the fact that the market incompleteness disappears in this case.

3.4 Power series expansions for the indifference price and hedge

We are interested in analysing the distribution of the terminal hedging error R_T. This is not possible in closed form, so our approach is to use the SDE (39) to simulate R over many asset price histories, and compute the distribution of terminal hedging error R_T. This programme was carried out in [24] and [26].

To simulate R via (39) efficiently, one may use analytic approximations for $p^{(n)}(t,y)$ and $p_y^{(n)}(t,y)$, in the form of power series expansions in powers of $a := -\alpha(1-\rho^2)n$. These arise from a Taylor expansion of the indifference pricing function

$$p^{(n)}(t,y) = \frac{1}{a}\log E^{Q^M}\left[\exp\left(ah(Y_T)\right)\middle|\, Y_t = y\right]. \qquad (40)$$

For a random variable X, recall that its cumulant generating function (CGF) is $\Psi_X(a) := \log E\exp(aX)$. Using linearity of the expectation operator, it is not hard to see that the CGF has a Taylor expansion of the form

$$\Psi_X(a) = \sum_{j=1}^{\infty}\frac{1}{j!}k_j(X)a^j,$$

where $k_j(X) \equiv k_j$ is the j^{th} *cumulant* of X. The cumulants are related to the central moments of X. For instance, writing

$$m_j(X) := E(X^j), \quad \mu_j(X) := E[(X-m_1)^j], \quad j\in\mathbb{N},$$

for the j^{th} raw and central moments, it is not hard to show that the first three cumulants are the mean, variance and skewness:

$$\begin{aligned} k_1(X) &= m_1(X), \\ k_2(X) &= \mu_2(X), \\ k_3(X) &= \mu_3(X). \end{aligned}$$

The first two cumulants being equal to the mean and variance implies that the low order approximation to the optimal hedge is a mean-variance hedging strategy, as pointed out by Kramkov and Sirbu [21].

Since the pricing function (40) is proportional to the cumulant generating function of the payoff under the minimal measure, it is easy to generate an analytic formula for the indifference pricing function. In [26], Monoyios gives the following representation.

Proposition 3 *The indifference pricing function $p^{(n)}(t,y)$ has the power series expansion*

$$p^{(n)}(t,y) = \sum_{j=1}^{5} \frac{1}{j!} k_j \left(h(Y_T)\right) a^{j-1} + O(a^5), \tag{41}$$

where $a = -\alpha(1-\rho^2)n$ and k_j is the j^{th} cumulant of the payoff under Q^M, conditional on $Y_t = y$. The expansion is valid for model parameters satisfying

$$E^{Q^M}\left[\exp(ah(Y_T))\right| Y_t = y] \leq 2. \tag{42}$$

This means one can produce an accurate perturbation series for $p^{(n)}(t,y)$, as a series of BS-type formulae, which can be differentiated term by term to give an analytic approximation for $p_y^{(n)}(t,y)$. In particular, the leading order term in the price expansion is Davis' [4] marginal price. Once again, this shows how the non-linear pricing rule given by the indifference price reduces to the linear pricing rule of the marginal price as the number of claims tends to zero.

The terms in the expansion depend ultimately on the moments $m_j := E^{Q^M}[h^j(Y_T)|Y_t = y]$, $j \in \mathbb{N}$ and (in the case of $p_y^{(n)}(t,y)$) on their partial derivatives $\partial m_j := \partial m_j/\partial y$, $j \in \mathbb{N}$. These are easy to compute (in the case of a put option we give some results shortly) since, under Q^M, and conditional upon $Y_t = y$, $\log Y_T$ is normally distributed: with $\mathrm{N}(m, \Sigma^2)$ denoting the normal probability law with mean m and variance Σ^2, we have

$$\begin{array}{rcl} \log Y_T & \sim & \mathrm{N}\left(\log y + b - \frac{1}{2}\Sigma^2, \Sigma^2\right), \\ b & = & \beta(\theta - \rho\lambda)(T-t), \\ \Sigma^2 & = & \beta^2(T-t). \end{array} \tag{43}$$

For the optimal hedging strategy, the explicit results are obtained by differentiating (41) with respect to y. If we denote by $\partial\kappa_j$ the partial derivative of κ_j with respect to y:

$$\partial\kappa_j \equiv \frac{\partial\kappa_j}{\partial y},$$

where κ_j denotes any of m_j, μ_j, k_j, then we have:

Corollary 2 *The partial derivative of the indifference price $p^{(n)}(t,y)$ with respect to y has the power series expansion*

$$\frac{\partial p^{(n)}}{\partial y}(t,y) = \sum_{j=1}^{5} \frac{1}{j!}(\partial k_j) a^{j-1} + O(a^5).$$

The partial derivatives of the cumulants are related to $\mu_j, \partial\mu_j$. For instance, up to $j=3$ we have

$$\begin{aligned} \partial k_1 &= \partial m_1, \\ \partial k_2 &= \partial \mu_2, \\ \partial k_3 &= \partial \mu_3. \end{aligned}$$

(See [26] for full details and more more formulae.)

The significance of the expansions is that they give easily computed closed form approximations for the indifference price and optimal hedge. In the specific case of a put option, we have the following formulae for the raw moments of the payoff under the minimal measure Q^M.

Lemma 1 *For a put option, $h(y) = (K-y)^+$, where $K>0$ is the strike price, the j^{th} moment $m_j := E^{Q^M}[h^j(Y_T)|Y_t = y]$, $j \in \mathbb{N}$, is given by*

$$m_j = \sum_{\ell=0}^{j} \left(\begin{array}{c} j \\ \ell \end{array} \right) (-y)^\ell K^{(j-\ell)} \exp\left[\ell\left(b + \frac{1}{2}(\ell-1)\Sigma^2\right)\right] \Phi(-d_1 - (\ell-1)\Sigma),$$

where $\Phi(\cdot)$ denotes the standard cumulative normal distribution function, and where

$$\begin{aligned} d_1 &= \frac{1}{\Sigma}\left[\log\left(\frac{y}{K}\right) + b + \frac{1}{2}\Sigma^2\right] \\ b &= \beta(\theta - \rho\lambda)(T-t), \\ \Sigma^2 &= \beta^2(T-t). \end{aligned}$$

Proof For the put payoff, we have, for $j \in \mathbb{N}$,

$$\begin{aligned} (h(Y_T))^j &= \left((K-Y_T)^+\right)^j \\ &= (K-Y_T)^j I_{\{Y_T \le K\}} \\ &= \sum_{\ell=0}^{j} \left(\begin{array}{c} j \\ \ell \end{array} \right) (-1)^\ell K^{(j-\ell)} Y_T^\ell I_{\{Y_T \le K\}}, \end{aligned}$$

where $I_{\{Y_T \le K\}}$ denotes the indicator function of the event $\{Y_T \le K\}$. Given the lognormal distribution (43) of Y_T, it is easy to show that

$$E^{Q^M}\left[Y_T^\ell I_{\{Y_T\le K\}} \middle| Y_t = y\right] = y^\ell \exp\left(\ell\left(b + \frac{1}{2}(\ell-1)\Sigma^2\right)\right) \Phi(-d_1 - (\ell-1)\Sigma),$$

from which the result follows.

□

Lemma 2 *Let $j \in \mathbb{N}$. For a put option payoff, $h(y) = (K-y)^+$, ∂m_j is given by*

$$\partial m_j = -\sum_{\ell=1}^{j} \left(\begin{array}{c} j \\ \ell \end{array} \right) (-y)^{(\ell-1)} K^{(j-\ell)} \exp\left(\ell\left(b + \frac{1}{2}(\ell-1)\Sigma^2\right)\right) \ell N(-d_1 - (\ell-1)\Sigma).$$

Proof This is a straightforward exercise in differentiation.

□

This power series expansions for $p^{(n)}(t,y)$ and $p_y^{(n)}(t,y)$ give a closed form and extremely accurate (see [24]) computation of the optimal price and hedging strategy, with the leading order term in the expansion for the price being the marginal price, $\hat{p}(t,y) = E^{Q^M}[h(Y_T)|Y_t = y]$, of the claim.

3.5 Optimal versus naive hedging

In [24, 26], a comparison was made between hedging a claim with the optimal strategy versus with the BS-style "naive" strategy (30) which takes S as a good proxy for Y.

In the BS-style hedge, let us repeat the calculation leading to the residual risk SDE (39), but with the claim traded at the BS price $P(0,Y_0) = \mathrm{BS}(0,Y_0)$ per claim and hedged using the $\rho \to 1$ limit of hedging formula (even though true value of ρ is *not* equal to 1). We then obtain the "naive" hedging error process R^N, following

$$dR_t^N = n\beta Y_t(\theta - \lambda)P_y(t,Y_t)dt + n\beta Y_t P_y(t,Y_t)[(\rho - 1)dB_t + \sqrt{1-\rho^2}dB_t^{\perp}].$$

Once again, we note that this is not riskless, but becomes so if the true value of ρ is indeed 1. The "naive" trader hopes this proves a good approximation.

For the case when the agent sells a put option ($n = -1$) on the non-traded asset, in [26] Monoyios generated optimal and naive hedging error distributions using 10,000 asset price histories. These showed that the optimal hedge error distribution has a higher mean, lower standard deviation, and a higher median, than the naive hedge error distribution. The increased median, in particular, showed how the relative frequency of profits over losses is increased when hedging optimally. We shall see some examples of this type of simulation in the next section, in the context of a partial information model.

Thus, the hedging strategy in (37) is, at first sight, superior to the BS-style hedge (30) But from (28) we see that the exponential hedge requires knowledge of λ, θ, which are impossible to estimate with any degree of accuracy (see Rogers [30] or Monoyios [26]). This can ruin the effectiveness of indifference hedging, as shown in [26].

It is therefore difficult to draw any meaningful conclusions on the effectiveness of utility-based hedging in this model without relaxing the assumption that the agent knows the true values of the drifts. This is the subject of the next section.

4 Partial information basis risk model

In the basis risk model of the previous section, we now assume the hedger does not know the values of the return parameters λ, θ, so these are considered to be random variables. Equivalently, the agent cannot observe the Brownian motions B, W driving the asset prices, so is required to use strategies adapted to the observation filtration $\hat{\mathbb{F}}$ generated by asset returns. This is a *partial information* scenario.

Partial information problems under various scenarios have been studied by a number of authors, usually in the context of optimal investment. Examples include Rogers [30], Björk, Davis and Landén [2], Nagai and Peng [29], and Monoyios [27], who treats the case of exponential hedging of basis risk, the subject of this section.

4.1 Choice of prior

We shall take the the two-dimensional random variable

$$U := \begin{pmatrix} \lambda \\ \theta \end{pmatrix}$$

to have a Gaussian distribution which will be updated as the agent attempts to filter the values of the drifts from asset observations during the hedging interval $[0, T]$.

The choice of Gaussian prior is motivated by the idea that the agent has some past observations of S, Y before time 0, uses these to obtain classical point estimates of the drifts, and the joint distribution of the estimators is used as the prior in a Bayesian framework. Ultimately, in order to obtain explicit solutions, we shall assume that the agent uses observations before time 0 of equal length for both assets. Throughout, we make the approximation that the asset price observations are continuous, so that σ, β, ρ are known from the quadratic variation and co-variation of S, Y. This is because our goal here is to focus on the severest problem of drift parameter uncertainty.

Consider an observer with data for S over a time interval of length t_S, and for Y over a window of length t_Y, who considers λ and θ as *constants*, and records the returns dS_t/S_t and dY_t/Y_t in order to estimate the values of the drifts. The best estimator of, say, λ is $\bar{\lambda}(t_S)$ given by

$$\begin{aligned} \bar{\lambda}(t_S) &= \frac{1}{t_S} \int_{t_0}^{t_0+t_S} \frac{dS_u}{S_u} \\ &= \lambda + \frac{B_{t_0+t_S}}{t_S} \\ &\sim \mathrm{N}\left(\lambda, \frac{1}{t_S}\right), \end{aligned}$$

The estimator of λ is normally distributed, with a similar computation for the estimator of θ. The estimator, $(\bar{\lambda}, \bar{\theta})$, of the (supposed constant) vector (λ, θ) is bivariate normal. Defining $\mathrm{v}_0 := 1/t_S$ and $\mathrm{w}_0 := 1/t_Y$ it is easily checked that

$$\begin{pmatrix} \bar{\lambda} \\ \bar{\theta} \end{pmatrix} \sim \mathrm{N}(M, C),$$

where the mean vector M and covariance matrix C are given by

$$M = \begin{pmatrix} \lambda \\ \theta \end{pmatrix}, \qquad C = \begin{pmatrix} \mathrm{v}_0 & \rho \min(\mathrm{v}_0, \mathrm{w}_0) \\ \rho \min(\mathrm{v}_0, \mathrm{w}_0) & \mathrm{w}_0 \end{pmatrix}. \tag{44}$$

With this in mind, we shall suppose that (λ, θ), now considered as a *random variable*, is bivariate normal according to

$$\lambda \sim \mathrm{N}(\lambda_0, \mathrm{v}_0), \quad \theta \sim \mathrm{N}(\theta_0, \mathrm{w}_0), \quad \mathrm{cov}(\lambda, \theta) = \mathrm{c}_0 := \rho \min(\mathrm{v}_0, \mathrm{w}_0).$$

This distribution will be updated via subsequent observations of

$$\xi_t := \frac{1}{\sigma}\int_0^t \frac{dS_u}{S_u} = \lambda t + B_t, \qquad \zeta_t := \frac{1}{\beta}\int_0^t \frac{dY_u}{Y_u} = \theta t + W_t$$

over the hedging interval $[0, T]$.

4.2 Kalman-Bucy filter

We are firmly within the realm of a two-dimensional Kalman filtering problem, which we treat as follows. Define the observation filtration by

$$\hat{\mathbb{F}} := (\hat{\mathcal{F}}_t)_{0 \le t \le T}, \qquad \hat{\mathcal{F}}_t = \sigma(\xi_s, \zeta_s; 0 \le s \le t).$$

The *observation process*, O, and unobservable *signal process*, U, are defined by

$$O := \begin{pmatrix} \xi_t \\ \zeta_t \end{pmatrix}_{0 \le t \le T}, \qquad U := \begin{pmatrix} \lambda \\ \theta \end{pmatrix},$$

satisfying the stochastic differential equations

$$dO_t = U dt + D d\mathbf{B}_t, \qquad dU = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where

$$D = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix}, \qquad \mathbf{B}_t = \begin{pmatrix} B_t \\ B_t^{\perp} \end{pmatrix}.$$

The optimal filter is $\hat{U}_t := E[U|\hat{\mathcal{F}}_t], 0 \le t \le T$, a two-dimensional process defining the best estimates of λ and θ given observations up to time $t \in [0, T]$:

$$\hat{U}_t \equiv \begin{pmatrix} \hat{\lambda}_t \\ \hat{\theta}_t \end{pmatrix} := \begin{pmatrix} E[\lambda|\hat{\mathcal{F}}_t] \\ E[\theta|\hat{\mathcal{F}}_t] \end{pmatrix}, \qquad \begin{pmatrix} \hat{\lambda}_0 \\ \hat{\theta}_0 \end{pmatrix} = \begin{pmatrix} \lambda_0 \\ \theta_0 \end{pmatrix}. \tag{45}$$

The solution to this filtering problem converts the partial information model to a full information model with random drifts, given in the following proposition. To avoid a proliferation of different symbols, we abuse notation and write, for example, $\hat{\lambda}_t \equiv \hat{\lambda}(t, S_t)$ when a process $\hat{\lambda}$ is a function of time and current stock price.

Proposition 4 *The partial information model is equivalent to a full information model in which the asset price dynamics in the observation filtration $\hat{\mathbb{F}}$ are*

$$\begin{aligned} dS_t &= \sigma S_t(\hat{\lambda}_t dt + d\hat{B}_t), && (46)\\ dY_t &= \beta Y_t(\hat{\theta}_t dt + d\hat{W}_t), && (47) \end{aligned}$$

where $\hat{B}, \hat{W}$ are $\hat{\mathbb{F}}$-Brownian motions with correlation ρ, and the random drifts $\hat{\lambda}, \hat{\theta}$ are $\hat{\mathbb{F}}$-adpated processes.

If λ and θ have common initial variance v_0, then $\hat{\lambda}, \hat{\theta}$ are given by

$$\begin{pmatrix} \hat{\lambda}_t \\ \hat{\theta}_t \end{pmatrix} = \begin{pmatrix} \lambda_0 \\ \theta_0 \end{pmatrix} + \int_0^t \mathrm{v}_u \begin{pmatrix} d\hat{B}_u \\ d\hat{W}_u \end{pmatrix}, \quad 0 \le t \le T, \tag{48}$$

where $(\mathrm{v}_t)_{0\le t\le T}$ is the deterministic function

$$\mathrm{v}_t = \frac{\mathrm{v}_0}{1+\mathrm{v}_0 t}.$$

Equivalently, $\hat{\lambda}, \hat{\theta}$ are given as functions of time and current asset price by

$$\hat{\lambda}_t = \hat{\lambda}(t, S_t) = \frac{\lambda_0 + \mathrm{v}_0 \xi_t}{1+\mathrm{v}_0 t}, \qquad \hat{\theta}_t = \hat{\theta}(t, Y_t) = \frac{\theta_0 + \mathrm{v}_0 \zeta_t}{1+\mathrm{v}_0 t}, \tag{49}$$

with

$$\xi_t = \frac{1}{\sigma}\log\left(\frac{S_t}{S_0}\right) + \frac{1}{2}\sigma t, \qquad \zeta_t = \frac{1}{\beta}\log\left(\frac{Y_t}{Y_0}\right) + \frac{1}{2}\beta t. \tag{50}$$

Proof By the Kalman-Bucy filter, for example Theorem V.9.2 in Fleming and Rishel [9], $\hat{U}$ satisfies the stochastic differential equation

$$d\hat{U}_t = \quad V_t \left(DD^T\right)^{-1}(dO_t - \hat{U}_t dt) \quad =: V_t \left(DD^T\right)^{-1} dN_t, \tag{51}$$

where $(N_t)_{0\le t\le T}$ is the innovations process, defined by

$$\begin{aligned} N_t &:= O_t - \int_0^t \hat{U}_s ds \\ &= \begin{pmatrix} \xi_t - \int_0^t \hat{\lambda}_s ds \\ \zeta_t - \int_0^t \hat{\theta}_s ds \end{pmatrix} \\ &=: \begin{pmatrix} \hat{B}_t \\ \hat{W}_t \end{pmatrix}, && (52) \end{aligned}$$

and classical filtering theory implies that $\hat{B}, \hat{W}$ are $\hat{\mathbb{F}}$-Brownian motions with correlation ρ. The deterministic matrix function V is the conditional variance-covariance matrix defined by

$$V_t := E\left[(U-\hat{U}_t)(U-\hat{U}_t)^{\mathrm{T}} \middle| \hat{\mathcal{F}}_t\right] = E\left[(U-\hat{U}_t)(U-\hat{U}_t)^{\mathrm{T}}\right],$$

(T denoting transpose) where the last equality follows because the error $U - \hat{U}_t$ is independent of $\hat{\mathcal{F}}_t$ (Theorem V.9.2 in [9] again).

Using (52), and writing dS_t in terms of $d\xi_t$, as in (26), gives the dynamics (46) of S in the observation filtration; (47) is established similarly.

The matrix $V = (V_t)_{0 \le t \le T}$ satisfies the Riccati equation

$$\frac{dV_t}{dt} = -V_t \left(DD^T\right)^{-1} V_t, \qquad V_0 = C,$$

where C is the covariance matrix in (44). Then $R_t := V_t^{-1}$ satisfies the Lyapunov equation

$$\frac{dR_t}{dt} = \left(DD^T\right)^{-1}.$$

Define the elements of the conditional covariance matrix by

$$V_t =: \begin{pmatrix} \mathrm{v}_t & \mathrm{c}_t \\ \mathrm{c}_t & \mathrm{w}_t \end{pmatrix}.$$

Then the filtering equation (51) is a pair of coupled stochastic differential equations:

$$\begin{aligned} \begin{pmatrix} d\hat{\lambda}_t \\ d\hat{\theta}_t \end{pmatrix} &= \frac{1}{1-\rho^2} \begin{pmatrix} \mathrm{v}_t - \rho \mathrm{c}_t & \mathrm{c}_t - \rho \mathrm{v}_t \\ \mathrm{c}_t - \rho \mathrm{w}_t & \mathrm{w}_t - \rho \mathrm{c}_t \end{pmatrix} \begin{pmatrix} d\xi_t - \hat{\lambda}_t dt \\ d\zeta_t - \hat{\theta}_t dt \end{pmatrix} \\ &= \frac{1}{1-\rho^2} \begin{pmatrix} \mathrm{v}_t - \rho \mathrm{c}_t & \mathrm{c}_t - \rho \mathrm{v}_t \\ \mathrm{c}_t - \rho \mathrm{w}_t & \mathrm{w}_t - \rho \mathrm{c}_t \end{pmatrix} \begin{pmatrix} d\hat{B}_t \\ d\hat{W}_t \end{pmatrix}. \end{aligned}$$

Solving the Lyapunov equation yields 3 equations for $\mathrm{v}_t, \mathrm{w}_t, \mathrm{c}_t$:

$$\begin{aligned} \frac{\mathrm{v}_t}{\mathrm{v}_t \mathrm{w}_t - \mathrm{c}_t^2} - \frac{\mathrm{v}_0}{\mathrm{v}_0 \mathrm{w}_0 - \mathrm{c}_0^2} &= \frac{t}{1-\rho^2}, \\ \frac{\mathrm{w}_t}{\mathrm{v}_t \mathrm{w}_t - \mathrm{c}_t^2} - \frac{\mathrm{w}_0}{\mathrm{v}_0 \mathrm{w}_0 - \mathrm{c}_0^2} &= \frac{t}{1-\rho^2}, \\ \frac{\mathrm{c}_t}{\mathrm{v}_t \mathrm{w}_t - \mathrm{c}_t^2} - \frac{\mathrm{c}_0}{\mathrm{v}_0 \mathrm{w}_0 - \mathrm{c}_0^2} &= \frac{\rho t}{1-\rho^2}, \end{aligned} \tag{53}$$

where we have written $\mathrm{c}_0 \equiv \rho \min(\mathrm{v}_0, \mathrm{w}_0)$ for brevity.

Now make the simplification $\mathrm{w}_0 = \mathrm{v}_0$. From the discussion on Section 4.1, we see that this corresponds to using past observations over the same length of time, $t_S = t_Y$, for both S and Y in fixing the prior. Then $\mathrm{c}_0 = \rho \mathrm{v}_0$, and the solution to the system of equations (53) gives the entries of the matrix V_t as

$$\mathrm{v}_t = \frac{\mathrm{v}_0}{1 + \mathrm{v}_0 t}, \quad \mathrm{w}_t = \mathrm{v}_t, \quad \mathrm{c}_t = \rho \mathrm{v}_t.$$

With this simplification, the equation for the optimal filter simplifies to

$$\begin{pmatrix} d\hat{\lambda}_t \\ d\hat{\theta}_t \end{pmatrix} = \mathrm{v}_t \begin{pmatrix} d\xi_t - \hat{\lambda}_t dt \\ d\zeta_t - \hat{\theta}_t dt \end{pmatrix} = \mathrm{v}_t \begin{pmatrix} d\hat{B}_t \\ d\hat{W}_t \end{pmatrix}, \tag{54}$$

which, along with the initial condition in (45), yields (48) and (49).

Finally, the expression in (50) for ξ_t follows easily from the solution of (26) for S:

$$\log\left(\frac{S_t}{S_0}\right) = \sigma\left(\xi_t - \frac{1}{2}\sigma t\right),$$

and a similar calculation gives the formula for ζ_t.

□

With Proposition 4 we may now treat the model as a full information model with random drift parameters $(\hat{\lambda}_t, \hat{\theta}_t)$, and this is done in the next section.

4.3 Indifference hedging with random drifts

On the stochastic basis $(\Omega, \hat{\mathcal{F}}, \hat{\mathbb{F}}, P)$, the wealth process associated with trading strategy $\pi := (\pi_t)_{0\le t\le T}$, an $\hat{\mathbb{F}}$-adapted process satisfying $\int_0^T \pi_t^2 dt < \infty$ a.s., is $X^\pi \equiv X := (X_t)_{0\le t\le T}$, satisfying

$$dX_t = \sigma\pi_t(\hat{\lambda}_t dt + d\hat{B}_t). \tag{55}$$

We use an exponential utility function, $U(x) = -\exp(-\alpha x)$, $x\in\mathbb{R}$, $\alpha > 0$. The primal value function $u^{(n)}$ is defined once again as the maximum expected utility of wealth at T from trading S and receiving n units of the claim on Y, when starting at time $t\in[0,T]$:

$$u^{(n)}(t,x,s,y) := \sup_{\pi\in\mathcal{A}} E[U(X_T + nh(Y_T))|X_t = x, S_t = s, Y_t = y], \tag{56}$$

where $\mathcal{A}$ denotes the set of admissible trading strategies. The dynamics of the state variables X, S, Y are given by (55) and (46,47). The set of admissible strategies is defined as in (10). Once again denote the optimal trading strategy by $\pi^* \equiv \pi^{*,n}$, and the optimal wealth process by $X^* \equiv X^{*,n}$.

The utility indifference price and hedge for a position in n claims are defined in the classical manner as earlier. In this case, we have that the indifference price per claim at $t\in[0,T]$, given $X_t = x, S_t = s, Y_t = y$, is $p^{(n)}$ given by

$$u^{(n)}(t, x - np^{(n)}(t,x,s,y), s, y) = u^{(0)}(t,x,s).$$

The optimal hedging strategy, $\pi^{\mathrm{H}} := \left(\pi_t^{\mathrm{H}}\right)_{0\le t\le T}$, is defined by

$$\pi_t^{\mathrm{H}} := \pi_t^{*,n} - \pi_t^{*,0}, \quad 0\le t\le T.$$

As before, with exponential utility the indifference price will not in fact depend on the initial cash wealth x, so we shall write $p^{(n)}(t,x,s,y) \equiv p^{(n)}(t,s,y)$ from now on.

For small positions in the claim, the marginal utility-based price of the claim at $t\in[0,T]$ is $\hat{p}(t,s,y)$ defined by

$$\hat{p}(t,s,y) := \lim_{n\to 0} p^{(n)}(t,s,y).$$

We have seen that with exponential utility the marginal price is also equivalent to the limit of the indifference price as risk aversion goes to zero. Under appropriate conditions (satisfied in this model) it is given by the expectation of the payoff under the optimal measure of the dual to the problem without the claim. As we shall see in the next section, in our case this measure will be the minimal martingale measure Q^M, and we shall obtain the representation $\hat{p}(t,s,y) = E^{Q^M}[h(Y_T)|S_t = s, Y_t = y]$.

4.4 The dual problem

We shall attack the primal utility maximisation problem (56) via its dual problem.

The class $\mathcal{M}$ of local martingale measures for this model are measures Q with density processes defined by

$$Z_t := \left.\frac{dQ}{dP}\right|_{\hat{\mathcal{F}}_t} = \mathcal{E}(-\hat{\lambda}\cdot\hat{B} - \psi\cdot\hat{B}^{\perp})_t, \quad 0 \le t \le T,$$

for integrands ψ satisfying $\int_0^T \psi_t^2 dt < \infty$ a.s. ($\int_0^T \hat{\lambda}_t^2 dt < \infty$ is not hard to show). For $\psi = 0$ we obtain the minimal martingale measure Q^M, with density process $Z_t^{Q^M} = \mathcal{E}(-\hat{\lambda}\cdot\hat{B})_t$, for $t \in [0,T]$.

The change of measure density $Z_t^{Q^M}$ satisfies $dZ_t^{Q^M} = -\hat{\lambda}_t Z_t^{Q^M} d\hat{B}_t$ and, since $\hat{\lambda}_t = \hat{\lambda}(t, S_t)$ is a function of S_t, then so is $Z_t^{Q^M}$. The relevance of this is that the dual value function will be a function of the current asset prices at any initial time, as we shall see.

Under $Q \in \mathcal{M}$, $(\hat{B}^Q, \hat{B}^{\perp,Q})$ is two-dimensional Brownian motion, where

$$d\hat{B}_t^Q := d\hat{B}_t^Q + \hat{\lambda}_t dt, \qquad d\hat{B}_t^{\perp,Q} := d\hat{B}_t^{\perp} + \psi_t dt.$$

Further, under $Q \in \mathcal{M}$, the asset prices and random drifts satisfy (with $\bar{\rho} := \sqrt{1-\rho^2}$)

$$\begin{aligned} dS_t &= \sigma S_t d\hat{B}_t^Q, \\ dY_t &= \beta Y_t[(\hat{\theta}_t - \rho\hat{\lambda}_t - \bar{\rho}\psi_t)dt + d\hat{W}_t^Q], \\ d\hat{\lambda}_t &= \mathrm{v}_t[-\hat{\lambda}_t dt + d\hat{B}_t^Q], \\ d\hat{\theta}_t &= \mathrm{v}_t[-(\rho\hat{\lambda}_t + \bar{\rho}\psi_t)dt + d\hat{W}_t^Q], \end{aligned}$$

where $\hat{W}^Q = \rho\hat{B}^Q + \bar{\rho}\hat{B}^{\perp,Q}$.

The *dual value function* $v \equiv v^{(n)}$ is defined by

$$v^{(n)}(t,\eta,s,y) := \inf_{Q\in\mathcal{M}} E\left[V\left(\eta\frac{Z_T}{Z_t}\right) + \eta\frac{Z_T}{Z_t} nh(Y_T)\right| S_t = s, Y_t = y\bigg],$$

where V is the convex conjugate of the utility function U.

The primal value function $u^{(n)}(t,x,s,y)$ is then recovered from the bidual relation

$$u^{(n)}(t,x,s,y) = \inf_{\eta>0}[v^{(n)}(t,\eta,s,y) + x\eta].$$

Expressing the density process $(Z_t)_{0\leq t\leq T}$ in terms of Q-Brownian motions and using the form of the convex conjugate function V it is easy to obtain the dual value function in the form

$$v^{(n)}(t,\eta,s,y) = V(\eta) + \frac{\eta}{\alpha}\inf_{Q\in\mathcal{M}} E^Q\left[\log\left(\frac{Z_T}{Z_t}\right) + \alpha n h(Y_T)\middle| S_t = s, Y_t = y\right], \quad (57)$$

with

$$\log\left(\frac{Z_T}{Z_t}\right) = -\int_t^T \hat{\lambda}_u d\hat{B}_u^Q - \int_t^T \psi_u d\hat{B}_u^{\perp,Q} + \frac{1}{2}\int_t^T \left(\hat{\lambda}_u^2 + \psi_u^2\right) du.$$

It is not difficult to see that $E^Q\int_0^t \hat{\lambda}_u^2 du < \infty$ for all $t\in[0,T]$. If, in addition,

$$E^Q\int_0^t \psi_u^2 du < \infty, \quad t\in[0,T], \quad (58)$$

then the stochastic integrals in (57) will have zero expectation. Denoting by $\mathcal{M}'$ the subset of $\mathcal{M}$ for which (58) holds, we clearly have

$$\begin{aligned} &\inf_{Q\in\mathcal{M}} E^Q\left[\log\left(\frac{Z_T}{Z_t}\right) + \alpha n h(Y_T)\middle| S_t = s, Y_t = y\right] \\ = &\inf_{Q\in\mathcal{M}'} E^Q\left[\log\left(\frac{Z_T}{Z_t}\right) + \alpha n h(Y_T)\middle| S_t = s, Y_t = y\right]. \end{aligned}$$

Then the dual value function decomposes as

$$v^{(n)}(t,\eta,s,y) = V(\eta) + \frac{\eta}{\alpha}\left(H(t,s) + G(t,s,y)\right),$$

where

$$H(t,s) := E^Q\left[\frac{1}{2}\int_t^T \hat{\lambda}_u^2 du\middle| S_t = s, Y_t = y\right],$$

and where $G(t,s,y)$ is the value function of a stochastic control problem:

$$G(t,s,y) := \inf_{\psi} E^Q\left[\frac{1}{2}\int_t^T \psi_u^2 du + \alpha n h(Y_T)\middle| S_t = s, Y_t = y\right]. \quad (59)$$

Note also that for $n = 0$, then $G = 0$, and we also have the dual value function for $n \neq 0$ given directly in terms of its counterpart for no claim on Y:

$$v^{(n)}(t,\eta,s,y) = v^{(0)}(t,\eta,s,y) + \frac{\eta}{\alpha}G(t,s,y),$$

which will be useful in obtaining a representation for the indifference price. The primal value function is then recovered from the bidual relation as

$$u^{(n)}(t,x,s,y) = u^{(0)}(t,x,s)\exp[-G(t,s,y)], \quad (60)$$

where

$$u^{(0)}(t,x,s) = U(x)\exp[-H(t,s)]. \quad (61)$$

Applying the definition of the indifference price then gives the following representation.

Theorem 6 *The indifference price at* $t \in [0, T]$ *is given by*

$$p^{(n)}(t, s, y) = \frac{1}{\alpha n} G(t, s, y),$$

where the value function G *is defined in (59).*

Remark 1 Theorem 6 is a special case of the general dual representation for the indifference price, Corollary 1, since the relative entropy $\mathcal{H}(Q, P)$ between a measure $Q \in \mathcal{M}$ and P is given by

$$\mathcal{H}(Q, P) = E^Q \left[\frac{1}{2} \int_0^T \left(\hat{\lambda}_t^2 + \psi_t^2 \right) dt \right].$$

The Hamilton-Jacobi-Bellman (HJB) equation for $G(t, s, y)$ is

$$G_t + \mathcal{A}_{S,Y}^{Q^M} G + \min_{\psi} \left[\frac{1}{2}\psi^2 - \beta\bar{\rho}\psi y G_y \right] = 0 \quad G(T, s, y) = \alpha n h(y),$$

where $\mathcal{A}_{S,Y}^{Q^M}$ is the generator of (S, Y) under the minimal measure:

$$\mathcal{A}_{S,Y}^{Q^M} G = \frac{1}{2} s^s G_{ss} + \beta(\hat{\theta} - \rho\hat{\lambda}) y G_y + \frac{1}{2}\beta^2 y^2 G_{yy} + \rho\sigma\beta s y G_{sy}.$$

The optimal control is $\psi_t^* \equiv \psi^*(t, S_t, Y_t)$ where

$$\psi^*(t, s, y) = \bar{\rho}\beta y G_y(t, s, y)$$

so the HJB equation is the semi-linear PDE

$$G_t + \mathcal{A}_{S,Y}^{Q^M} G - \frac{1}{2}(1 - \rho^2)\beta^2 y^2 G_y^2 = 0.$$

Hence, the indifference price $p^{(n)}$ satisfies

$$p_t^{(n)} + \mathcal{A}_{S,Y}^{Q^M} p^{(n)} - \frac{1}{2}\alpha n(1 - \rho^2)\beta^2 y^2 \left(p_y^{(n)} \right)^2 = 0. \quad p^{(n)}(T, s, y) = h(y)$$

The optimal hedging strategy is obtained easily, since the HJB equation for primal value function gives the optimal trading strategy $\pi^{*,n}$ in terms of derivatives of $G(t, s, y)$, and hence in terms of derivatives of the indifference price, and we have the following theorem.

Theorem 7 *The optimal hedging strategy is to hold* $(\Delta_t^{\mathrm{H}})_{0 \le t \le T}$ *shares of* S *at time* $t \in [0, T]$*, given by*

$$\Delta_t^{\mathrm{H}} = -n \left(p_s^{(n)}(t, S_t, Y_t) + \rho \frac{\beta}{\sigma} \frac{Y_t}{S_t} p_y^{(n)}(t, S_t, Y_t) \right).$$

Note that, compared with the full information case, the optimal hedging strategy contains an additional term involving a derivative with respect to the stock price variable. This in turn has resulted from the extra dimension of the associated control problems. The drift parameter uncertainty has led to additional risk and hence to an extra hedging term to counteract the added risk.

Proof of Theorem 7 The HJB equation for the primal value function is

$$u_t^{(n)} + \max_{\pi} \mathcal{A}_{X,S,Y} u^{(n)} = 0,$$

where $\mathcal{A}_{X,S,Y}$ is the generator of (X,S,Y) under P. Performing the maximisation over π yields the optimal Markov control as $\pi_t^{*,n} = \pi^{*,n}(t, X_t^{*,n}, S_t, Y_t)$, where

$$\pi^{*,n}(t,x,s,y) = -\left(\frac{\hat{\lambda} u_x^{(n)} + \sigma s u_{xs}^{(n)} + \rho\beta y u_{xy}^{(n)}}{\sigma u_{xx}^{(n)}}\right),$$

and where the arguments of the functions on the right-hand-side are omitted for brevity. For the case $n=0$ there is no dependence on y in the value function $u^{(0)}$, and we have $\pi_t^{*,0} = \pi^{*,0}(t, X_t^{*,0}, S_t)$, where

$$\pi^{*,0}(t,x,s) = -\left(\frac{\hat{\lambda} u_x^{(0)} + \sigma s u_{xs}^{(0)}}{\sigma u_{xx}^{(0)}}\right).$$

Applying the definition of the optimal hedging strategy along with the representations (60) and (61) for the value functions, gives the result.

□

4.4.1 Linear approximation for the indifference price

To obtain analytic results and hence conduct a simulation study of the effectiveness of the optimal hedging strategy, we may approximate the indifference price by the marginal price. For $n=0$ the indifference price PDE becomes linear, and the Feynman-Kac theorem gives the marginal price as follows.

Corollary 3 *The marginal price at $t \in [0,T]$ is given by*

$$\hat{p}(t,s,y) = E^{Q^M}[h(Y_T) | S_t = s, Y_t = y].$$

This is perfectly consistent with the general result of Theorem 5 since the minimal entropy measure Q^E coincides with Q^M, as can be seen from Remark 1.

The marginal price (and hence the optimal trading strategy) can be computed in analytic form since, under Q^M, $\log Y_T$ is Gaussian. We have the following result.

Proposition 5 *Under Q^M, conditional on $S_t = s, Y_t = y$,*

$$\begin{aligned}
\log Y_T &\sim \mathrm{N}(m, \Sigma^2) \\
m &= \log y + \beta\left(\hat{\theta}(t,y) - \rho\hat{\lambda}(t,s) - \frac{1}{2}\beta\right)(T-t) \\
\Sigma^2 &= \left[1 + (1-\rho^2)\mathrm{v}_t (T-t)\right]\beta^2 (T-t)
\end{aligned}$$

Proof This is established by computing the SDEs for Y and for $\hat{\theta}_t - \rho\hat{\lambda}_t$ under Q^M. Indeed, applying the Itô formula to $\log Y_t$ under Q^M, we obtain, for $t < T$,

$$\log Y_T = \log Y_t + \beta \int_t^T \left(\hat{\theta}_u - \rho\hat{\lambda}_u\right) du - \frac{1}{2}\beta^2(T-t) + \beta \int_t^T d\hat{W}_u^{Q^M}, \tag{62}$$

where $\hat{W}^{Q^M}$ is a Brownian motion under Q^M. The dynamics of $\hat{\theta}_t - \rho\hat{\lambda}_t$ under Q^M are

$$d(\hat{\theta}_t - \rho\hat{\lambda}_t) = \bar{\rho}\mathrm{v}_t d\hat{B}_t^{\perp,Q^M},$$

where $\hat{B}^{\perp,Q^M}$ is a Q^M-Brownian motion perpendicular to that driving the stock, and related to $\hat{W}^{Q^M}$ by $\hat{W}^{Q^M} = \rho\hat{B}^{Q^M} + \bar{\rho}\hat{B}^{\perp,Q^M}$, where $\hat{B}^{Q^M}$ is the Brownian motion driving S. Hence, for $u > t$, after changing the order of integration in a double integral, we obtain

$$\int_t^T \left(\hat{\theta}_u - \rho\hat{\lambda}_u\right) du = \left(\hat{\theta}_t - \rho\hat{\lambda}_t\right)(T-t) + \bar{\rho}\int_t^T \mathrm{v}_u(T-u)d\hat{B}_u^{\perp,Q^M}.$$

This can be inserted into (62) to yield the desired result.

□

We are thus able to obtain BS-style formulae for the price and hedge. For a put option of strike K we obtain the following explicit formulae for the marginal price and the associated optimal hedging strategy, where Φ denotes the standard cumulative normal distribution function.

Corollary 4 *With m and Σ as in Proposition 5, define $b \equiv b(t,s,y)$ by*

$$m = \log y + b - \frac{1}{2}\Sigma^2.$$

Then the marginal price at time $t \in [0,T]$ of a put option with payoff $(K - Y_T)^+$ is $\hat{p}(t, S_t, Y_t)$, where

$$\begin{aligned} \hat{p}(t,s,y) &= K\Phi(-d_1 + \Sigma) - ye^b\Phi(-d_1), \\ d_1 &= \frac{1}{\Sigma}\left[\log\left(\frac{y}{K}\right) + b + \frac{1}{2}\Sigma^2\right]. \end{aligned}$$

The optimal hedging strategy given by Theorem 7 with $\hat{p}$ as an approximation to the indifference price is $\hat{\Delta}_t \equiv \hat{\Delta}(t, S_t, Y_t)$, where

$$\hat{\Delta}(t,s,y) = n\rho\frac{\beta}{\sigma}\frac{y}{s}e^b\Phi(-d_1).$$

4.5 Performance of the optimal strategy

To assess the performance of the optimal strategy based on the marginal price, we conduct the following simulation experiment.

Using chosen values for the "true" drifts λ, θ, we generate asset price paths S, Y over a time frame t_0 (1 year) and use this "data" to estimate the asset drifts, and so set a prior distribution at time 0. We then set initial asset prices S_0, Y_0 and generate a price history over $[0, T]$.

We suppose a put option of strike K is sold at time 0 for $\hat{p}(0, S_0, Y_0)$ and optimally hedged over $[0, T]$ ($T = 1$year), incorporating updating from filtering, and using daily rebalancing. In this way, we generate a terminal hedging error. We repeat this procedure of setting a prior using past "data" and then hedging over $[0, T]$, over many paths (with the same values of S_0, Y_0 in each simulation) to generate a terminal hedging error distribution. Note that we used a new set of "past data" to set the prior on each simulation. The idea is to allow for occasions where the mean of prior distribution is of variable quality in relation to the true values of the drifts.

We repeat the hedging error computation over the same asset price histories using the BS-style hedge (30), and also with a hedge in the absence of filtering, where we used the initial estimates of the asset drifts to compute the hedge throughout the hedging time frame. This uses the hedge in (37) and the approximation formulae from Monoyios [26].

The parameters were set as below. The initial BS price and delta are denoted by BS_0 and Δ_0^{BS}. The notation $\bar{p}_0$ denotes the average marginal price that the put was sold for (recall that each simulation run results in a different initial prior). The average mean for the prior distribution of λ is denoted $\bar{\lambda}_0$ (and similarly for θ); "NF" denotes "no filtering".

$$
\begin{array}{lllll}
\lambda = 0.5, & \sigma = 0.22, & \theta = 0.45, & \beta = 0.18, & \rho = 0.85 \\
S_0 = 100, & Y_0 = 100, & K = 100 & & \\
\mathrm{BS}_0 = 7.17, & \Delta_0^{\mathrm{BS}} = -0.46 & & & \\
\bar{p}_0 = 7.20, & \bar{\Delta}_0 = -0.32 & & & \\
\bar{p}_0^{\mathrm{NF}} = 7.33, & \bar{\Delta}_0^{\mathrm{NF}} = -0.47 & & & \\
\bar{\lambda}_0 = 0.11, & \bar{\theta}_0 = 0.09. & & &
\end{array}
$$

The results are shown in Figure 1 and Table 1. They clearly show that the optimal hedge with learning produces a hedging error distribution with higher mean, lower standard deviation and, significantly, a higher median (all as percentages of the initial option premium), than either the BS hedge or the hedge without learning. Thus, the frequency of profits over losses is increased by the optimal hedging program incorporating learning.

Varying some parameters slightly gave the results in Figure 2 and Table 2. The results are still favourable, even if one sells the option for a lower value than the BS price, showing that the improvement in hedging performance is not due to starting with a higher wealth in the initial hedging portfolio.

The conclusion is that optimal hedging combined with a filtering algorithm to deal with drift parameter uncertainty can indeed give improved hedging performance over methods which take S as a perfect proxy for Y.

Table 1: Hedging error statistics for Figure 1 (as percentage of premium)

	Mean	SD	Median
Optimal Hedge	2.09	62.8	12.3
BS Hedge	-17.6	80.5	-11.6
Unfiltered Hedge	-15.7	79.4	-10.1

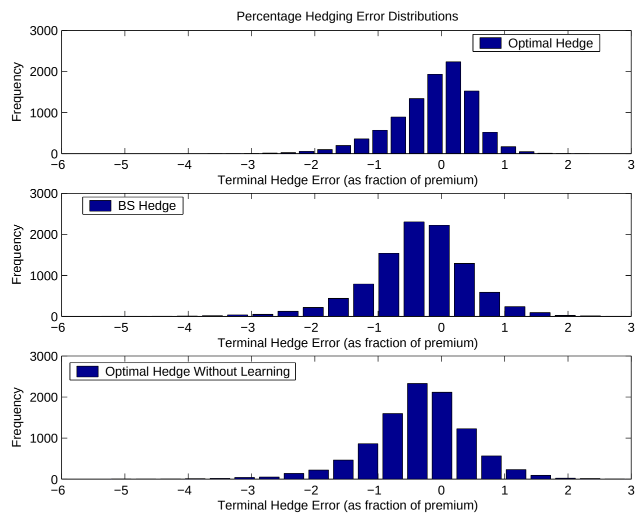

Figure 1: Percentage hedging error distribution

Table 2: Hedging error statistics for Figure 2 (as percentage of premium). The parameters are the same as those for Table 1 except for $\lambda = 0.4$, $\beta = 0.23$, $\rho = 0.9$. In this case the BS price was 9.16, the average optimal price was 8.99 and the average price without filtering was 9.12.

	Mean	SD	Median
Optimal Hedge	7.09	51.8	13.1
BS Hedge	5.57	52.8	10.0
Unfiltered Hedge	5.16	53.1	9.60

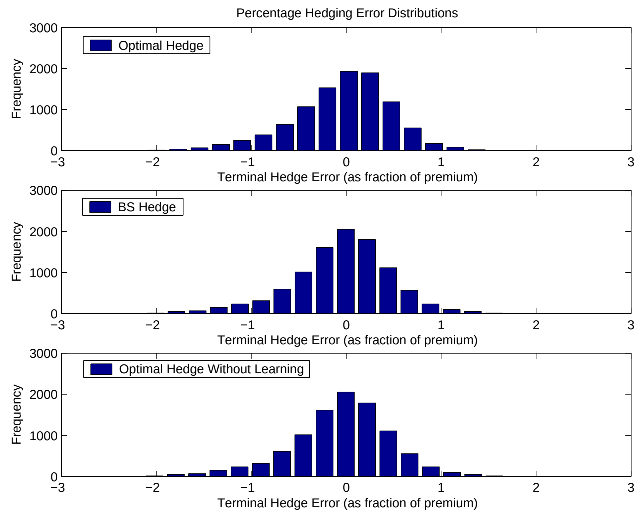

Figure 2: Percentage hedging error distribution

Conclusion

In this chapter we analysed utility-based pricing and hedging in incomplete security markets. We have derived probabilistic representations for indifference prices. In general these are non-linear pricing rules reducing to linear pricing rules as the number of claims becomes small. We computed prices and optimal hedging strategies in basis risk models, under full and partial information scenarios. these showed how optimal hedging can indeed outperform hedging methods based on complete market approximations. This is therefore of relevance to practitioners as well as to academics. The take-up of utility-based pricing by practitioners has been somewhat limited, given the need to specify a utility function. It is possible that the results of this chapter can show the potential benefits of applying such techniques in practice.

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