Optimal investment and hedging under partial and inside information

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Abstract

This article concerns optimal investment and hedging for agents who must use trading strategies which are adapted to the filtration generated by asset prices, possibly augmented with some inside information related to the future evolution of an asset price. The price evolution and observations are taken to be continuous, so the partial (and, when applicable, inside) information scenario is characterised by asset price processes with an unknown drift parameter, which is to be filtered from price observations. We first give an exposition of filtering theory, leading to the Kalman-Bucy filter. We outline the dual approach to portfolio optimisation, which is then applied to the Merton optimal investment problem when the agent does not know the drift parameter of the underlying stock. This is taken to be a random variable with a Gaussian prior distribution, which is updated via the Kalman filter. This results in a model with a stochastic drift process adapted to the observation filtration, and which can be treated as a full information problem, and an explicit solution to the optimal investment problem is possible. We also consider the same problem when the agent has noisy knowledge at time 0 of the terminal value of the Brownian motion driving the stock. Using techniques of enlargement of filtration to accommodate the insider's additional knowledge, followed by filtering the asset price drift, we are again able to obtain an explicit solution. Finally we treat an incomplete market hedging problem. A claim on a non-traded asset is hedged using a correlated traded asset. We summarise the full information case, then treat the partial information scenario in which the hedger is uncertain of the true values of the asset price drifts. After filtering, the resulting problem with random drifts is solved in the case that each asset's prior distribution has the same variance, resulting in analytic approximations for the optimal hedging strategy.

1 Introduction

This article examines some problems of optimal investment, and of optimal hedging of a contingent claim in an incomplete market, when the agent's information set is restricted to stock price observations, possibly augmented by some additional information related to the terminal value of a stock price.

In classical models of financial mathematics, one usually specifies a probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$, and then writes down some stochastic process $S = (S_t)_{0 \leq t \leq T}$ for an asset price, such that S is adapted to the filtration $\mathbb{F}$. A typical example would be the Black-Scholes (henceforth, BS) model of a stock price, following the geometric Brownian motion

$$dS_t = \sigma S_t(\lambda dt + dB_t), \tag{1}$$

where B is a $(P, \mathbb{F})$-Brownian motion and the volatility $\sigma > 0$ and the Sharpe ratio λ are assumed to be known constants. Of course, this is a strong assumption that an agent is assumed to be

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able to observe the Brownian motion process B, as well as the stock price process S. We refer to this as a *full information* scenario. In this case, an agent uses $\mathbb{F}$-adapted trading strategies in S, which is an $\mathbb{F}$-adapted process with known drift and diffusion coefficients.

We shall frequently relax the full information assumption in this article. We shall assume that the agent can only observe the stock price process, and not the Brownian motion B, and so the parameters σ, λ are not known with certainty. The agent's trading strategies must therefore be adapted to the *observation filtration* $\widehat{\mathbb{F}} := (\widehat{\mathcal{F}}_t)_{0 \le t \le T}$ generated by S. We refer to this as a *partial information* scenario.

In this case, the parameter λ would be regarded as an unknown constant whose value needs to be determined from price data. In principle, one would also have to apply this philosophy to the volatility σ, but we shall make the approximation that price observations are continuous, so that σ can be computed from the quadratic variation $[S]_t$ of the stock price, since we have

$$[S]_t = \sigma^2 S_t^2 t, \quad 0 \le t \le T.$$

One way to model the uncertainty in our knowledge of the (supposed constant) parameter λ is to take a so-called Bayesian approach. This means we consider λ to be an $\mathcal{F}_0$-measurable random variable with a given initial distribution (the *prior* distribution). The prior distribution initialises the probability law of λ conditional on $\widehat{\mathcal{F}}_0$, and this is updated in the light of new price information, that is, as the observation filtration $\widehat{\mathbb{F}}$ evolves. (In the case that λ is some unknown process $(\lambda_t)_{0 \le t \le T}$ (as opposed to an unknown constant), then we would consider it to be some $\mathbb{F}$-adapted process such that its starting value λ_0 has a given prior distribution conditional on $\widehat{\mathcal{F}}_0$.)

This is an example of a *filtering* problem: to compute the best estimate of a random variable given observations up to time $t \in [0, T]$, and hence given the sigma algebra $\widehat{\mathcal{F}}_t, t \in [0, T]$. In the case of the BS model (1), where we model λ as an $\mathcal{F}_0$-measurable random variable, we are interested in computing the conditional expectation

$$\widehat{\lambda}_t := E[\lambda | \widehat{\mathcal{F}}_t], \quad 0 \le t \le T.$$

We shall see that the effect of filtering is that the model (1) may be replaced by a model specified on the filtered probability space $(\Omega, \widehat{\mathcal{F}}_T, \widehat{\mathbb{F}}, P)$ and written as

$$dS_t = \sigma S_t (\widehat{\lambda}_t dt + d\widehat{B}_t),$$

where $\widehat{B}$ is a $(P, \widehat{\mathbb{F}})$-Brownian motion. This model may now be treated as a full information model, since both $\widehat{B}$ and $\widehat{\lambda}$ are $\widehat{\mathbb{F}}$-adapted processes. The price we have paid for restoring a full information scenario is that the constant parameter λ has been replaced by a random process $\widehat{\lambda}$. The procedure by which a partial information model is replaced with a tractable full information model under the observation filtration is typically only achievable in special circumstances, such as Gaussian prior distributions and certain linearity properties in the relation between the observable and unobservable processes, as we shall see in the next section.

In the rest of the article, we first give, in Section 2, an exposition of filtering theory (along the lines of Rogers and Williams [32] Chapter VI.8, which draws on the seminal work of Fujisaki et al [7]), culminating in the linear filtering case, the Kalman-Bucy filter [11]. In Section 3.1 we describe the dual (or martingale) approach to portfolio optimisation (see Karatzas [13] for example), that we will use frequently in what follows.

In Section 3 we apply the results to the Merton problem [20, 21] of optimal investment, which seeks a trading strategy to maximise expected utility of terminal wealth. We explicitly solve the problem for a stock with a Gaussian drift process. The partial information case studied by, among others, Rogers [31], as well as the classical full information case, are special cases of this.

In Section 4 we solve the Merton optimal investment problem when the agent is assumed to have some additional information in the form of knowledge of the value of a random variable I, which represents noisy information on the underlying Brownian motion at time T. Further examples of models with both inside information and parameter uncertainty can be found in Danilova, Monoyios and Ng [3].

Finally, in Section 5 we consider the hedging of a claim in an incomplete market setting under partial information. Specifically, we shall consider a *basis risk* model involving the optimal hedging of a contingent claim on a non-tradeable asset Y using a traded stock S, correlated with Y, when the hedger is restricted to trading strategies in S that are adapted to the observation filtration $\widehat{\mathbb{F}}$ generated by the asset prices.

In the full information case the asset prices are correlated log-Brownian motions given by

$$dS_t = \sigma S_t(\lambda dt + dB_t), \quad dY_t = \beta Y_t(\theta dt + dW_t),$$

where the Brownian motions B, W are correlated with correlation $\rho \in [-1, 1]$. The parameters $\sigma > 0$, λ, $\beta > 0$ and θ are assumed constant.

This market is complete when the correlation is perfect, but incomplete otherwise. A number of papers, such as Henderson [8] Monoyios [22, 23] and Musiela and Zariphopoulou [27], have used exponential indifference valuation methods to hedge the claim in an optimal manner in a full information scenario. We outline these results before moving on to the partial information case, where we assume the hedger does not know with certainty the drifts of S and Y. Analytic approximations for prices and hedging strategies are given. Further work on this topic can be found in Monoyios [24, 26].

2 Filtering theory

Filtering problems concern estimating (in a manner to be made precise shortly) something about an unobserved stochastic process Ξ given observations of a related process Λ. In particular, one seeks the conditional expectation $E[\Xi_t | \widehat{\mathcal{F}}_t], 0 \leq t \leq T$, where $\widehat{\mathbb{F}} := (\widehat{\mathcal{F}}_t)_{0 \leq t \leq T}$ is the filtration generated by Λ. This problem was solved for linear systems in continuous time by Kalman and Bucy [11]. Subsequent work sought generalisations to systems with nonlinear dynamics, see Zakai [33] for instance. Kailath [10] developed the so-called innovations approach to linear filtering, which formulated the problem in the context of martingale theory. This approach to nonlinear filtering was given a definitive treatment by Fujisaki, Kallianpur and Kunita [7]. Textbook treatments can be found Kallianpur [12], Lipster and Shiryaev [17, 18] and Rogers and Williams [32], Chapter VI.8, whose treatment is closest to the one used below, and which follows the program of Fujisaki, Kallianpur and Kunita [7].

The setting is a probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$. All processes are assumed to be $\mathbb{F}$-adapted. Note that $\mathbb{F}$ is not the observation filtration. Let us call $\mathbb{F}$ the *background filtration*. We consider two processes, both taken to be one-dimensional (for simplicity):

- a *signal process* $\Xi = (\Xi_t)_{0 \leq t \leq T}$ which is not directly observable;

- an *observation process* $\Lambda = (\Lambda_t)_{0 \leq t \leq T}$, which is observable and somehow correlated with Ξ, so that by observing Λ we can say something about the distribution of Ξ.

Let $\widehat{\mathbb{F}} := (\widehat{\mathcal{F}}_t)_{0 \leq t \leq T}$ denote the observation filtration generated by Λ. That is,

$$\widehat{\mathcal{F}}_t := \sigma(\Lambda_s; 0 \leq s \leq t), \quad 0 \leq t \leq T.$$

The *filtering problem* is to *compute the conditional distribution* of the signal $\Xi_t, t \in [0, T]$, given observations up to that time. Or, equivalently, to compute the conditional expectation

$$E[f(\Xi_t) | \widehat{\mathcal{F}}_t], \quad 0 \leq t \leq T,$$

where $f : \mathbb{R} \to \mathbb{R}$ is some test function.

To proceed further, we need to specify some particular model for the observation process (followed later by more structure on the signal process).

2.1 Observation model

Let $B = (B_t)_{0 \leq t \leq T}$ be an $\mathbb{F}$-Brownian motion, let $H = (H_t)_{0 \leq t \leq T}$ be an $\mathbb{F}$-adapted process satisfying

$$E \int_0^T H_t^2 dt < \infty, \tag{2}$$

and we shall assume the observation process Λ is of the form

$$\Lambda_t = \int_0^t H_s ds + B_t, \quad 0 \leq t \leq T, \tag{3}$$

The typical situation will be where $H_t = h(t, \Xi_t)$, a deterministic function of time and the current signal value. In general, H and Ξ will be suitably correlated with each other and with the process B. A specialised situation (that we shall later focus on) is the *linear* case when $h(t,x) = G(t)x$, with $G(\cdot)$ a deterministic function. Then $H_t = G(t)\Xi_t$ and the observation process stochastic differential equation (SDE) is

$$d\Lambda_t = G(t)\Xi_t dt + dB_t, \quad \Lambda_0 = 0, \quad \text{(linear observation model)}.$$

2.2 Innovations process

Introduce the notation

$$\widehat{\phi}_t := E[\phi_t | \widehat{\mathcal{F}}_t], \quad 0 \leq t \leq T,$$

for any process ϕ. Define the $\widehat{\mathbb{F}}$-adapted *innovations process*

$$N_t := \Lambda_t - \int_0^t \widehat{H}_s ds, \quad 0 \leq t \leq T. \tag{4}$$

Proposition 1. *The innovations process N is an $\widehat{\mathbb{F}}$-Brownian motion.*

Proof. From (3) and (4) we have

$$N_t = \int_0^t (H_s - \widehat{H}_s) ds + B_t.$$

With $s \leq t$, we have

$$
\begin{aligned}
E[N_t | \widehat{\mathcal{F}}_s] - N_s &= E\left[\left. \int_s^t (H_u - \widehat{H}_u) du + B_t - B_s \right| \widehat{\mathcal{F}}_s \right] \\
&= E\left[\left. \int_s^t \left(E[H_u | \widehat{\mathcal{F}}_u] - \widehat{H}_u \right) du \right| \widehat{\mathcal{F}}_s \right] + E\left[E(B_t - B_s | \mathcal{F}_s) | \widehat{\mathcal{F}}_s \right] = 0,
\end{aligned}
$$

using the Tower property of conditional expectation. So N is is continuous $\widehat{\mathbb{F}}$-martingale with quadratic variation $[N]_t = [B]_t = t$, so N is an $\widehat{\mathbb{F}}$-Brownian motion. $\qquad \square$

2.2.1 The Innovations Conjecture

Denote by $\mathbb{F}^N := (\mathcal{F}_t^N)_{0 \leq t \leq T}$ the filtration generated by N, so $\mathcal{F}_t^N := \sigma(N_s; 0 \leq s \leq t)$. Since N is $\widehat{\mathbb{F}}$-adapted, we have $(\mathcal{F}_t^N) \subseteq (\widehat{\mathcal{F}}_t)$. For *linear* systems, we shall see that in fact we have

$$\mathcal{F}_t^N = \widehat{\mathcal{F}}_t, \quad 0 \leq t \leq T, \tag{5}$$

so that in this case the observations and the innovations represent the same information (because there is an invertible map that derives one from the other). The "innovations conjecture" (of Kailath) is that the identity (5) holds in general, but we now know that this is *not* the case (though it is true when H and B are independent [1]). However, the following positive and very useful result *is* true.

Theorem 1. *Every local $\widehat{\mathbb{F}}$-martingale M admits a representation of the form*

$$M_t = M_0 + \int_0^t \Phi_s dN_s, \quad 0 \le t \le T,$$

where Φ is $\widehat{\mathbb{F}}$-adapted and $\int_0^T \Phi_t^2 dt < \infty$ a.s. If M happens to be a square-integrable martingale, then Φ can be chosen so that $E \int_0^T \Phi_t^2 dt < \infty$.

To prove this result, we shall use the following well-known result on representation of local martingales with respect to a Brownian filtration. See Theorems 3.4.2 and 3.4.15 in Karatzas and Shreve [15].

Theorem 2 (Local martingale representation)**.** *Let W be a Brownian motion and let $\mathbb{F}^W$ denote its natural filtration. Every local martingale M with respect to $\mathbb{F}^W$ admits a representation of the form*

$$M_t = M_0 + \int_0^t b_s dW_s, \quad 0 \le t < \infty,$$

for an $\mathbb{F}^W$-adapted process b satisfying $\int_0^t b_s^2 ds < \infty$ almost surely for every $0 < t < \infty$. In particular, every such M has continuous sample paths. If M happens to be a square-integrable martingale ($EM_t^2 < \infty, \forall t \ge 0$), then b can be chosen so that $E \int_0^t b_s^2 ds < \infty$ for every $0 < t < \infty$.

Note that, if only the innovations conjecture (5) were true in general, then Theorem 1 would follow directly from Theorem 2. As the innovations conjecture is not true in general, we shall prove the theorem by performing a measure change that turns Λ into a Brownian motion, then apply Theorem 2, then invert the change of measure to revert back to the innovations process N.

Proof of Theorem 1. We carry this out only in the case of bounded H, to present the ideas with the minimum of technical fuss. We make some remarks on how to deal with the non-bounded case after the proof.

If H is bounded, then so is $\widehat{H}$, so the process

$$Z_t := \mathcal{E}(-\widehat{H} \cdot N)_t = \exp\left(-\int_0^t \widehat{H}_s dN_s - \frac{1}{2} \int_0^t \widehat{H}_s^2 ds\right), \quad 0 \le t \le T, \tag{6}$$

is a $(P, \widehat{\mathbb{F}})$-martingale. By the Girsanov Theorem, since N is a $(P, \widehat{\mathbb{F}})$-Brownian motion, the process

$$N_t + \int_0^t \widehat{H}_s ds = \Lambda_t, \quad 0 \le t \le T,$$

that is, the observation process, is a $(\widetilde{P}, \widehat{\mathbb{F}})$-Brownian motion, where the probability measure $\widetilde{P}$ is defined on $(\Omega, \widehat{\mathbb{F}})$ by

$$\left.\frac{d\widetilde{P}}{dP}\right|_{\widehat{\mathcal{F}}_t} = Z_t, \quad 0 \le t \le T.$$

Notice that the inverse likelihood ratio is

$$\begin{aligned}
\left.\frac{dP}{d\widetilde{P}}\right|_{\widehat{\mathcal{F}}_t} = \Gamma_t := \frac{1}{Z_t} &= \exp\left(\int_0^t \widehat{H}_s dN_s + \frac{1}{2} \int_0^t \widehat{H}_s^2 ds\right) \\
&= \exp\left(\int_0^t \widehat{H}_s d\Lambda_s - \frac{1}{2} \int_0^t \widehat{H}_s^2 ds\right) \\
&= \mathcal{E}(\widehat{H} \cdot \Lambda)_t.
\end{aligned}$$

The SDEs for Z, Γ are therefore

$$Z_t = 1 - \int_0^t Z_s \widehat{H}_s dN_s, \quad \Gamma_t = 1 + \int_0^t \Gamma_s \widehat{H}_s d\Lambda_s. \tag{7}$$

Using the so-called *Bayes rule*, for $s \leq t$ and an $\widehat{\mathcal{F}}_t$-measurable random variable X:

$$\widetilde{E}[X|\widehat{\mathcal{F}}_s] = \frac{1}{Z_s}E[Z_t X|\widehat{\mathcal{F}}_s],$$

we find that, since M is a local $(P,\widehat{\mathbb{F}})$-martingale, then ΓM is a local $(\widetilde{P},\widehat{\mathbb{F}})$-martingale, as we now show. With $(\tau_n)_{n=1}^{\infty}$ a localising sequence[1] then $(M_t^{(n)}) := (M_{t\wedge\tau_n})$ is a $(P,\widehat{\mathbb{F}})$-martingale. So, with $s \leq t$, we have

$$\widetilde{E}[\Gamma_t M_t^{(n)}|\widehat{\mathcal{F}}_s] = \frac{1}{Z_s}E[Z_t\Gamma_t M_t^{(n)}|\widehat{\mathcal{F}}_s] = \Gamma_s E[M_t^{(n)}|\widehat{\mathcal{F}}_s] = \Gamma_s M_s^{(n)},$$

so that ΓM is a local $(\widetilde{P},\widehat{\mathbb{F}})$-martingale, as claimed.

An application of the martingale representation theorem, Theorem 2, gives a representation for ΓM of the form

$$\Gamma_t M_t = \Gamma_0 M_0 + \int_0^t \Psi_s d\Lambda_s = \Gamma_0 M_0 + \int_0^t \Psi_s(dN_s + \widehat{H}_s ds), \tag{8}$$

for some process Ψ satisfying $\int_0^T \Psi_t^2 dt < \infty$ a.s. Now from (8), (7) and the integration by parts formula,[2] we obtain

$$
\begin{aligned}
M_t &= (\Gamma_t M_t)Z_t \\
&= (\Gamma_0 M_0)Z_0 + \int_0^t (\Gamma_s M_s)dZ_s + \int_0^t Z_s d(\Gamma_s M_s) + [\Gamma M, Z]_t \\
&= M_0 + \int_0^t \Gamma_s M_s(-Z_s\widehat{H}_s dN_s) + \int_0^t Z_s\Psi_s(dN_s + \widehat{H}_s ds) - \int_0^t Z_s\widehat{H}_s\Psi_s d[N]_s \\
&= M_0 + \int_0^t (Z_s\Psi_s - M_s\widehat{H}_s)dN_s \\
&= M_0 + \int_0^t \Phi_s dN_s,
\end{aligned}
$$

for $\Phi = Z\Psi - M\widehat{H}$.

$\square$

Remark 1 (If H is not necessarily bounded). An examination of the above proof shows that the boundedness of H (and hence of $\widehat{H}$) was needed so that the exponential local martingale Z was actually a martingale, and so could be used to define the equivalent probability measure $\widetilde{P}$. If H is not bounded, and merely satisfies the integrability condition (2), then a little more care is needed in defining the relevant equivalent probability measure. Here is an outline of the procedure. See Rogers and Williams [32], Chapter VI.8, for more details.

One first fixes $n \in \mathbb{N}$ and defines the $\widehat{\mathbb{F}}$ stopping time

$$T_n := \inf\left\{t \geq 0 : \left|\int_0^t \widehat{H}_s dN_s\right| + \int_0^t \widehat{H}_s^2 ds = n\right\} \wedge T.$$

(Notice that the sequence $(T_n)_{n=1}^{\infty} \to T$ as $n \to \infty$.) Then with Z defined by (6), we have that the stopped process

$$Z_t^{(n)} := Z_{t\wedge T_n}, \quad 0 \leq t \leq T_n,$$

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[1] A localising sequence is defined as follows. Given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,\infty)}, P)$, an adapted process $(X_t, \mathcal{F}_t)_{t\in[0,\infty)}$ is a *local martingale* if there exists an increasing sequence $(\tau_n)_{n=1}^{\infty}$ of $(\mathcal{F}_t)$-stopping times with $\lim_{n\to\infty} \tau_n = \infty$ a.s. such that the *stopped process* $(X_t^{(n)}, \mathcal{F}_t)_{t\in[0,\infty)}$, defined by $X_t^{(n)} := X_{t\wedge\tau_n}$, is a martingale for all $n \geq 1$. The sequence $(\tau_n)_{n=1}^{\infty}$ is called a localising sequence.

[2] For any two processes X, Y,

$$X_t Y_t = X_0 Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + [X, Y]_t.$$

is a positive $(P, \widehat{\mathbb{F}})$-martingale so can be used to define an equivalent probability measure $\widetilde{P}^{(n)}$ on $(\Omega, \widehat{\mathcal{F}}_{T_n})$ by

$$\frac{d\widetilde{P}^{(n)}}{dP} = Z_{T_n}^{(n)}.$$

One now applies the Girsanov Theorem to deduce that the process

$$\Lambda_t^{(n)} := N_t + \int_0^{t \wedge T_n} \widehat{H}_s ds, \quad 0 \le t \le T_n,$$

is a $(\widetilde{P}^{(n)}, \widehat{\mathbb{F}})$-Brownian motion. Defining the inverse likelihood ratio

$$\Gamma_t^{(n)} := \left. \frac{dP}{d\widetilde{P}^{(n)}} \right|_{\widehat{\mathcal{F}}_t}, \quad 0 \le t \le T_n,$$

one shows that $(\Gamma_t^{(n)} M_t)_{0 \le t \le T_n}$ is a $(\widetilde{P}^{(n)}, \widehat{\mathbb{F}})$-local martingale, and consequently that it has a stochastic integral representation with respect to $\Lambda^{(n)}$, and hence that $(M_t)_{0 \le t \le T_n}$ has a stochastic integral representation with respect to N. This procedure works for each fixed $n \in \mathbb{N}$. Letting $n \to \infty$ completes the proof.

To proceed further, we now need yet more structure, this time on the signal process Ξ.

2.3 Signal process model

We take the signal process to be of the form

$$\Xi_t = \Xi_0 + \int_0^t b(s, \Xi_s) ds + \int_0^t \sigma(s, \Xi_s) dW_s, \quad 0 \le t \le T, \tag{9}$$

where W is a $(P, \mathbb{F})$-Brownian motion independent of the $\mathcal{F}_0$-measurable random variable Ξ_0, and correlated with B in the observation model (3) according to

$$[W, B]_t = \rho t, \quad 0 \le t \le T, \quad \rho \in [-1, 1].$$

The functions $b : [0, T] \times \mathbb{R} \to \mathbb{R}$ and $\sigma : [0, T] \times \mathbb{R} \to \mathbb{R}$ are assumed to satisfy the Lipschitz and linear growth conditions

$$|b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| \le K|x - y|, \quad \forall x, y \in \mathbb{R},$$
$$|b(t, x)| + |\sigma(t, x)| \le K(1 + |x|), \quad \forall x \in \mathbb{R},$$

for some real $K > 0$. Then there exists a unique process Ξ that satisfies (9).

Let $f \in C_0^2(\mathbb{R})$ be a twice continuously differentiable function with compact support[3]. The generator of Ξ is $\mathcal{A}_t$ given by

$$\mathcal{A}_t f(x) = b(t, x) f'(x) + \frac{1}{2} \sigma^2(t, x) f''(x).$$

For brevity we use the notation

$$f_t \equiv f(\Xi_t), \quad (\mathcal{A}f)_t \equiv \mathcal{A}_t f(\Xi_t), \quad 0 \le t \le T.$$

By Itô's formula we have that

$$M_t^f := f_t - f_0 - \int_0^t (\mathcal{A}f)_s ds = \int_0^t \sigma(s, \Xi_s) f'(\Xi_s) dW_s \tag{10}$$

is an $\mathbb{F}$-local martingale, and in fact a square-integrable martingale if f is of compact support. We assume this is the case. The cross-variation of M^f with B is

$$[M^f, B]_t = \int_0^t \sigma(s, \Xi_s) f'(\Xi_s) d[W, B]_s = \int_0^t \rho \sigma(s, \Xi_s) f'(\Xi_s) ds =: \int_0^t \alpha_s^f ds, \tag{11}$$

where we have defined the process α^f by

$$\alpha_t^f := \rho \sigma(t, \Xi_t) f'(\Xi_t), \quad 0 \le t \le T.$$

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[3]So f is zero outside of a compact set.

2.4 Fundamental filtering equation

The fundamental filtering theorem is the following.

Theorem 3. *For the observation and signal process models of (3) and (9) we have, for every $f \in C_0^2(\mathbb{R})$, and with $f_t \equiv f(\Xi_t)$, using the notation $\widehat{\phi}_t := E[\phi_t|\widehat{\mathcal{F}}_t]$ for any process ϕ, the fundamental filtering equation*

$$\widehat{f}_t = \widehat{f}_0 + \int_0^t (\widehat{\mathcal{A}f})_s ds + \int_0^t \left(\widehat{f_s H_s} - \widehat{f}_s \widehat{H}_s + \widehat{\alpha}_s^f \right) dN_s, \quad 0 \le t \le T. \tag{12}$$

The proof will require the following lemma.

Lemma 1. *Consider two $\mathbb{F}$-adapted processes P, C with $E|P_t| < \infty, \forall t \in [0,T]$ and $E \int_0^T |C_t| dt < \infty$. If $J_t := P_t - \int_0^t C_s ds$ is an $\mathbb{F}$-martingale, then*

$$\widehat{J}_t := \widehat{P}_t - \int_0^t \widehat{C}_s ds \quad \text{is an } \widehat{\mathbb{F}}\text{-martingale.}$$

Proof. For $s \le t$, writing $\int_0^t C_u du = \int_0^s C_u du + \int_s^t C_u du$ and using the fact that J is an $\mathbb{F}$-martingale, we have

$$E\left[P_t - \int_0^s C_u du - \int_s^t C_u du \,\Big|\, \mathcal{F}_s \right] = P_s - \int_0^s C_u du$$

$$\Rightarrow \quad E\left[P_t - \int_s^t C_u du \,\Big|\, \mathcal{F}_s \right] = P_s, \tag{13}$$

which we shall use shortly.

Now consider

$$
\begin{aligned}
E[\widehat{J}_t|\widehat{\mathcal{F}}_s] &= E\left[\widehat{P}_t - \int_0^t \widehat{C}_u du \,\Big|\, \widehat{\mathcal{F}}_s \right] \\
&= E\left[\widehat{P}_t - \int_0^s \widehat{C}_u du - \int_s^t \widehat{C}_u du \,\Big|\, \widehat{\mathcal{F}}_s \right] \\
&= E\left[\widehat{P}_t - \int_s^t \widehat{C}_u du \,\Big|\, \widehat{\mathcal{F}}_s \right] - \int_0^s \widehat{C}_u du \\
&= E\left[E[P_t|\widehat{\mathcal{F}}_t] - \int_s^t E[C_u|\widehat{\mathcal{F}}_u] du \,\Big|\, \widehat{\mathcal{F}}_s \right] - \int_0^s \widehat{C}_u du \\
&= E[P_t|\widehat{\mathcal{F}}_s] - \int_s^t E[C_u|\widehat{\mathcal{F}}_s] du - \int_0^s \widehat{C}_u du \\
&= E[E[P_t|\mathcal{F}_s]\widehat{\mathcal{F}}_s] - \int_s^t E[E[C_u|\mathcal{F}_s]\widehat{\mathcal{F}}_s] du - \int_0^s \widehat{C}_u du \\
&= E\left[E\left[P_t - \int_s^t C_u du \,\Big|\, \mathcal{F}_s \right] \Big|\, \widehat{\mathcal{F}}_s \right] - \int_0^s \widehat{C}_u du \\
&= E[P_s|\widehat{\mathcal{F}}_s] - \int_0^s \widehat{C}_u du \quad \text{(using (13))} \\
&= \widehat{P}_s - \int_s^t \widehat{C}_u du \\
&= \widehat{J}_s.
\end{aligned}
$$

$\square$

Proof of Theorem 3. Recall from (10) that

$$f_t = f_0 + \int_0^t (\mathcal{A}f)_s ds + M_t^f, \tag{14}$$

8

where $M_t^f = \int_0^t \sigma(\Xi_s)f'(\Xi_s)dW_s$ is an $\mathbb{F}$-martingale. So, using Lemma 1 and Theorem 1 we have

$$\widehat{M}_t^f := \widehat{f}_t - \widehat{f}_0 - \int_0^t (\widehat{\mathcal{A}f})_s ds = \widehat{\mathbb{F}}\text{-martingale} =: \int_0^t \Phi_s dN_s, \qquad (15)$$

for a suitable $\widehat{\mathbb{F}}$-adapted process Φ such that $E\int_0^T \Phi_t^2 dt < \infty$. We want to compute Φ, to establish that

$$\Phi_t = \widehat{f_t H_t} - \widehat{f}_t \widehat{H}_t + \widehat{\alpha}_t^f, \quad 0 \le t \le T. \qquad (16)$$

This will be accomplished by computing $E[f_t\Lambda_t|\widehat{\mathcal{F}}_t] = \widehat{f}_t\Lambda_t$ in two ways and comparing the results.

On the one hand, using (14), (3), (11) and the integration by parts formula,

$$\begin{aligned}
f_t\Lambda_t &= f_0\Lambda_0 + \int_0^t f_s d\Lambda_s + \int_0^t \Lambda_s df_s + [f, \Lambda]_t \\
&= \int_0^t f_s(H_s ds + dB_s) + \int_0^t \Lambda_s\left((\mathcal{A}f)_s ds + dM_s^f\right) + [M^f, B]_t \\
&= \int_0^t \left(f_s H_s + \Lambda_s(\mathcal{A}f)_s + \alpha_s^f\right) ds + \mathbb{F}\text{-martingale.}
\end{aligned}$$

So by Lemma 1,

$$\widehat{f_t\Lambda_t} = \widehat{f}_t\Lambda_t = \int_0^t \left(\widehat{f_s H_s} + \Lambda_s(\widehat{\mathcal{A}f})_s + \widehat{\alpha}_s^f\right) ds + \widehat{\mathbb{F}}\text{-martingale.} \qquad (17)$$

On the other hand, from (15), (4) and the integration by parts formula, we obtain

$$\begin{aligned}
\widehat{f}_t\Lambda_t &= \widehat{f}_0\Lambda_0 + \int_0^t \widehat{f}_s d\Lambda_s + \int_0^t \Lambda_s d\widehat{f}_s + [\widehat{f}, \Lambda]_t \\
&= \int_0^t \widehat{f}_s(\widehat{H}_s ds + dN_s) + \int_0^t \Lambda_s\left((\widehat{\mathcal{A}f})_s ds + \Phi_s dN_s\right) + \int_0^t \Phi_s ds \\
&= \int_0^t \left(\widehat{f}_s\widehat{H}_s + \Lambda_s(\widehat{\mathcal{A}f})_s + \Phi_s\right) ds + \widehat{\mathbb{F}}\text{-martingale.}
\end{aligned} \qquad (18)$$

Comparing (17) and (18), the difference between the bounded variation parts is a continuous martingale of bounded variation, so is constant, and is null at zero, so is identically zero, and therefore (16) holds.

$\square$

2.4.1 Linear observations

Take $H_t = h(t, \Xi_t) = G(t)\Xi_t$ and $f(x) = x^k, k = 1, 2, \ldots$. Then we obtain from (12):

$$\widehat{\Xi}_t = \widehat{\Xi}_0 + \int_0^t \widehat{b(s, \Xi_s)}ds + \int_0^t \left[G(s)\left(\widehat{\Xi_s^2} - (\widehat{\Xi}_s)^2\right) + \rho\widehat{\sigma(s, \Xi_s)}\right] dN_s, \quad (k = 1), \qquad (19)$$

$$\begin{aligned}
\widehat{\Xi_t^k} &= \widehat{\Xi_0^k} + k\int_0^t \left(\widehat{b(s, \Xi_s)\Xi_s^{k-1}} + \frac{1}{2}(k-1)\widehat{\sigma^2(s, \Xi_s)\Xi_s^{k-2}}\right) ds \\
&\quad + \int_0^t \left[G(s)\left(\widehat{\Xi_s^{k+1}} - \widehat{\Xi}_s\widehat{\Xi_s^k}\right) + k\rho\widehat{\sigma(s, \Xi_s)\Xi_s^{k-1}}\right] dN_s, \quad k = 2, 3, \ldots.
\end{aligned} \qquad (20)$$

Equations (19) and (20) convey the complexity of non-linear filtering. To solve the equation for the kth conditional moment, one needs to know the $(k+1)^{\text{th}}$ conditional moment as well as $E[g(s, \Xi_s)|\widehat{\mathcal{F}}_s]$ for $g(s, x) = b(s, x)x^{k-1}$, $g(s, x) = \sigma^2(s, x)x^{k-2}$, $g(s, x) = \sigma(s, x)x^{k-1}$. This means the computation of conditional moments cannot be achieved by induction on k and the problem is inherently infinite dimensional except in the linear case.

2.4.2 Linear observations and linear signal

Now take $h(t,x) = G(t)x$ as before, and $b(t,x) = A(t)x$, $\sigma(t,x) = C(t)$, for deterministic functions $A(\cdot), C(\cdot)$, and suppose that the signal process has a Gaussian initial distribution. Hence the signal and observation processes follow

$$
\begin{aligned}
d\Xi_t &= A(t)\Xi_t dt + C(t)dW_t, \quad \Xi_0 \sim \mathrm{N}(\mu, \mathrm{v}), \\
d\Lambda_t &= G(t)\Xi_t dt + dB_t, \quad \Lambda_0 = 0,
\end{aligned}
$$

with Ξ_0 independent of B and of W, and where $\mathrm{N}(\mu, \mathrm{v})$ denotes the normal probability law with mean μ and variance v. The two-dimensional process (Ξ, Λ) is then Gaussian, so the conditional distribution of Ξ_t given the sigma-field $\widehat{\mathcal{F}}_t = \sigma(\Lambda_s; 0 \le s \le t)$ generated by the Λ will also be normal (and so, in particular, is completely characterised by its mean and variance), with mean

$$
\widehat{\Xi}_t := E[\Xi_t | \widehat{\mathcal{F}}_t]
$$

and variance

$$
V_t := \mathrm{var}[\Xi_t | \widehat{\mathcal{F}}_t] = E[(\Xi_t - \widehat{\Xi}_t)^2 | \widehat{\mathcal{F}}_t] = \widehat{\Xi_t^2} - \left(\widehat{\Xi}_t\right)^2.
$$

Notice that the initial values are

$$
\widehat{\Xi}_0 = E[\Xi_0 | \widehat{\mathcal{F}}_0] = E\Xi_0 = \mu,
$$

and

$$
V_0 = E[(\Xi_0 - \widehat{\Xi}_0)^2 | \widehat{\mathcal{F}}_0] = E[(\Xi_0 - \mu)^2] = \mathrm{var}(\Xi_0) = \mathrm{v}.
$$

The problem then boils down to finding an algorithm for computing the sufficient statistics $\widehat{\Xi}_t, V_t$ from their initial values $\widehat{\Xi}_0 = \mu$, $V_0 = \mathrm{v}$.

Now, from (19) we obtain, along with the initial condition $\widehat{\Xi}_0 = \mu$, the SDE

$$
d\widehat{\Xi}_t = A(t)\widehat{\Xi}_t dt + [G(t)V_t + \rho C(t)] dN_t, \quad \widehat{\Xi}_0 = \mu. \tag{21}
$$

From (20) with $k = 2$ we obtain

$$
d\widehat{\Xi_t^2} = \left(C^2(t) + 2A(t)\widehat{\Xi_t^2}\right) dt + \left[G(t)\left(\widehat{\Xi_t^3} - \widehat{\Xi}_t \widehat{\Xi_t^2}\right) + 2\rho C(t)\widehat{\Xi}_t\right] dN_t, \quad \widehat{\Xi_0^2} - \mu^2 = \mathrm{v}. \tag{22}
$$

But for a normal random variable $X \sim \mathrm{N}(m, s^2)$, we have

$$
E[X^3] = m(m^2 + 3s^2),
$$

whence

$$
\widehat{\Xi_t^3} = E[\Xi_t^3 | \widehat{\mathcal{F}}_t] = E[\Xi_t | \widehat{\mathcal{F}}_t]\left(\left(E[\Xi_t | \widehat{\mathcal{F}}_t]\right)^2 + 3\mathrm{var}[\Xi_t | \widehat{\mathcal{F}}_t]\right) = \widehat{\Xi}_t \left[\left(\widehat{\Xi}_t\right)^2 + 3V_t\right],
$$

and therefore

$$
\widehat{\Xi_t^3} - \widehat{\Xi}_t \widehat{\Xi_t^2} = \widehat{\Xi}_t \left[\left(\widehat{\Xi}_t\right)^2 + 3V_t - \widehat{\Xi_t^2}\right] = 2V_t \widehat{\Xi}_t.
$$

Using this, (21), (22) and the Itô formula, we obtain

$$
\begin{aligned}
dV_t &= d\left[\widehat{\Xi_t^2} - \left(\widehat{\Xi}_t\right)^2\right] \\
&= \left(C^2(t) + 2A(t)\widehat{\Xi_t^2}\right) dt + \left[G(t)\left(2V_t \widehat{\Xi}_t\right) + 2\rho C(t)\widehat{\Xi}_t\right] dN_t - 2\widehat{\Xi}_t d\widehat{\Xi}_t - d[\widehat{\Xi}]_t,
\end{aligned}
$$

which simplifies to the *non-stochastic* Riccati equation

$$
\frac{dV_t}{dt} = (1 - \rho^2)C^2 + 2\left(A(t) - \rho C(t)G(t)\right) V_t = G^2(t)V_t^2, \quad V_0 = \mathrm{v}. \tag{23}
$$

In other words, the conditional variance V_t is a *deterministic* function of t, and given by the solution of (23). Thus, there is in fact only one sufficient statistic, the conditional mean $\widehat{\Xi}_t$ which satisfies the linear SDE (21), which is the celebrated *Kalman-Bucy filter*. We summarise all this below.

Theorem 4 (One-dimensional Kalman-Bucy filter). *On a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, with $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$, let $\Xi = (\Xi_t)_{0 \leq t \leq T}$ be an $\mathbb{F}$-adapted signal process satisfying*

$$d\Xi_t = A(t)\Xi_t dt + C(t)dW_t,$$

and let $\Lambda = (\Lambda_t)_{0 \leq t \leq T}$ be an $\mathbb{F}$-adapted observation process satisfying

$$d\Lambda_t = G(t)\Xi_t dt + dB_t, \quad \Lambda_0 = 0,$$

where W, B are $\mathbb{F}$-Brownian motions with correlation ρ, and the coefficients $A(\cdot), C(\cdot), G(\cdot)$ are deterministic functions satisfying

$$\int_0^T \left(|A(t)| + C^2(t) + G^2(t) \right) dt < \infty.$$

Define the observation filtration $\widehat{\mathbb{F}} := (\widehat{\mathcal{F}}_t)_{0 \leq t \leq T}$ by

$$\widehat{\mathcal{F}}_t := \sigma(\Lambda_s; 0 \leq s \leq t).$$

Suppose Ξ_0 is an $\mathcal{F}_0$-measurable random variable, and that the distribution of Ξ_0 is Gaussian with mean μ and variance v, independent of W and B. Then the conditional expectation $\widehat{\Xi}_t := E[\Xi_t | \widehat{\mathcal{F}}_t]$ for $0 \leq t \leq T$ satisfies

$$d\widehat{\Xi}_t = A(t)\widehat{\Xi}_t dt + [G(t)V_t + \rho C(t)] \, dN_t, \quad \widehat{\Xi}_0 = \mu,$$

where $N = (N_t)_{0 \leq t \leq T}$ is the innovations process, an $\widehat{\mathbb{F}}$-Brownian motion satisfying

$$dN_t = d\Lambda_t - G(t)\widehat{\Xi}_t dt,$$

and $V_t = \mathrm{var}[\Xi_t | \widehat{\mathcal{F}}_t]$, for $0 \leq t \leq T$, is the conditional variance, which is independent of $\widehat{\mathcal{F}}_t$ and satisfies the deterministic Riccati equation

$$\frac{dV_t}{dt} = (1 - \rho^2)C^2(t) + 2 \left[A(t) - \rho C(t)G(t) \right] V_t - G^2(t)V_t^2, \quad V_0 = v.$$

Remark 2 (Validity of innovations conjecture for linear systems). It is now straightforward to confirm the validity of the innovations conjecture $\mathcal{F}_t^N = \widehat{\mathcal{F}}_t, 0 \leq t \leq T$, for linear systems. The solution $\widehat{\Xi}$ of the SDE (21) is adapted to the filtration $\mathbb{F}^N$ of the driving Brownian motion N, so $(\mathcal{F}_t^{\widehat{\Xi}}) \subseteq (\mathcal{F}_t^N)$, where $\mathcal{F}_t^{\widehat{\Xi}} = \sigma(\widehat{\Xi}_s; 0 \leq s \leq t)$. So from the relation $\Lambda_t = N_t + \int_0^s G(s)\widehat{\Xi}_s$ we see that Λ is $\mathbb{F}^N$-adapted, i.e. $(\widehat{\mathcal{F}}_t) \subseteq (\mathcal{F}_t^N)$. Because the reverse inclusion $(\mathcal{F}_t^N) \subseteq (\widehat{\mathcal{F}}_t)$ always holds, the two filtrations are the same.

2.5 Multi-dimensional Kalman-Bucy filter

A multi-dimensional version of the Kalman-Bucy filter can be derived along similar lines to the one-dimensional case. We state the result below.

Theorem 5. *Consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, with $\mathbb{F} = (\mathcal{F})_{0 \leq t \leq T}$, and two $\mathbb{F}$-adapted processes Ξ, Λ as given below.*

Let $\Xi = (\Xi_t)_{0 \leq t \leq T}$ be an n-dimensional signal process satisfying

$$d\Xi_t = A(t)\Xi_t dt + C(t)dW_t, \quad \Xi_0 \sim \mathrm{N}(\mu, v), \quad \textit{(linear signal)}, \tag{24}$$

where $\Xi_0 \sim \mathrm{N}(\mu, v)$ denotes an n-dimensional $\mathcal{F}_0$-measurable Gaussian vector with mean $\mu \in \mathbb{R}^n$ and covariance matrix $v \in \mathbb{R}^n \times \mathbb{R}^n$, independent of the d-dimensional Brownian motion W, and where $A(t) \in \mathbb{R}^n \times \mathbb{R}^n$, $C(t) \in \mathbb{R}^n \times \mathbb{R}^d$.

Let $\Lambda = (\Lambda_t)_{0 \leq t \leq T}$ be an m-dimensional observation process satisfying

$$d\Lambda_t = G(t)\Xi_t dt + D(t)dB_t, \quad \Lambda_0 = 0, \quad \textit{(linear observations)},$$

where $G(t) \in \mathbb{R}^m \times \mathbb{R}^n$, $C(t) \in \mathbb{R}^m \times \mathbb{R}^k$, and B is a k-dimensional Brownian motion independent of W and Ξ_0.

We assume that A, C, G, D are bounded on bounded intervals, that DD^T is non-singular, and that $(D(t)D(t)^T)^{-1}$ is bounded on every bounded t-interval.

Let $\widehat{\mathbb{F}} = (\widehat{\mathcal{F}}_t)_{0 \le t \le T}$ denote the observation filtration generated by Λ, so that $\widehat{\mathcal{F}}_t = \sigma(\Lambda_s; 0 \le s \le t)$.

The conditional expectation vector $\widehat{\Xi}_t := E[\Xi_t | \widehat{\mathcal{F}}_t], 0 \le t \le T$, satisfies the SDE

$$
\begin{aligned}
d\widehat{\Xi}_t &= A(t)\widehat{\Xi}_t dt + V_t G^T(t) \left(D(t)D^T(t) \right)^{-1} \left(d\Lambda_t - G(t)\widehat{\Xi}_t dt \right), \quad \widehat{\Xi}_0 = \mu, \\
&= A(t)\widehat{\Xi}_t dt + V_t G^T(t) \left(D(t)D^T(t) \right)^{-1} dN_t, \quad \widehat{\Xi}_0 = \mu,
\end{aligned}
\tag{25}
$$

where N is the innovations process, defined by

$$
N_t := \Lambda_t - \int_0^t G(s)\widehat{\Xi}_s ds, \quad 0 \le t \le T,
$$

and satisfying

$$
N_t = \int_0^t D(s)d\widehat{B}_s,
\tag{26}
$$

where $\widehat{B}$ is a standard k-dimensional $\widehat{\mathbb{F}}$-Brownian motion.

The error $\Xi_t - \widehat{\Xi}_t$ is independent of $\widehat{\mathcal{F}}_t$, and the error covariance

$$
V_t := E[(\Xi_t - \widehat{\Xi}_t)(\Xi_t - \widehat{\Xi}_t)^T | \widehat{\mathcal{F}}_t] = \text{var}[\Xi_t | \widehat{\mathcal{F}}_t],
$$

satisfies the deterministic matrix Riccati equation

$$
\frac{dV_t}{dt} = A(t)V_t + V_t A^T(t) - V_t G^T(t)(D(t)D^T(t))^{-1}G(t)V_t + C(t)C^T(t), \quad V_0 = \text{v}.
$$

Notice that:

- by (26) we can rewrite (25) as

$$
d\widehat{\Xi}_t = A(t)\widehat{\Xi}_t dt + V_t G^T(t) \left(D(t)D^T(t) \right)^{-1} D(t)d\widehat{B}_t, \quad \widehat{\Xi}_0 = \mu,
$$

which is a linear SDE of the same type as (24);

- since $\Xi, \widehat{\Xi}$ satisfy (24), (25) and Ξ_0 is Gaussian, then $\Xi_t, \widehat{\Xi}_t$ are Gaussian vectors for each t, and the error $\Xi_t - \widehat{\Xi}_t$ is also Gaussian: $\Xi_t - \widehat{\Xi}_t$ has mean 0 and covariance V_t, and $\text{Law}(\Xi_t | \widehat{\mathcal{F}}_t) = \text{N}(\widehat{\Xi}_t, V_t)$.

3 Optimal investment problems with random drift

3.1 Portfolio optimisation via convex duality

We wish to apply the filtering theory developed in the previous section to portfolio optimisation and optimal hedging problems when the agent does not know the drift parameters of the underlying assets. The filtering approach will lead to portfolio problems in which the assets follow SDEs with random drift parameters. The dual approach to portfolio optimisation is well suited to such problems, so in this section we outline this approach in a complete market, and summarise the results for an incomplete market. The dual approach to portfolio problems is well documented by Karatzas [13] to which the reader is referred for more details and further references.

Consider an agent with a continuous, differentiable, increasing, concave utility function $U : \mathbb{R}^+ \to \mathbb{R}$. Define the *convex conjugate* $\widetilde{U} : \mathbb{R}^+ \to \mathbb{R}$ of U by

$$
\widetilde{U}(\eta) := \sup_{x \in \mathbb{R}^+} [U(x) - x\eta], \quad \eta > 0.
\tag{27}
$$

Then $\widetilde{U}$ is a decreasing, continuously differentiable, convex function, satisfying the inequality

$$\widetilde{U}(\eta) \geq U(x) - x\eta, \quad \text{with equality iff } x = x^* \text{ such that } U'(x^*) = \eta. \tag{28}$$

In other words, the supremum in (27) is achieved by $x = x^*$ satisfying

$$U'(x^*) = \eta \Leftrightarrow x^* = I(\eta), \tag{29}$$

where I is the inverse of U', so that $U'(I(\eta)) = I(U'(\eta)) = \eta$. Then $\widetilde{U}(\eta)$ may be written as

$$\widetilde{U}(\eta) = U(I(\eta)) - \eta I(\eta). \tag{30}$$

Differentiating (30) gives

$$\widetilde{U}'(\eta) = -I(\eta), \tag{31}$$

so the marginal utility is the inverse of minus the gradient of the convex conjugate:

$$U'(-\widetilde{U}'(x)) = x.$$

We note that the defining duality relation (27) is equivalent to the bidual relation

$$U(x) = \inf_{\eta \in \mathbb{R}^+} [\widetilde{U}(\eta) + x\eta], \quad x > 0, \tag{32}$$

since this gives that the value of η achieving the above infimum is η^* satisfying

$$\widetilde{U}'(\eta^*) = -x,$$

or, by (31),

$$I(\eta^*) = x \Leftrightarrow U'(x) = \eta^*,$$

which is equivalent to (29). Note also that (32) implies (28).

We are interested in solving an optimal portfolio problem for an agent in a complete market with a single stock whose price process is a continuous semimartingale. To be precise, on an a probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$, suppose a stock price $S = (S_t)_{0 \leq t \leq T}$ follows

$$dS_t = \sigma_t S_t (\lambda_t dt + dB_t),$$

where $\sigma = (\sigma_t)_{0 \leq t \leq T}$ and $\lambda = (\lambda_t)_{0 \leq t \leq T}$ are $\mathbb{F}$-adapted processes, and $B = (B_t)_{0 \leq t \leq T}$ is an $\mathbb{F}$-Brownian motion. For simplicity, we take the interest rate to be zero.

The wealth process $X = (X_t)_{0 \leq t \leq T}$ associated with a self-financing portfolio involving S is given by

$$dX_t = \sigma_t \theta_t X_t (\lambda_t dt + dB_t), \quad X_0 = x,$$

where the process $\theta = (\theta_t)_{0 \leq t \leq T}$ represents the proportion of wealth placed in the stock, and constitutes the agent's trading strategy. Define the set $\mathcal{A}$ of admissible trading strategies as those whose wealth process satisfies $X_t \geq 0$ a.s. for all $t \in [0, T]$.

The unique martingale measure $Q \sim P$ on $\mathcal{F}_T$ is defined by

$$\frac{dQ}{dP} = Z_T, \tag{33}$$

where $Z = (Z_t)_{0 \leq t \leq T}$ is the exponential local martingale defined by

$$Z_t := \mathcal{E}(-\lambda \cdot B)_t, \quad 0 \leq t \leq T,$$

satisfying

$$dZ_t = -\lambda_t Z_t dB_t, \quad Z_0 = 1.$$

We shall assume that

$$E \exp\left(\frac{1}{2} \int_0^T \lambda_t^2 dt\right) < \infty,$$

13

so that Z is indeed a martingale and Q is indeed a probability measure equivalent to P.

Under Q, the process B^Q defined by

$$B_t^Q := B_t + \int_0^t \lambda_s ds, \quad 0 \leq t \leq T,$$

is a Brownian motion. The Q-dynamics of S, X are

$$dS_t = \sigma_t S_t dB_t^Q, \quad dX_t = \sigma_t \theta_t X_t dB_t^Q.$$

In particular, the solution of the SDE for X, given $X_0 = x$, is

$$X_t = x\mathcal{E}(\sigma\theta \cdot B^Q)_t, \quad 0 \leq t \leq T.$$

We assume that

$$E^Q \exp\left(\frac{1}{2}\int_0^T \sigma_t^2 \theta_t^2 dt\right) < \infty,$$

so that X is a Q-martingale, satisfying

$$E^Q[X_t|\mathcal{F}_s] = X_s, \quad 0 \leq s \leq t \leq T.$$

In particular, with $t = T$, $s = 0$ and $X_0 = x$,

$$E^Q X_T = x,$$

or

$$E[Z_T X_T] = x, \tag{34}$$

which we shall regard as a *constraint* on the terminal wealth X_T. This is the foundation of the dual approach to portfolio optimisation, namely to enforce the martingale constraint on the wealth process.

The *primal* portfolio problem (also called the *primal* problem) is, given $X_0 = x$, to maximise expected utility of wealth at time T:

$$u(x) := \sup_{\theta \in \mathcal{A}} EU(X_T), \tag{35}$$

subject to (34).

The *dual* value function is $\tilde{u} : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$\tilde{u}(\eta) := E\widetilde{U}\left(\eta\frac{dQ}{dP}\right), \quad \eta > 0.$$

Then the main result on portfolio optimisation via convex duality for this model is as follows.

Theorem 6. *1. The primal and dual value functions $u(x)$ and $\tilde{u}(\eta)$ are conjugate:*

$$\tilde{u}(\eta) = \sup_{x \in \mathbb{R}^+}[u(x) - x\eta], \qquad u(x) = \inf_{\eta > 0}[\tilde{u}(\eta) + x\eta],$$

so that $u'(x) = \eta$ (equivalently, $\tilde{u}'(\eta) = -x$);

2. The optimal terminal wealth in (35) is X_T^* satisfying

$$U'(X_T^*) = \eta\frac{dQ}{dP}, \quad equivalently, \quad X_T^* = I\left(\eta\frac{dQ}{dP}\right).$$

Proof. With $dQ/dP = Z_T$ given by (33), consider the maximisation of the objective functional $EU(X_T)$ subject to the constraint $E[Z_T X_T] = x$, via the Lagrangian

$$
\begin{aligned}
L(X_T, \eta) &:= EU(X_T) + \eta \left(x - E\left[Z_T X_T\right] \right) \\
&= \eta x + E\left[U(X_T) - \eta Z_T X_T\right] \\
&\leq \eta x + E\widetilde{U}\left(\eta Z_T\right), \quad \text{[by (28)]} \\
&= \eta x + \tilde{u}(\eta),
\end{aligned}
\tag{36}
$$

with equality in (36) if and only if $X_T = X_T^*$ given by

$$
U'(X_T^*) = \eta Z_T \Leftrightarrow X_T^* = I\left(\eta Z_T\right).
\tag{37}
$$

Since maximising the left-hand-side of (36) gives the primal value function $u(x)$, we conclude that (37) identifies the optimal terminal wealth, which proves part 2 of the theorem.

The value of the multiplier η is needed to completely specify X_T^*, and this is fixed by inserting (37) into the constraint $E[Z_T X_T^*] = x$, giving

$$
E[Z_T I(\eta Z_T)] = x,
$$

or

$$
\chi(\eta) = x,
\tag{38}
$$

where we have defined the function

$$
\chi(\eta) := E[Z_T I(\eta Z_T)], \quad \eta > 0.
$$

Define $\Upsilon(\cdot) := \chi^{-1}(\cdot)$ as the inverse of the function $\chi(\cdot)$. Then, inverting (38) we write η as

$$
\eta = \Upsilon(x).
\tag{39}
$$

Now, differentiating the defining relation $\tilde{u}(\eta) = E\widetilde{U}(\eta Z_T)$ gives

$$
\tilde{u}'(\eta) = E[Z_T \widetilde{U}'(\eta Z_T)] = -E[Z_T I(\eta Z_T)],
$$

since $I(\cdot) = -\widetilde{U}'(\cdot)$. In other words, $\chi(\cdot)$ and $\tilde{u}'(\cdot)$ are related by

$$
\tilde{u}'(\eta) = -\chi(\eta),
$$

which together with (38) implies

$$
\tilde{u}'(\eta) = -x.
$$

Using (37) and (39) we write X_T^* as

$$
X_T^* = I[\Upsilon(x) Z_T].
\tag{40}
$$

Now define the function

$$
G(\eta) := EU[I(\eta Z_T)], \quad \eta > 0.
$$

Using (40), write $u(x)$ as

$$
u(x) = EU(X_T^*) = EU[I(\Upsilon(x) Z_T)],
$$

or

$$
u(x) = G[\Upsilon(x)].
\tag{41}
$$

Note also that

$$
\begin{aligned}
G(\eta) - \eta \chi(\eta) &= E\left[U(I(\eta Z_T)) - \eta Z_T I(\eta Z_T)\right] \\
&= E\widetilde{U}(\eta Z_T), \quad \text{[by (30)]},
\end{aligned}
$$

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or

$$\tilde{u}(\eta) = G(\eta) - \eta\chi(\eta), \quad \eta > 0. \tag{42}$$

Moreover, by (41) we have $u[\chi(\eta)] = G[\Upsilon(\chi(\eta))]$, or, since $\Upsilon(\cdot), \chi(\cdot)$ are inverses of each other,

$$u[\chi(\eta)] = G(\eta). \tag{43}$$

Now, inequality (36) holds for arbitrary $x \in \mathbb{R}^+, \theta \in \mathcal{A}, \eta > 0$, so we have in particular, taking the optimal strategy $\theta^* \in \mathcal{A}$ on the left-hand-side of (36),

$$u(x) \le \eta x + \tilde{u}(\eta), \quad \forall x \in \mathbb{R}^+,$$

whence

$$\sup_{x \in \mathbb{R}^+} [u(x) - x\eta] \le \tilde{u}(\eta), \quad \forall \eta > 0. \tag{44}$$

On the other hand, (42) gives

$$
\begin{aligned}
\tilde{u}(\eta) &= G(\eta) - \eta\chi(\eta) \\
&= u[\chi(\eta)] - \eta\chi(\eta) \quad [\text{by (43)}] \\
&\le \sup_{x \in \mathbb{R}^+} [u(x) - x\eta], \quad \forall \eta > 0.
\end{aligned}
\tag{45}
$$

So (44) and (45) imply

$$\tilde{u}(\eta) = \sup_{x \in \mathbb{R}^+} [u(x) - x\eta],$$

which establishes part 1 of the theorem, and the proof is complete.

$\square$

3.2 Incomplete markets

Similar duality theorems have been developed for incomplete market situations, and also when the agent has a random terminal endowment, possibly in the form of a contingent claim. For the incomplete market case, see the seminal paper by Karatzas et al [14] for markets with continuous price processes, and Kramkov and Schachermayer [16] for the case with general semimartingale price processes. For problems involving a terminal random endowment in the form of an $\mathcal{F}_T$-measurable random variable, contributions have been made by (among others) Hugonnier and Kramkov [9], Owen [28] and by Delbaen et al [6] for an agent with an exponential utility function. We shall use the results of [6] in Section 5, when we examine the exponential hedging of a contingent claim in a basis risk model.

For an incomplete market, in which the set $\mathcal{M}$ of martingale measures is no longer a singleton, the significant change is that the dual value function is then defined by

$$\tilde{u}(\eta) := \inf_{Q \in \mathcal{M}} E\widetilde{U}\left(\eta\frac{dQ}{dP}\right). \tag{46}$$

The form of the duality theorem for an incomplete market is similar to Theorem 6, but with the unique martingale measure Q of the complete market replaced by the optimal *dual minimiser* Q^* that achieves the infimum in (46). We formalise this in Theorem 7 below, whose proof is omitted. See [13], for instance, for details in an Itô process setting.

Theorem 7. *In an incomplete market model with martingale measures $Q \in \mathcal{M}$, define the primal value function by*

$$u(x) := \sup_{\theta \in \mathcal{A}} EU(X_T^{(\theta)}), \quad x \in \mathrm{dom}(U),$$

where $X_T^{(\theta)}$ denotes the terminal wealth generated from using a trading strategy θ from the admissible set $\mathcal{A}$.

Define the dual value function $\tilde{u}$ by (46). Then:

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1. $u(x)$ and $\tilde{u}(\eta)$ are conjugate:

$$\tilde{u}(\eta) = \sup_{x \in \text{dom}(U)} [u(x) - x\eta], \qquad u(x) = \inf_{\eta > 0} [\tilde{u}(\eta) + x\eta],$$

so that $u'(x) = \eta$ (equivalently, $\tilde{u}'(\eta) = -x$);

2. The optimal terminal wealth X_T^* and optimal dual minimiser Q^* are unique and related by satisfying

$$U'(X_T^*) = \eta \frac{dQ^*}{dP}, \quad \text{equivalently}, \quad X_T^* = I\left(\eta \frac{dQ^*}{dP}\right);$$

3.3 Optimal investment with Gaussian drift process

We wish to apply filtering theory and the martingale approach to portfolio optimisation to the classical optimal portfolio problem of Merton [20, 21], in the case that the agent does not know the drift parameter of the stock. As we shall see, this will involve a portfolio problem in which the market price of risk of the stock is a Gaussian process. Hence we first describe the solution to such a problem.

Suppose a stock price $S = (S_t)_{0 \le t \le T}$ follows the process

$$dS_t = \sigma S_t (\lambda_t dt + dB_t),$$

on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}, P)$, with B an $\mathbb{F}$-Brownian motion and λ an $\mathbb{F}$-adapted process following

$$\lambda_t = \lambda_0 + \int_0^t \mathrm{w}_s dB_s, \quad \mathrm{w}_t = \frac{\mathrm{w}_0}{1 + \mathrm{w}_0 t}, \quad 0 \le t \le T, \tag{47}$$

for constants λ_0, w_0.

The self-financing wealth process X from trading S is given by

$$dX_t = \sigma \theta_t X_t (\lambda_t dt + dB_t), \quad X_0 = x, \tag{48}$$

where the trading strategy $\theta = (\theta_t)_{0 \le t \le T}$ is the proportion of wealth invested in stock. We define the set $\mathcal{A}$ of admissible strategies as those satisfying $\int_0^T \theta_t^2 dt < \infty$ almost surely, such that $X_t \ge 0$ almost surely for all $t \in [0, T]$.

The value function is

$$u(x) := \sup_{\theta \in \mathcal{A}} E[U(X_T) | \mathcal{F}_0] \tag{49}$$

where $U(x)$ is the power utility function given by

$$U(x) = \frac{x^\gamma}{\gamma}, \quad 0 < \gamma < 1. \tag{50}$$

Theorem 8. *Assume that*

$$-1 < \mathrm{w}_0 T < \frac{1 - \gamma}{\gamma}.$$

Then the value function (49) is given by

$$u(x) = \frac{x^\gamma}{\gamma} C^{1-\gamma}, \tag{51}$$

where C is given by

$$C = \left(\frac{(1 + \mathrm{w}_0 T)^q}{1 + q \mathrm{w}_0 T}\right)^{1/2} \exp\left(-\frac{1}{2} \frac{q(1-q)\lambda_0^2 T}{1 + q\mathrm{w}_0 T}\right), \quad q = -\frac{\gamma}{1 - \gamma}. \tag{52}$$

The optimal trading strategy θ^* achieving the supremum in (49) is given by

$$\theta_t^* = \frac{\lambda_t}{\sigma(1 - \gamma)} \left(\frac{1}{1 + q\mathrm{w}_t(T - t)}\right), \quad 0 \le t \le T. \tag{53}$$

Proof. Let Q denote the unique martingale measure for this market. The change of measure martingale $Z := (Z_t)_{0 \le t \le T}$ is given by

$$Z_t := \left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} = \mathcal{E}(-\lambda \cdot B)_t, \quad 0 \le t \le T,$$

and satisfies the SDE

$$dZ_t = -\lambda_t Z_t dB_t, \quad Z_0 = 1. \tag{54}$$

Notice that

$$\lim_{w_0 \to 0} Z_t = \mathcal{E}(-\lambda_0 B)_t = \exp\left(-\lambda_0 B_t - \frac{1}{2}\lambda_0^2 t\right). \tag{55}$$

We may write $Z_t = f(t, \lambda_t)$ where $f : [0,T] \times \mathbb{R} \to \mathbb{R}^+$ is a smooth function, and apply Itô's formula along with the SDE (47) for λ to give

$$dZ_t = \left[f_t(t, \lambda_t) + \frac{1}{2}w_t^2 f_{xx}(t, \lambda_t)\right] dt + w_t f_x(t, \lambda_t) dB_t, \tag{56}$$

with subscripts of f denoting partial derivatives. Equating (54) and (56) yields the partial differential equations for f

$$w_t f_x(t, x) = -x f(t, x),$$
$$f_t(t, x) + \frac{1}{2}w_t^2 f_{xx}(t, x) = 0,$$

with $f(0, \cdot) = Z_0 = 1$. The solution to these equations gives Z_t in the form

$$Z_t = \left(\frac{w_0}{w_t}\right)^{1/2} \exp\left[-\frac{1}{2}\left(\frac{\lambda_t^2}{w_t} - \frac{\lambda_0^2}{w_0}\right)\right], \quad 0 \le t \le T \tag{57}$$

Note that this function is actually well-defined even for $w_0 \to 0$. It is not hard to check that (57) reduces to (55) in the limit $w_0 \to 0$.

For power utility, the convex conjugate $\widetilde{U}$ of then utility function is given by

$$\widetilde{U}(\eta) = -\frac{\eta^q}{q}, \quad q = -\frac{\gamma}{1 - \gamma}, \quad \eta > 0. \tag{58}$$

The dual value function is defined by

$$\tilde{u}(\eta) := E[\widetilde{U}(\eta Z_T)|\mathcal{F}_0], \quad \eta > 0.$$

Using (58) we obtain

$$\tilde{u}(\eta) = -\frac{\eta^q}{q}C,$$

where

$$C := E\left[Z_T^q | \mathcal{F}_0\right]. \tag{59}$$

From Theorem 6, the primal and dual value functions are conjugate, which yields that the primal value function is indeed given by (51), with C defined by (59). It therefore remains to show that C is indeed equal to the expression in (52) and that the optimal strategy is given by (53).

Once again using Theorem 6, the optimal terminal wealth X_T^*, attained by adopting the strategy that achieves the supremum in (49), is given by

$$X_T^* = -\widetilde{U}'(u'(x)Z_T).$$

Hence, using the form (51) for u, we obtain

$$X_T^* = \frac{x}{C}(Z_T)^{-(1-q)}.$$

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The optimal wealth process X^* is a $(Q, \mathbb{F})$-martingale, so

$$X_t^* = E^Q[X_T^*|\mathcal{F}_t] = \frac{1}{Z_t}E[Z_T X_T^*|\mathcal{F}_t] = \frac{x}{CZ_t}E[Z_T^q|\mathcal{F}_t], \quad 0 \le t \le T. \tag{60}$$

So, to compute explicit formulae for $C = E[Z_T^q|\mathcal{F}_0]$ and the optimal wealth process (from which the optimal trading strategy will be derived), we need to evaluate the conditional expectation $E[Z_T^q|\mathcal{F}_t], 0 \le t \le T$.

From (47), for $t \le T$, and conditional on $\mathcal{F}_t$, λ_T is normally distributed according to

$$\text{Law}(\lambda_T|\mathcal{F}_t) = \text{N}(\lambda_t, w_t - w_T), \quad 0 \le t \le T.$$

For a normally distributed random variable $Y \sim \text{N}(m, s^2)$, we have

$$E \exp(cY^2) = \frac{1}{\sqrt{1 - 2cs^2}} \exp\left(\frac{cm^2}{1 - 2cs^2}\right),$$

so that, given the explicit expression (57) for Z_t, both C and the right-hand-side of (60) can be computed in closed form. We find that C is indeed given by (52). Notice that $1 + qw_0T > 0$ and $1 + w_0T > 0$ due to the conditions on w_0T, thus the solution is well defined.

For the optimal wealth process, we obtain the formula

$$X_t^* = x \left(\frac{\Psi_t}{\Psi_0}\right)^{\frac{1}{2}} \exp\left(\frac{1}{2}(1 - q)(\Phi_t - \Phi_0)\right), \quad 0 \le t \le T, \tag{61}$$

where

$$\Psi_t := \frac{w_t}{1 + qw_t(T - t)}, \quad \Phi_t := \frac{\lambda_t^2}{w_t(1 + qw_t(T - t))}, \quad 0 \le t \le T.$$

To compute the optimal trading strategy θ^*, we apply the Itô formula to (61), using the SDE for λ and noting that the derivative of w_t is given by

$$\frac{dw_t}{dt} = -w_t^2.$$

We compare the coefficient of dB_t in dX_t^* with that in (48) for the case of the optimal wealth process. This gives (53). $\qquad \square$

3.3.1 Classical Merton problem

In the limit $w_0 \to 0$, the drift of the stock becomes the constant λ_0, and Theorem 8 gives the solution to the classical full information Merton optimal investment problem for a stock with constant market price of risk λ_0 and volatility σ. In this case it is easy to check that the value function (51) becomes

$$u(x) = \frac{x^\gamma}{\gamma} \exp\left(\frac{1}{2}\frac{\gamma}{1 - \gamma}\lambda_0^2 T\right),$$

and the optimal trading strategy (53) becomes

$$\theta_t^* = \frac{\lambda_0}{\sigma(1 - \gamma)}, \quad 0 \le t \le T.$$

That is, the Merton investor keeps a constant proportion of wealth invested in the stock, as is well known.

3.4 Merton problem with uncertain drift

We can now solve the Merton problem when the agent has uncertainty over the true value of the drift parameter. Optimal investment models under partial information have been considered by many authors. We refer the reader to Rogers [31], Björk, Davis and Landén [2], and Platen and Runggaldier [30], for example.

A stock price process $S = (S_t)_{0 \leq t \leq T}$ follows

$$dS_t = \sigma S_t(\lambda dt + dB_t), \tag{62}$$

on a complete probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $\mathbb{F} := (\mathcal{F}_t)_{0 \leq t \leq T}$, with $B = (B_t)_{0 \leq t \leq T}$ an $\mathbb{F}$-Brownian motion. In a partial information model with continuous stock price observations, an agent must use $\widehat{\mathbb{F}}$-adapted trading strategies, where where $\widehat{\mathbb{F}} := (\widehat{\mathcal{F}}_t)_{0 \leq t \leq T}$ is the observation filtration, defined by

$$\widehat{\mathcal{F}}_t := \sigma(\xi_s; 0 \leq s \leq t) = \sigma(S_s; 0 \leq s \leq t).$$

Then σ is known from the quadratic variation of S, but λ is an unknown constant, and hence modelled as an $\mathcal{F}_0$-measurable random variable. We assume the distribution of λ is Gaussian, $\lambda \sim N(\lambda_0, v_0)$, independent of B.

Let us define the process $\xi = (\xi_t)_{0 \leq t \leq T}$, by

$$\xi_t := \frac{1}{\sigma} \int_0^t \frac{dS_u}{S_u} = \lambda t + B_t. \tag{63}$$

The process ξ will be considered as the observation process in a filtering framework, corresponding to noisy observations of λ, with B representing the noise.

We are faced with a Kalman-Bucy type filtering problem whose unobservable signal process is the market price of risk λ. The signal process SDE is

$$d\lambda = 0, \tag{64}$$

and the observation process SDE is (63).

We apply Theorem 4 to the signal process λ in (64) and observation process ξ in (63). Then the optimal filter

$$\widehat{\lambda}_t := E[\lambda | \widehat{\mathcal{F}}_t], \quad 0 \leq t \leq T,$$

satisfies

$$d\widehat{\lambda}_t = v_t d\widehat{B}_t, \quad \widehat{\lambda}_0 = \lambda_0, \tag{65}$$

where

$$v_t := E[(\lambda - \widehat{\lambda}_t)^2 | \widehat{\mathcal{F}}_t], \quad 0 \leq t \leq T,$$

is the conditional variance. This satisfies the Riccati equation

$$\frac{dv_t}{dt} = -v_t^2, \tag{66}$$

with initial value v_0, so that

$$v_t = \frac{v_0}{1 + v_0 t}, \quad 0 \leq t \leq T. \tag{67}$$

The process $\widehat{B}$ is an $\widehat{\mathbb{F}}$-Brownian motion, the innovations process, satisfying

$$d\widehat{B}_t = d\xi_t - \widehat{\lambda}_t dt. \tag{68}$$

Using this in (65), the optimal filter can also be written in terms of the observable ξ as

$$\widehat{\lambda}_t = \frac{\lambda_0 + v_0 \xi_t}{1 + v_0 t}, \quad 0 \leq t \leq T. \tag{69}$$

The effect of the filtering is that the agent is now investing in a stock with dynamics given by $dS_t = \sigma S_t d\xi_t$ which, using (68), becomes

$$dS_t = \sigma S_t(\widehat{\lambda}_t dt + d\widehat{B}_t). \tag{70}$$

Our agent has a power utility function (50) and may invest a portion of his wealth in shares and the remaining wealth in a cash account with zero interest rate (for simplicity). The ($\widehat{\mathbb{F}}$-adapted) wealth process X^0 then follows

$$dX_t^0 = \sigma \theta_t^0 X_t^0(\widehat{\lambda}_t dt + d\widehat{B}_t), \quad X_0^0 = x, \tag{71}$$

where θ_t^0 is the proportion of wealth invested in shares at time $t \in [0, T]$, an $\widehat{\mathbb{F}}$-adapted process satisfying $\int_0^T \left(\theta_t^0\right)^2 dt < \infty$ almost surely, and such that $X_t^0 \geq 0$ almost surely for all $t \in [0, T]$. Denote by $\mathcal{A}_0$ the set of such admissible strategies.

The objective is to maximise expected utility of terminal wealth over the $\widehat{\mathbb{F}}$-adapted admissible strategies. The value function is

$$u_0(x) := E[U(X_T^0)|\widehat{\mathcal{F}}_0].$$

This may now be treated as a full information problem, with state dynamics given by (71).

We see from equations (65), (67) and (70), that the solution to the partial information optimal portfolio problem is given by Theorem 8, when we replace the process λ of Theorem 8 by $\widehat{\lambda}$, and replace $(w_t)_{0 \leq t \leq T}$ by $(v_t)_{0 \leq t \leq T}$. We have therefore proved the following result.

Theorem 9 (Merton problem with uncertain drift). *In a complete market with stock price process S given by (62), suppose an agent is restricted to using stock price adapted strategies to maximise expected utility of terminal wealth, with power utility function given by (50). Suppose further that the agent's prior distribution for λ is Gaussian, according to*

$$\mathrm{Law}(\lambda|\widehat{\mathcal{F}}_0) = \mathrm{N}(\lambda_0, v_0),$$

and assume that

$$-1 < v_0 T < \frac{1 - \gamma}{\gamma}.$$

Then the agent's value function is given by

$$u_0(x) = \frac{x^\gamma}{\gamma} C_0^{1-\gamma}.$$

where

$$C_0 = \left[\frac{(1 + v_0 T)^q}{1 + q v_0 T}\right]^{1/2} \exp\left[-\frac{1}{2}\frac{q(1-q)\lambda_0^2 T}{1 + q v_0 T}\right], \quad q = -\frac{\gamma}{1 - \gamma},$$

The optimal trading strategy is $\theta^{0,*} = (\theta_t^{0,*})_{0 \leq t \leq T}$, given by

$$\theta_t^{0,*} = \frac{\widehat{\lambda}_t}{\sigma(1-\gamma)}\left(\frac{1}{1 + q v_t(T-t)}\right), \quad 0 \leq t \leq T,$$

where $\widehat{\lambda} = (\widehat{\lambda}_t)_{0 \leq t \leq T}$ satisfies (65) and v_t is given by (67).

The classical Merton strategy is thus altered in two ways: the constant λ is replaced by its filtered estimate $\widehat{\lambda}_t$, and the risky asset proportion is decreased by the factor $(1 + q v_t(T-t))^{-1}$. We note that the more risk averse the investor, the less likely he is to invest in shares, and as $t \to T$, the optimal strategy gets closer and closer to the Merton rule.

4   Investment with inside information and drift uncertainty

We again consider the Merton optimal investment problem in which the agent does not know the stock price drift, but now with the added feature that the agent has some additional information at time zero, represented by noisy knowledge of the terminal value B_T of the Brownian motion driving the stock. We refer the reader to Danilova, Monoyios and Ng [3] for further examples, such as when the additional information involves noisy knowledge of the terminal stock price. The work in this section and in [3] extends the classical inside information model of Pikovsky and Karatzas [29] by considering the situation where the insider does not know the stock's appreciation rate. The agent must use strategies that are adapted to the stock price filtration, but enlarged by the additional information. We must therefore utilise a filtering algorithm which computes the best estimate of the drift, given stock price observations and the additional information. The usual Kalman-Bucy equations hold in this scenario, but with modified initial conditions reflecting the additional information.

The market is the same one as in Section 3.4, with a single stock whose price process S follows (62), repeated below:

$$dS_t = \sigma S_t(\lambda dt + dB_t), \tag{72}$$

on a complete probability space $(\Omega, \mathcal{F}, P)$ equipped with a background filtration $\mathbb{F} := (\mathcal{F}_t)_{0 \leq t \leq T}$, with B an $\mathbb{F}$-Brownian motion. We shall again allow for uncertainty in the value of λ, so consider it to be an $\mathcal{F}_0$-measurable random variable. Once again we take the interest rate to be zero.

As before, we define the observation process $\xi = (\xi_t)_{0 \leq t \leq T}$ by (63), repeated below:

$$\xi_t := \int_0^t \frac{dS_s}{\sigma S_s} = \lambda t + B_t, \quad 0 \leq t \leq T, \tag{73}$$

and the filtration generated by ξ is again denoted by $\widehat{\mathbb{F}} = (\widehat{\mathcal{F}}_t)_{0 \leq t \leq T}$. Since the background filtration $\mathbb{F}$ contains the Brownian filtration and also the sigma-field generated by λ, we have $\widehat{\mathcal{F}}_t \subseteq \mathcal{F}_t$, for all $t \in [0, T]$.

Also as before, the uncertainty in the $\mathcal{F}_0$-measurable random variable λ is modelled by assuming that its prior distribution conditional on $\widehat{\mathcal{F}}_0$ is Gaussian, according to

$$\text{Law}(\lambda | \widehat{\mathcal{F}}_0) = \text{N}(\lambda_0, \text{v}_0), \quad \text{independent of } B, \tag{74}$$

for given constants λ_0, v_0.[4]

In contrast to earlier, the utility-maximising agent will not only have access to $\widehat{\mathbb{F}}$ in order to estimate λ and implement an optimal strategy, but will be able to augment $\widehat{\mathbb{F}}$ with some additional information, represented by knowledge of a random variable I.

Our procedure in this section is to first enlarge the background filtration $\mathbb{F}$ with the information carried by the random variable I. Denote the enlarged filtration by $\mathbb{F}^{\sigma(I)} = (\mathcal{F}_t^{\sigma(I)})_{0 \leq t \leq T}$, with

$$\mathcal{F}_t^{\sigma(I)} := \mathcal{F}_t \vee \sigma(I), \quad 0 \leq t \leq T.$$

By starting with an enlarged background filtration and then considering the optimal investment problem with uncertain drift, we aim to incorporate the insider's additional information in the estimation of the unknown market price of risk λ.

The next step is to write the stock price SDE (72) in terms of quantities adapted to $\mathbb{F}^{\sigma(I)}$. As $\mathbb{F}$ contains the Brownian filtration, we apply classical initial enlargement results (see, for instance, Mansuy and Yor [19]). There exists an $\mathbb{F}^{\sigma(I)}$-adapted process ν, the *information drift*, such that the Brownian motion B decomposes according to

$$B_t := B_t^I + \int_0^t \nu_s ds, \quad 0 \leq t \leq T, \tag{75}$$

where B^I is an $\mathbb{F}^{\sigma(I)}$-Brownian motion. We shall characterise the information drift via Lemma 2 shortly.

---

[4]One way to choose λ_0, v_0 would be to use past data before time zero to obtain a point estimate of λ, and to use the distribution of the estimator as the prior, as in Monoyios [24] and Section 5 of this article.

Using (75), the stock price dynamics (72) is written in terms of $\mathbb{F}^{\sigma(I)}$-adapted processes, to give

$$dS_t = \sigma S_t \left(\lambda_t^I dt + dB_t^I \right), \tag{76}$$

where

$$\lambda_t^I := \lambda + \nu_t, \quad 0 \le t \le T,$$

is $\mathbb{F}^{\sigma(I)}$-adapted. If the insider happened to know the value of λ, then we would interpret (76) as his stock price SDE, with a stochastic market price of risk λ^I, on the filtered probability space $(\Omega, \mathcal{F}_T^{\sigma(I)}, \mathbb{F}^{\sigma(I)}, P)$.

We study a problem where the inside information consists of noisy Brownian inside information. In other words, we take I to be given by

$$I := aB_T + (1-a)\epsilon, \quad 0 < a < 1, \tag{77}$$

and where ϵ is a standard normal random variable independent of B and λ.

Define the insider's observation filtration $\widehat{\mathbb{F}}^{\sigma(I)} = (\widehat{\mathcal{F}}_t^{\sigma(I)})_{0 \le t \le T}$ by

$$\widehat{\mathcal{F}}_t^{\sigma(I)} := \sigma(I, \xi_s; 0 \le s \le t), \quad 0 \le t \le T.$$

We now incorporate the insider's uncertainty in the knowledge of λ by treating it as an $\mathcal{F}_0^{\sigma(I)}$-measurable Gaussian random variable with distribution conditional on $\widehat{\mathcal{F}}_0$ given by (74). In this example, λ is independent of I, so its distribution conditional on $\widehat{\mathcal{F}}_0^{\sigma(I)}$ is unaltered from that in (74):

$$\mathrm{Law}(\lambda | \widehat{\mathcal{F}}_0^{\sigma(I)}) = \mathrm{Law}(\lambda | \widehat{\mathcal{F}}_0) = \mathrm{N}(\lambda_0, \mathrm{v}_0). \tag{78}$$

Treating λ^I as an unobservable signal process, we shall see that λ^I will satisfy a linear SDE with respect to $\mathbb{F}^{\sigma(I)}$. The Kalman-Bucy filter then allows the insider to infer the conditional expectation

$$\widehat{\lambda}_t^I := E[\lambda_t^I | \widehat{\mathcal{F}}_t^{\sigma(I)}], \quad 0 \le t \le T, \tag{79}$$

that is, the best estimate of the signal λ^I based on the insider's observation filtration $\widehat{\mathbb{F}}^{\sigma(I)}$, which turns out to be a Gaussian process, fully characterised by the filtering algorithm. The initial condition for the optimal filter incorporates the inside information, and the SDE for the filter augments this with the stock price observations. This will convert the partial information model (76) to a full information model on the filtered probability space $(\Omega, \widehat{\mathcal{F}}_T^{\sigma(I)}, \widehat{\mathbb{F}}^{\sigma(I)}, P)$ with the stock price following

$$dS_t = \sigma S_t(\widehat{\lambda}_t^I dt + d\widehat{B}_t^I), \tag{80}$$

where $\widehat{B}^I$ is an $\widehat{\mathbb{F}}^{\sigma(I)}$-Brownian motion. Finally, once we have the full information model (80), we are able to compute the maximum utility via duality.

Denote the agent's $\widehat{\mathbb{F}}^{\sigma(I)}$-adapted wealth process by $X^I = (X_t^I)_{0 \le t \le T}$, with trading strategy $\theta^I = (\theta_t^I)_{0 \le t \le T}$, the proportion of wealth invested in the stock, an $\widehat{\mathbb{F}}^{\sigma(I)}$-adapted process satisfying $\int_0^T \left(\theta_t^I \right)^2 dt < \infty$ almost surely, such that $X_t^I \ge 0$ almost surely for all $t \in [0, T]$. Denote by $\mathcal{A}_I$ the set of such admissible strategies.

The value function for this problem is

$$u_I(x) := \sup_{\theta^I \in \mathcal{A}_I} E[U(X_T^I) | \widehat{\mathcal{F}}_0^{\sigma(I)}], \quad x > 0, \tag{81}$$

where U is the power utility function (50). We emphasise that the objective function in (81) is conditioned on $\widehat{\mathcal{F}}_0^{\sigma(I)}$.

Define the modulated terminal time T_a by

$$T_a := T + \left(\frac{1-a}{a} \right)^2, \tag{82}$$

which will appear in our results. Then the solution to this problem is as follows.

Theorem 10. *Assume that*

$$\frac{T}{T_a} - 1 < \mathrm{v}_0 T < \frac{T}{T_a} + \frac{1-\gamma}{\gamma}.$$

Define the function $\mathrm{v}^I : [0,T] \to \mathbb{R}$ *by*

$$\mathrm{v}_t^I := \frac{\mathrm{v}_0^I}{1 + \mathrm{v}_0^I t}, \quad \mathrm{v}_0^I := \mathrm{v}_0 - \frac{1}{T_a}, \quad 0 \le t \le T. \tag{83}$$

Then the process $\widehat{\lambda}^I$ *in (79) is given by*

$$\widehat{\lambda}_t^I = \lambda_0 + \frac{I}{aT_a} + \int_0^t \mathrm{v}_s^I d\widehat{B}_s^I, \quad 0 \le t \le T, \tag{84}$$

where I *is defined in (77) and* T_a *in (82). The value function of the insider with knowledge of* I *at time zero is given by*

$$u_I(x) = \frac{x^\gamma}{\gamma} C_I^{1-\gamma}, \tag{85}$$

where C_I *is the* $\widehat{\mathcal{F}}_0^I$*-measurable random variable given by*

$$C_I = \left(\frac{(1 + \mathrm{v}_0^I T)^q}{1 + q\mathrm{v}_0^I T} \right)^{1/2} \exp\left(-\frac{1}{2} \frac{q(1-q)(\widehat{\lambda}_0^I)^2 T}{1 + q\mathrm{v}_0^I T} \right), \quad q = -\frac{\gamma}{1-\gamma}.$$

The insider's optimal trading strategy is $\theta^{I,*} = (\theta_t^{I,*})_{0 \le t \le T}$*, given by*

$$\theta_t^{I,*} = \frac{\widehat{\lambda}_t^I}{\sigma(1-\gamma)} \left(\frac{1}{1 + q\mathrm{v}_t^I(T-t)} \right), \quad 0 \le t \le T.$$

Of course, the value function (85) depends explicitly on I, through its dependence on $\widehat{\lambda}_0^I$. We note the similarity in the structure of the solution to this problem with that of the Merton problem with uncertain drift and no inside information. The function v^I plays a similar role to the function v in the conventional partial information problem. It turns out that v^I is related to (but not identical to) the variance of λ^I conditional on $\widehat{\mathbb{F}}^I$, as we shall see.

4.1 Computing the information drift

The first result we need in order to prove Theorem 10 is a lemma that gives an explicit formula for the information drift in (75). Recall that we begin with a background filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$ that includes the Brownian filtration and the sigma-field generated by λ. We enlarge $\mathbb{F}$ with the information carried by the random variable I. Define, for a bounded Borel function $f : \mathbb{R} \to \mathbb{R}$, the process $(\pi_t(f))_{0 \le t \le T}$ as the continuous version of the martingale $(E[f(I)|\mathcal{F}_t])_{0 \le t \le T}$:

$$\pi_t(f) := E[f(I)|\mathcal{F}_t], \quad 0 \le t \le T.$$

There then exists a predictable family of measures $(\mu_t(dx))_{0 \le t \le T}$ such that

$$\pi_t(f) = \int_{\mathbb{R}} f(x)\mu_t(dx).$$

For fixed $t \in [0,T]$, the measure $\mu_t(dx)$ is the conditional distribution of I given $\mathcal{F}_t$. Suppose I is such that there exists a density function $g(t,x,y)$ for each $t \in [0,T]$, and such that

$$\pi_t(f) = \int_{\mathbb{R}} f(x)\mu_t(dx) = \int_{\mathbb{R}} f(x)g(t,x,B_t)dx. \tag{86}$$

The enlargement decomposition formula is given by the following lemma.

Lemma 2. *Suppose that I is continuous random variable with conditional (on $\mathcal{F}_t$) distribution given by $g(t, x, B_t)$. Assume also that this distribution satisfies the following conditions:*

$$\int_{\mathbb{R}} |g_y(t, x, y)| dx < \infty, \quad \int_{\mathbb{R}} \left| \frac{g_y(t, x, y)}{g(t, x, y)} \right| dx < \infty,$$

for a.e. $t \in [0, T]$ and a.e. $y \in \mathbb{R}$. Then the $\mathbb{F}$- Brownian motion B decomposes with respect to the enlarged filtration $\mathbb{F}^{\sigma(I)}$ according to

$$B_t = B_t^I + \int_0^t \nu_s ds, \quad 0 \le t \le T,$$

where B^I is an $\mathbb{F}^{\sigma(I)}$-Brownian motion. The information drift ν is given by

$$\nu_t = \frac{g_y(t, I, B_t)}{g(t, I, B_t)}, \quad 0 \le t \le T.$$

Proof. Let f be a test function. Introduce the $\mathbb{F}$-predictable process $(\dot{\pi}_t(f))_{0 \le t \le T}$ such that

$$\pi_t(f) = Ef(I) + \int_0^t \dot{\pi}_s(f) dB_s,$$

which exists by the representation property of Brownian martingales as stochastic integrals with respect to B. There exists a predictable family of measures $(\dot{\mu}_t(dx))_{0 \le t \le T}$ such that

$$\dot{\pi}_t(f) = \int_{\mathbb{R}} f(x) \dot{\mu}_t(dx),$$

such that for each $t \in [0, T]$ the measure $\dot{\mu}_t(dx)$ is absolutely continuous with respect to $\mu_t(dx)$. Define $\alpha(t, x)$ by

$$\dot{\mu}_t(dx) = \alpha(t, x) \mu_t(dx).$$

Now suppose we have a continuous $\mathbb{F}$-martingale M given by

$$M_t = \int_0^t m_s dB_s, \quad 0 \le t \le T.$$

By Theorem 1.6 in Mansuy and Yor [19], there exists an $\mathbb{F}^{\sigma(I)}$-local martingale M^I such that

$$M_t = M_t^I + \int_0^t \alpha(s, I) d[M, B]_s,$$

provided that, almost surely,

$$\int_0^t |\alpha(s, I)| d[M, B]_s < \infty.$$

In particular, if $\int_0^t |\alpha(s, I)| ds < \infty$ almost surely, then B decomposes as

$$B_t = B_t^I + \int_0^t \alpha(s, I) ds, \quad 0 \le t \le T,$$

with B^I an $\mathbb{F}^{\sigma(I)}$-Brownian motion.

From the definition of $\alpha(t, x)$ we have

$$\dot{\pi}_t(f) = \int_{\mathbb{R}} f(x) \alpha(t, x) \mu_t(dx) = \int_{\mathbb{R}} f(x) \alpha(t, x) g(t, x, B_t) dx.$$

Hence,

$$d\pi_t(f) = \dot{\pi}_t(f) dB_t = \left(\int_{\mathbb{R}} f(x) \alpha(t, x) g(t, x, B_t) dx \right) dB_t,$$

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so that

$$d[\pi(f), M]_t = \left(\int_{\mathbb{R}} f(x) \alpha(t, x) g(t, x, B_t) dx \right) d[B, M]_t. \tag{87}$$

But from the defining representation (86), the right-hand side of which is a smooth function of B_t, the Itô formula gives

$$d[\pi(f), M]_t = \left(\int_{\mathbb{R}} f(x) g_y(t, x, B_t) dx \right) d[B, M]_t, \tag{88}$$

and comparing (87) with (88) yields the result.

$$\square$$

Proof of Theorem 10. For I given by (77), the conditional distribution of I given $\mathcal{F}_t$, for $t \leq T$, is

$$\mathrm{N}(aB_t, a^2(T - t) + (1 - a)^2) = \mathrm{N}(aB_t, a^2(T_a - t)),$$

where T_a is defined in (82). Hence the conditional density is

$$g(t, x, B_t) = \frac{1}{a\sqrt{2\pi(T_a - t)}} \exp\left[-\frac{1}{2} \frac{(x - aB_t)^2}{a^2(T_a - t)} \right].$$

So by Lemma 2, the information drift is

$$\nu_t = \frac{I - aB_t}{a(T_a - t)}, \quad 0 \leq t \leq T. \tag{89}$$

Using the information drift in (89) we write the stock price SDE (72) in terms of $\mathbb{F}^{\sigma(I)}$-adapted processes, to obtain (76), where the $\mathbb{F}^{\sigma(I)}$-adapted market price of risk λ^I is given by

$$\lambda_t^I := \lambda + \nu_t = \lambda + \frac{I - aB_t}{a(T_a - t)} =: h(t, B_t), \quad 0 \leq t \leq T,$$

and where $h : [0, T] \times \mathbb{R} \to \mathbb{R}$ is defined by

$$h(t, x) := \lambda + \frac{I - ax}{a(T_a - t)}.$$

Applying the Itô's formula and using $dB_t = \nu_t dt + dB_t^I$, we obtain

$$d\lambda_t^I = -\frac{1}{T_a - t} dB_t^I, \quad \lambda_0^I = \lambda + \frac{I}{aT_a}. \tag{90}$$

With ξ being the returns process in (73), we have

$$d\xi_t = \lambda_t^I dt + dB_t^I. \tag{91}$$

We now regard λ as an unknown constant, and hence a random variable, whose distribution conditional on $\widehat{\mathcal{F}}_0^{\sigma(I)}$ is given by (78). Then we regard $(\lambda_t^I)_{0 \leq t \leq T}$ as an unobservable signal process following (90), and ξ as an observation process following (91), in a filtering framework to give the best estimate of λ_t^I conditional on $\widehat{\mathcal{F}}_t^{\sigma(I)}$.

Using (78), we can write down the initial distribution of λ_0^I given $\widehat{\mathcal{F}}_0^{\sigma(I)}$:

$$\mathrm{Law}(\lambda_0^I | \widehat{\mathcal{F}}_0^{\sigma(I)}) = \mathrm{Law}\left(\lambda + \frac{I}{aT_a} \,\bigg|\, \widehat{\mathcal{F}}_0^{\sigma(I)} \right) = \mathrm{N}\left(\lambda_0 + \frac{I}{aT_a}, \mathrm{v}_0 \right).$$

This defines the prior distribution of the signal process λ^I. Of course, since I is $\widehat{\mathcal{F}}_0^{\sigma(I)}$-measurable, it does not contribute to the initial variance.

The Kalman-Bucy filter, Theorem4, is directly applicable, and yields that the optimal filter

$$\widehat{\lambda}_t^I := E[\lambda_t^I | \widehat{\mathcal{F}}_t^{\sigma(I)}], \quad 0 \leq t \leq T,$$

satisfies the SDE

$$d\widehat{\lambda}_t^I = \left(V_t^I - \frac{1}{T_a - t}\right)d\widehat{B}_t^I, \quad \widehat{\lambda}_0^I = \lambda_0 + \frac{I}{aT_a}, \tag{92}$$

where $\widehat{B}^I$ is the innovations process, an $\widehat{\mathbb{F}}^{\sigma(I)}$-Brownian motion defined by

$$\widehat{B}_t^I := \xi_t - \int_0^t \widehat{\lambda}_s^I ds, \quad 0 \le t \le T, \tag{93}$$

and V_t^I is the conditional variance of λ_t^I:

$$V_t^I := E\left[\left.\left(\lambda_t^I - \widehat{\lambda}_t^I\right)^2\right| \widehat{\mathcal{F}}_t^{\sigma(I)}\right], \quad 0 \le t \le T,$$

which satisfies

$$\frac{dV_t^I}{dt} = \frac{2}{T_a - t}V_t^I - \left(V_t^I\right)^2, \quad V_0^I = v_0.$$

If we define

$$v_t^I := V_t^I - \frac{1}{T_a - t}, \quad 0 \le t \le T,$$

then (92) becomes

$$d\widehat{\lambda}_t^I = v_t^I d\widehat{B}_t^I, \quad \widehat{\lambda}_0^I = \lambda_0 + \frac{I}{aT_a}. \tag{94}$$

Note that (94) is of the same form as (47) with w_t replaced by v_t^I and with B_t replaced by $\widehat{B}_t^I$. Indeed, v_t^I plays the role of an 'effective variance', satisfying the Riccati equation (66), with a modified initial condition:

$$\frac{dv_t^I}{dt} = -\left(v_t^I\right)^2, \quad v_0^I = v_0 - \frac{1}{T_a}.$$

The solution to this equation is then given by (83), and the solution to (94) is then (84).

Using (93) in the SDE (94), the optimal filter may also be written explicitly in terms of the observable ξ, as

$$\widehat{\lambda}_t^I = \frac{\widehat{\lambda}_0^I + v_0^I \xi_t}{1 + v_0^I t}, \quad 0 \le t \le T.$$

This is of the same form as (69), with λ_0 replaced by $\widehat{\lambda}_0^I$ and v_0 replaced by v_0^I.

The effect of the filtering is that the agent is now investing in a stock with dynamics given by $dS_t = \sigma S_t d\xi_t$ which, using (93), becomes (80). The $\widehat{\mathbb{F}}^{\sigma(I)}$-adapted wealth process X^I then follows

$$dX_t^I = \sigma \theta_t^I X_t^I (\widehat{\lambda}_t^I dt + d\widehat{B}_t^I), \quad X_0^I = x,$$

where θ^I is the $\widehat{\mathbb{F}}^{\sigma(I)}$-adapted trading strategy. The theorem then follows immediately from making the replacements

$$w \to v^I, \quad \lambda \to \widehat{\lambda}^I,$$

in Theorem 8.

$\square$

It can be shown that the additional information increases the insider's utility over the regular agent: see [3] for this and other effects of the inside information.

5 Optimal hedging of basis risk with partial information

In this section we analyse the hedging of a contingent claim in a basis risk model, a tractable example of an incomplete market, first under a full information assumption, and then under a partial information scenario. Basis risk models involve a claim on a non-traded asset, which is hedged using a correlated traded asset. They were first studied systematically by Davis [5]

(whose preprint on the subject originated in 2000) who used a dual approach to derive approximations for indifference prices. Subsequently, Henderson [8], and Musiela and Zariphopoulou [27] derived an expectation representation (given in Theorem 11) for the value function of the utility maximisation problem involving a random endowment of the claim. This was used by Monoyios [22] to derive accurate analytic approximations for indifference prices and hedging strategies. In simulation experiments, Monoyios showed that exponential indifference hedging could outperform the BS approximation of taking the traded asset as a good proxy for the non-traded asset. Unfortunately, the utility-based hedge requires knowledge of the drift parameters of the assets. These are hard to estimate accurately, as shown by Rogers [31] and Monoyios [23], who showed that drift parameter mis-estimation could ruin the effectiveness of the optimal hedge. Finally, in [24, 26] Monoyios developed a filtering algorithm to deal with the drift parameter uncertainty, and showed that with this added ingredient, utility-based hedging was indeed effective, even in the face of parameter uncertainty. We shall describe some of these results in this section.

5.1 Basis risk model: full information case

In a full information model, the setting is a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{0 \leq t \leq T}, P)$, where the filtration $\mathbb{F}$ is the P-augmentation of that generated by a two-dimensional Brownian motion $(B, B^\perp)$. A traded stock price $S := (S_t)_{0 \leq t \leq T}$ follows a log-Brownian process given by

$$dS_t = \sigma S_t(\lambda dt + dB_t) =: \sigma S_t d\xi_t, \tag{95}$$

where $\sigma > 0$ and λ are known constants. For simplicity, the interest rate is taken to be zero. The process ξ in (95) defined by $d\xi_t := \lambda dt + dB_t$ will subsequently play a role as one component of an observation process in a partial information model, when λ will be treated as a random variable rather than as a known constant.

A non-traded asset price $Y := (Y_t)_{0 \leq t \leq T}$ follows the correlated log-Brownian motion

$$dY_t = \beta Y_t(\theta dt + dW_t) =: \beta Y_t d\zeta_t, \tag{96}$$

with $\beta > 0$ and θ known constants. The Brownian motion W is correlated with B according to

$$[B, W]_t = \rho t, \quad W = \rho B + \sqrt{1 - \rho^2} B^\perp, \quad \rho \in [-1, 1],$$

and the process ζ, given by $d\zeta_t := \theta dt + dW_t$, will act as the second component of an observation process in a partial information model, when θ will be considered a random variable. We shall henceforth refer to the Sharpe ratios λ (respectively, θ) as the drift of S (respectively, Y), for brevity.

A European contingent claim pays the non-negative random variable $h(Y_T)$ at time T, where $h : \mathbb{R}^+ \to \mathbb{R}^+$. In what follows we shall consider utility maximisation problems with the additional random terminal endowment $nh(Y_T)$, for $n \in \mathbb{R}$. We assume the random endowment $nh(Y_T)$ is continuous and bounded below, with finite expectation under any martingale measure.

An agent may trade the stock in a self-financing fashion, leading to the portfolio wealth process $X = (X_t)_{0 \leq t \leq T}$ satisfying

$$dX_t = \sigma \pi_t(\lambda dt + dB_t),$$

where $\pi := (\pi_t)_{0 \leq t \leq T}$ is the wealth in the stock, representing the agent's trading strategy, satisfying $\int_0^T \pi_t^2 dt < \infty$ almost surely.

5.1.1 Perfect correlation case

This market is incomplete for $|\rho| \neq 1$. If the correlation is perfect, however, the market becomes complete and perfect hedging is possible, as we now show.

The minimal martingale measure Q^M has density process with respect to P given by

$$\left. \frac{dQ^M}{dP} \right|_{\mathcal{F}_t} = \mathcal{E} \left(-\lambda \cdot B \right)_t, \quad 0 \leq t \leq T.$$

Under Q^M, (S, Y) follow

$$
\begin{aligned}
dS_t &= \sigma S_t dB_t^{Q^M}, \\
dY_t &= \beta\left(\theta - \rho\lambda\right) Y_t dt + \beta Y_t dW_t^{Q^M}, \quad (97)
\end{aligned}
$$

where B^{Q^M}, W^{Q^M} are correlated Brownian motions under Q^M. The stock price S is a local Q^M-martingale, but this is not the case for the non-traded asset, unless we have the perfect correlation case, $\rho = 1$. In this case Y is effectively a traded asset (as Y_t is then a deterministic function of S_t), so the Q^M-drift of Y vanishes. Therefore, given σ, β, when $\rho = 1$ the Sharpe ratios λ, θ are equal:

$$
\theta = \lambda.
$$

In this case the market becomes complete, and perfect hedging is possible. It is easy to show that with $\rho = 1$, so that $W = B$, we have

$$
Y_t = Y_0 \left(\frac{S_t}{S_0}\right)^{\beta/\sigma} e^{ct}, \quad c = \frac{1}{2}\sigma\beta\left(1 - \frac{\beta}{\sigma}\right).
$$

Let the claim price process be $v(t, Y_t), 0 \le t \le T$, where $v : [0, T] \times \mathbb{R}^+ \to \mathbb{R}^+$ is smooth enough to apply the Itô formula, so that

$$
dv(t, Y_t) = \left[v_t(t, Y_t) + \mathcal{A}^Y v(t, Y_t)\right] dt + \beta Y_t v_y(t, Y_t) dW_t,
$$

where $\mathcal{A}^Y$ is the generator of the process Y in (96). The replication conditions are

$$
X_t = v(t, Y_t), \quad 0 \le t \le T, \quad dX_t = dv(t, Y_t).
$$

Standard arguments then show that to perfectly hedge the claim one must hold Δ_t shares of S at $t \in [0, T]$, given by

$$
\Delta_t = \frac{\beta}{\sigma} \frac{Y_t}{S_t} \frac{\partial v}{\partial y}(t, Y_t), \quad 0 \le t \le T, \quad (98)
$$

and the claim pricing function $v(t, y)$ satisfies

$$
v_t(t, y) + \beta(\theta - \lambda) y v_y(t, y) + \frac{1}{2}\beta^2 y^2 v_{yy}(t, y) = 0, \quad v(T, y) = h(y).
$$

But with $\rho = 1$, $\theta = \lambda$, so we get the BS partial differential equation (PDE), and hence

$$
v(t, Y_t) = \mathrm{BS}(t, Y_t), \quad 0 \le t \le T,
$$

where $\mathrm{BS}(t, y)$ denotes the BS option pricing formula at time t, with underlying asset price y.

Therefore, a position in n claims is hedged by $\Delta_t^{(\mathrm{BS})}$ units of S at $t \in [0, T]$, where

$$
\Delta_t^{(\mathrm{BS})} = -n\frac{\beta}{\sigma} \frac{Y_t}{S_t} \frac{\partial}{\partial y} \mathrm{BS}(t, Y_t; \beta), \quad 0 \le t \le T, \quad (99)
$$

and where $\mathrm{BS}(t, y; \beta)$ denotes the BS formula at time t for underlying asset price y and volatility β. From our perspective, the salient feature of (99) is that the perfect hedge does not require knowledge of the values of the drifts λ, θ.

5.1.2 Incomplete case

Now suppose the correlation is not perfect, so that the market is incomplete. We embed the problem in a utility maximisation framework in a manner that is by now classical. Let the agent have risk preferences expressed via the exponential utility function

$$
U(x) = -\exp(-\alpha x), \quad x \in \mathbb{R}, \quad \alpha > 0.
$$

The agent maximises expected utility of terminal wealth at time T, with a random endowment of n units of claim payoff:

$$J(t, x, y; \pi) = E[U(X_T + nh(Y_T))|X_t = x, Y_t = y].$$

The value function is $u^{(n)}(t, x, y) \equiv u(t, x, y)$, defined by

$$
\begin{aligned}
u(t, x, y) &:= \sup_{\pi \in A} J(t, x, y; \pi), &&(100)\\
u(T, x, y) &= U(x + nh(y)).
\end{aligned}
$$

Denote the optimal trading strategy that achieves the supremum in (100) by $\pi^* \equiv \pi^{*,n}$, and denote the optimal wealth process by $X^* \equiv X^{*,n}$.

The following definitions of utility-based price and hedging strategy are now standard.

Definition 1 (Indifference price). The indifference price per claim at $t \in [0, T]$, given $X_t = x, Y_t = y$, $p(t, x, y) \equiv p^{(n)}(t, x, y)$, is defined by

$$u^{(n)}(t, x - np^{(n)}(t, x, y), y) = u^{(0)}(t, x, y)$$

We allow for possible dependence on t, x, y of $p^{(n)}$ in the above definition, but with exponential preferences it turns out that there is no dependence on x.

Definition 2 (Optimal hedging strategy). The optimal hedging strategy for n units of the claim is $\pi^H := (\pi_t^H)_{0 \le t \le T}$ given by

$$\pi_t^H := \pi_t^{*,n} - \pi_t^{*,0}, \quad 0 \le t \le T.$$

We have the following representation for the value function and indifference price.

Theorem 11. *The value function $u \equiv u^{(n)}$ and indifference price $p \equiv p^{(n)}$, given $X_t = x, Y_t = y$ for $t \in [0, T]$, are given by*

$$
\begin{aligned}
u^{(n)}(t, x, y) &= -e^{-\alpha x - \frac{1}{2}\lambda^2(T-t)} \left[F(t, Y_t)\right]^{1/(1-\rho^2)}, \\
F(t, y) &= E^{Q^M}\left[\exp\left(-\alpha(1 - \rho^2)nh(Y_T)\right)\big| Y_t = y\right], &&(101)\\
p^{(n)}(t, y) &= -\frac{1}{\alpha(1 - \rho^2)n} \log F(t, y). &&(102)
\end{aligned}
$$

Proof. The Hamilton-Jacobi-Bellman (HJB) equation for the value function u is

$$u_t + \sigma \sup_{\pi} \left(\lambda \pi u_x + \frac{1}{2}\sigma \pi^2 u_{xx} + \rho \beta \pi y u_{xy}\right) + \mathcal{A}^Y u = 0.$$

Performing the maximisation gives the optimal feedback control as $\Pi^*(t, x, y)$, where the function $\Pi^* : [0, T] \times \mathbb{R} \times \mathbb{R}^+$ is given by

$$\Pi^*(t, x, y) := -\left(\frac{\lambda u_x + \rho \beta y u_{xy}}{\sigma u_{xx}}\right).$$

The optimal trading strategy π^* is then given by $\pi_t^* = \Pi^*(t, X_t^*, Y_t)$. Substituting the optimal Markov control back into the Bellman equation gives the HJB PDE

$$u_t + \mathcal{A}^Y u - \frac{(\lambda u_x + \rho \beta y u_{xy})^2}{2u_{xx}} = 0.$$

The function $F(t, y)$ in (101) satisfies the linear PDE

$$F_t + \beta(\theta - \rho\lambda)F_y + \frac{1}{2}\beta^2 y^2 F_{yy} = 0, \quad F(T, y) = \exp(-\alpha(1 - \rho^2)nh(y)),$$

by virtue of the Feynman-Kac theorem. It is then straightforward to verify that u as given in the theorem solves the above HJB equation, and the definition of the indifference price gives the formula (102).

$\square$

Theorem 12. *The optimal hedging strategy for a position in n claims is to hold Δ_t^I shares at $t \in [0, T]$, given by*

$$\Delta_t^I = -n\rho \frac{\beta}{\sigma} \frac{Y_t}{S_t} \frac{\partial p^{(n)}}{\partial y}(t, Y_t), \quad 0 \le t \le T. \tag{103}$$

Proof. From Theorem 11 the value function may be written in terms of the indifference price as

$$u^{(n)}(t, x, y) = -\exp\left(-\alpha(x + np^{(n)}(t, y)) - \frac{1}{2}\lambda^2(T - t)\right).$$

Applying Definition 2 gives the result.

$\square$

Notice that, given the PDE satisfied by F, the indifference pricing function $p(t, y) \equiv p^{(n)}(t, y)$ satisfies

$$p_t + \beta(\theta - \rho\lambda)yp_y + \frac{1}{2}\beta^2 y^2 p_{yy} - \frac{1}{2}\beta^2 y^2 \alpha n(1 - \rho^2)(p_y)^2 = 0.$$

Then, if $\rho = 1$, we recover the perfect delta hedge (98), and that the claim price then satisfies the BS PDE.

In [22, 23] the hedging strategy in (103) is shown to be superior to the BS-style hedge (99), in terms of the terminal hedging error distribution produced by selling the claim at the appropriate price (the indifference price or the BS price) and investing the proceeds in the corresponding hedging portfolio. But from (97) we see that the exponential hedge requires knowledge of λ, θ, which are impossible to estimate accurately (see Rogers [31] or Monoyios [23]). This can ruin the effectiveness of indifference hedging, as shown in [23]. It is therefore dubious to draw any meaningful conclusions on the effectiveness of utility-based hedging in this model without relaxing the assumption that the agent knows the true values of the drifts.

5.2 Partial information case

Now we assume the hedger does not know the values of the return parameters λ, θ, so these are considered to be random variables. Equivalently, the agent cannot observe the Brownian motions B, W driving the asset prices, so is required to use strategies adapted to the observation filtration $\widehat{\mathbb{F}}$ generated by asset returns.

5.2.1 Choice of prior

We take the the two-dimensional random variable

$$\Xi := \begin{pmatrix} \lambda \\ \theta \end{pmatrix}$$

to have a Gaussian distribution which will be updated as the agent attempts to filter the values of the drifts from asset observations during the hedging interval $[0, T]$.

The choice of Gaussian prior is motivated by the idea that the agent has some past observations of S, Y before time 0, uses these to obtain classical point estimates of the drifts, and the joint distribution of the estimators is used as the prior in a Bayesian framework. Ultimately, in order to obtain explicit solutions, we shall assume that the agent uses observations before time 0 of equal length for both assets. In setting the prior this way, we make the approximation that the asset price observations are continuous, so that σ, β, ρ are known from the quadratic variation and co-variation of S, Y. This is because our goal here is to focus on the severest problem of drift parameter uncertainty.

So, consider, for the moment, an observer with data for S over a time interval of length t_S, and for Y over a window of length t_Y, who considers λ and θ as *constants*, and records the returns dS_t/S_t and dY_t/Y_t in order to estimate the values of the drifts. The best estimator of

λ is $\bar{\lambda}(t_S)$ given by

$$
\begin{aligned}
\bar{\lambda}(t_S) &= \frac{1}{t_S} \int_{t_0}^{t_0+t_S} \frac{dS_u}{\sigma S_u} \\
&= \lambda + \frac{B_{t_0+t_S}}{t_S} \\
&\sim N\left(\lambda, \frac{1}{t_S}\right).
\end{aligned}
$$

The estimator of λ is normally distributed, with a similar computation for the estimator of θ. The estimator, $(\bar{\lambda}, \bar{\theta})$, of the (supposed constant) vector (λ, θ) is bivariate normal. Defining $v_0 := 1/t_S$ and $w_0 := 1/t_Y$ it is easily checked that

$$
\begin{pmatrix} \bar{\lambda} \\ \bar{\theta} \end{pmatrix} \sim N(M, C_0),
$$

where the mean vector M and covariance matrix C_0 are given by

$$
M = \begin{pmatrix} \lambda \\ \theta \end{pmatrix}, \qquad C_0 = \begin{pmatrix} v_0 & \rho \min(v_0, w_0) \\ \rho \min(v_0, w_0) & w_0 \end{pmatrix}. \tag{104}
$$

With this in mind, we shall suppose that (λ, θ), now considered as a *random variable*, is bivariate normal according to

$$
\lambda \sim N(\lambda_0, v_0), \quad \theta \sim N(\theta_0, w_0), \quad \text{cov}(\lambda, \theta) = c_0 := \rho \min(v_0, w_0),
$$

for some chosen values λ_0, θ_0, typically obtained from past data prior to time zero. This distribution will be updated via subsequent observations of

$$
\xi_t := \frac{1}{\sigma} \int_0^t \frac{dS_u}{S_u} = \lambda t + B_t, \qquad \zeta_t := \frac{1}{\beta} \int_0^t \frac{dY_u}{Y_u} = \theta t + W_t,
$$

over the hedging interval $[0, T]$.

5.2.2 Two-dimensional Kalman-Bucy filter

We are firmly within the realm of a two-dimensional Kalman filtering problem, which we treat as follows. Define the observation filtration by

$$
\widehat{\mathbb{F}} := (\widehat{\mathcal{F}}_t)_{0 \le t \le T}, \qquad \widehat{\mathcal{F}}_t = \sigma(\xi_s, \zeta_s; 0 \le s \le t).
$$

The observation process, Λ, and unobservable signal process, Ξ, are defined by

$$
\Lambda := \begin{pmatrix} \xi_t \\ \zeta_t \end{pmatrix}_{0 \le t \le T}, \qquad \Xi := \begin{pmatrix} \lambda \\ \theta \end{pmatrix},
$$

satisfying the stochastic differential equations

$$
d\Lambda_t = \Xi dt + D d\mathbf{B}_t, \qquad d\Xi = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
$$

where

$$
D = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix}, \qquad \mathbf{B}_t = \begin{pmatrix} B_t \\ B_t^\perp \end{pmatrix}.
$$

The optimal filter is $\widehat{\Xi}_t := E[\Xi | \widehat{\mathcal{F}}_t], 0 \le t \le T$, a two-dimensional process defining the best estimates of λ and θ given observations up to time $t \in [0, T]$:

$$
\widehat{\Xi}_t \equiv \begin{pmatrix} \widehat{\lambda}_t \\ \widehat{\theta}_t \end{pmatrix} := \begin{pmatrix} E[\lambda | \widehat{\mathcal{F}}_t] \\ E[\theta | \widehat{\mathcal{F}}_t] \end{pmatrix}, \qquad \begin{pmatrix} \widehat{\lambda}_0 \\ \widehat{\theta}_0 \end{pmatrix} = \begin{pmatrix} \lambda_0 \\ \theta_0 \end{pmatrix}. \tag{105}
$$

The solution to this filtering problem converts the partial information model to a full information model with random drifts, given in the following proposition. To avoid a proliferation of symbols, we abuse notation and write $\widehat{\lambda}_t \equiv \widehat{\lambda}(t, S_t)$ and $\widehat{\theta} \equiv \widehat{\theta}(t, Y_t)$ for processes $\widehat{\lambda}, \widehat{\theta}$ that will turn out to be functions of time and current asset price.

Proposition 2. *The partial information model is equivalent to a full information model in which the asset price dynamics in the observation filtration $\widehat{\mathbb{F}}$ are*

$$dS_t = \sigma S_t(\widehat{\lambda}_t dt + d\widehat{B}_t), \tag{106}$$

$$dY_t = \beta Y_t(\widehat{\theta}_t dt + d\widehat{W}_t), \tag{107}$$

where $\widehat{B}, \widehat{W}$ are $\widehat{\mathbb{F}}$-Brownian motions with correlation ρ, and the random drifts $\widehat{\lambda}, \widehat{\theta}$ are $\widehat{\mathbb{F}}$-adapted processes.

If λ and θ have common initial variance v_0, then $\widehat{\lambda}, \widehat{\theta}$ are given by

$$\begin{pmatrix} \widehat{\lambda}_t \\ \widehat{\theta}_t \end{pmatrix} = \begin{pmatrix} \lambda_0 \\ \theta_0 \end{pmatrix} + \int_0^t v_s \begin{pmatrix} d\widehat{B}_s \\ d\widehat{W}_s \end{pmatrix}, \quad 0 \le t \le T, \tag{108}$$

where $(v_t)_{0 \le t \le T}$ is the deterministic function

$$v_t := \frac{v_0}{1 + v_0 t}.$$

Equivalently, $\widehat{\lambda}, \widehat{\theta}$ are given as functions of time and current asset price by

$$\widehat{\lambda}_t = \widehat{\lambda}(t, S_t) = \frac{\lambda_0 + v_0 \xi_t}{1 + v_0 t}, \qquad \widehat{\theta}_t = \widehat{\theta}(t, Y_t) = \frac{\theta_0 + v_0 \zeta_t}{1 + v_0 t}, \tag{109}$$

with

$$\xi_t = \frac{1}{\sigma} \log\left(\frac{S_t}{S_0}\right) + \frac{1}{2}\sigma t, \qquad \zeta_t = \frac{1}{\beta} \log\left(\frac{Y_t}{Y_0}\right) + \frac{1}{2}\beta t. \tag{110}$$

Proof. By the Kalman-Bucy filter, Theorem 5, $\widehat{\Xi}$ satisfies the stochastic differential equation

$$d\widehat{\Xi}_t = C_t \left(DD^T\right)^{-1} \left(d\Lambda_t - \widehat{\Xi}_t dt\right) =: C_t \left(DD^T\right)^{-1} dN_t, \tag{111}$$

where $(N_t)_{0 \le t \le T}$ is the innovations process, defined by

$$\begin{aligned} N_t &:= \Lambda_t - \int_0^t \widehat{\Xi}_s ds \\ &= \begin{pmatrix} \xi_t - \int_0^t \widehat{\lambda}_s ds \\ \zeta_t - \int_0^t \widehat{\theta}_s ds \end{pmatrix} \\ &=: \begin{pmatrix} \widehat{B}_t \\ \widehat{W}_t \end{pmatrix}, \end{aligned} \tag{112}$$

and $\widehat{B}, \widehat{W}$ are $\widehat{\mathbb{F}}$-Brownian motions with correlation ρ. The deterministic matrix function C_t is the conditional variance-covariance matrix defined by

$$C_t := E\left[(\Xi - \widehat{\Xi}_t)(\Xi - \widehat{\Xi}_t)^T \middle| \widehat{\mathcal{F}}_t \right] = E\left[(\Xi - \widehat{\Xi}_t)(\Xi - \widehat{\Xi}_t)^T \right],$$

(T denoting transpose) where the last equality follows because the error $\Xi - \widehat{\Xi}_t$ is independent of $\widehat{\mathcal{F}}_t$.

Using (112), and writing dS_t in terms of $d\xi_t$, as in (95), gives the dynamics (106) of S in the observation filtration; (107) is established similarly.

The matrix $C = (C_t)_{0 \le t \le T}$ satisfies the Riccati equation

$$\frac{dC_t}{dt} = -C_t \left(DD^T \right)^{-1} C_t,$$

with C_0 given in (104). Then $R_t := C_t^{-1}$ satisfies the Lyapunov equation

$$\frac{dR_t}{dt} = \left(DD^T \right)^{-1}.$$

Define the elements of the conditional covariance matrix by

$$C_t =: \left(\begin{array}{cc} v_t & c_t \\ c_t & w_t \end{array} \right).$$

Then the filtering equation (111) is a pair of coupled stochastic differential equations:

$$\begin{aligned}
\left(\begin{array}{c} d\widehat{\lambda}_t \\ d\widehat{\theta}_t \end{array} \right) &= \frac{1}{1 - \rho^2} \left(\begin{array}{cc} v_t - \rho c_t & c_t - \rho v_t \\ c_t - \rho w_t & w_t - \rho c_t \end{array} \right) \left(\begin{array}{c} d\xi_t - \widehat{\lambda}_t dt \\ d\zeta_t - \widehat{\theta}_t dt \end{array} \right) \\
&= \frac{1}{1 - \rho^2} \left(\begin{array}{cc} v_t - \rho c_t & c_t - \rho v_t \\ c_t - \rho w_t & w_t - \rho c_t \end{array} \right) \left(\begin{array}{c} d\widehat{B}_t \\ d\widehat{W}_t \end{array} \right).
\end{aligned}$$

Solving the Lyapunov equation yields 3 equations for v_t, w_t, c_t:

$$\begin{aligned}
\frac{v_t}{v_t w_t - c_t^2} - \frac{v_0}{v_0 w_0 - c_0^2} &= \frac{t}{1 - \rho^2}, \\
\frac{w_t}{v_t w_t - c_t^2} - \frac{w_0}{v_0 w_0 - c_0^2} &= \frac{t}{1 - \rho^2}, \\
\frac{c_t}{v_t w_t - c_t^2} - \frac{c_0}{v_0 w_0 - c_0^2} &= \frac{\rho t}{1 - \rho^2},
\end{aligned} \tag{113}$$

where we have written $c_0 \equiv \rho \min(v_0, w_0)$ for brevity.

Now make the simplification $w_0 = v_0$. From the discussion in Section 5.2.1, we see that this corresponds to using past observations over the same length of time, $t_S = t_Y$, for both S and Y in fixing the prior. Then $c_0 = \rho v_0$, and the solution to the system of equations (113) gives the entries of the matrix C_t as

$$v_t = \frac{v_0}{1 + v_0 t}, \quad w_t = v_t, \quad c_t = \rho v_t.$$

With this simplification, the equation for the optimal filter simplifies to

$$\left(\begin{array}{c} d\widehat{\lambda}_t \\ d\widehat{\theta}_t \end{array} \right) = v_t \left(\begin{array}{c} d\xi_t - \widehat{\lambda}_t dt \\ d\zeta_t - \widehat{\theta}_t dt \end{array} \right) = v_t \left(\begin{array}{c} d\widehat{B}_t \\ d\widehat{W}_t \end{array} \right),$$

which, along with the initial condition in (105), yields (108) and (109).

Finally, the expressions in (110) for ξ_t, ζ_t follow directly from the solutions of (95) and (96) for S and Y.

$\square$

Armed with Proposition 2 we may now treat the model as a full information model with random drift parameters $(\widehat{\lambda}_t, \widehat{\theta}_t)$, and this is done in the next section.

5.2.3 Optimal hedging with random drifts

On the stochastic basis $(\Omega, \widehat{\mathcal{F}}, \widehat{\mathbb{F}}, P)$, the wealth process associated with trading strategy $\pi := (\pi_t)_{0 \le t \le T}$, an $\widehat{\mathbb{F}}$-adapted process satisfying the integrability condition $\int_0^T \pi_t^2 dt < \infty$ a.s., is $X = (X_t)_{0 \le t \le T}$, satisfying

$$dX_t = \sigma \pi_t (\widehat{\lambda}_t dt + d\widehat{B}_t). \tag{114}$$

The class $\mathcal{M}$ of local martingale measures for this model consists of measures Q with density processes defined by

$$Z_t := \frac{dQ}{dP}\bigg|_{\widehat{\mathcal{F}}_t} = \mathcal{E}(-\widehat{\lambda} \cdot \widehat{B} - \psi \cdot \widehat{B}^\perp)_t, \quad 0 \le t \le T, \tag{115}$$

for integrands ψ satisfying $\int_0^t \psi_s^2 ds < \infty$ a.s., for all $t \in [0,T]$ (it is not hard to show that $\int_0^t \widehat{\lambda}_s^2 ds < \infty, 0 \le t \le T$). For $\psi = 0$ we obtain the minimal martingale measure Q^M.

Under $Q \in \mathcal{M}$, $(\widehat{B}^Q, \widehat{B}^{\perp,Q})$ is two-dimensional Brownian motion, where

$$d\widehat{B}_t^Q := d\widehat{B}_t^Q + \widehat{\lambda}_t dt, \qquad d\widehat{B}_t^{\perp,Q} := d\widehat{B}_t^\perp + \psi_t dt,$$

and the asset prices and random drifts satisfy

$$\begin{aligned}
dS_t &= \sigma S_t d\widehat{B}_t^Q, \\
dY_t &= \beta Y_t[(\widehat{\theta}_t - \rho\widehat{\lambda}_t - \sqrt{1-\rho^2}\psi_t)dt + d\widehat{W}_t^Q], \\
d\widehat{\lambda}_t &= \mathrm{v}_t[-\widehat{\lambda}_t dt + d\widehat{B}_t^Q], \\
d\widehat{\theta}_t &= \mathrm{v}_t[-(\rho\widehat{\lambda}_t + \sqrt{1-\rho^2}\psi_t)dt + d\widehat{W}_t^Q],
\end{aligned}$$

where $\widehat{W}^Q = \rho\widehat{B}^Q + \sqrt{1-\rho^2}\widehat{B}^{\perp,Q}$.

The relative entropy between $Q \in \mathcal{M}$ and P is defined by

$$\begin{aligned}
\mathcal{H}(Q,P) &:= E\left[\frac{dQ}{dP}\log\frac{dQ}{dP}\right] \\
&= E^Q\left[-\int_0^T \widehat{\lambda}_t d\widehat{B}_t^Q - \int_0^T \psi_t d\widehat{B}_t^{\perp,Q} + \frac{1}{2}\int_0^T \left(\widehat{\lambda}_t^2 + \psi_t^2\right)dt\right].
\end{aligned}$$

Using the Q-dynamics of $\widehat{\lambda}_t$ it is straightforward to establish that $E^Q \int_0^t \widehat{\lambda}_s^2 ds < \infty$ for all $t \in [0,T]$. If, in addition, we have the integrability condition

$$E^Q \int_0^t \psi_s^2 ds < \infty, \quad 0 \le t \le T, \tag{116}$$

then

$$\mathcal{H}(Q,P) = E^Q\left[\frac{1}{2}\int_0^T \left(\widehat{\lambda}_t^2 + \psi_t^2\right)dt\right] < \infty. \tag{117}$$

In this case we write $Q \in \mathcal{M}_f$, where $\mathcal{M}_f$ denotes the set of martingale measures Q with finite relative entropy with respect to P, and we define $\mathcal{H}(Q,P) := \infty$ otherwise. From (117) we note that the minimal entropy measure Q^E is given by

$$\mathcal{H}(Q^E,P) = E^Q\left[\frac{1}{2}\int_0^T \widehat{\lambda}_t^2 dt\right],$$

corresponding to $\psi \equiv 0$ in (117). This means that the minimal martingale measure and the minimal entropy measure in this model coincide: $Q^E = Q^M$.

For an initial time $t \in [0,T]$, we define the conditional entropy between $Q \in \mathcal{M}$ and P by

$$H_t(Q,P) := E\left[\frac{Z_T}{Z_t}\log\left(\frac{Z_T}{Z_t}\right)\bigg|\widehat{\mathcal{F}}_t\right], 0 \le t \le T, \tag{118}$$

satisfying $H_0(Q,P) \equiv \mathcal{H}(Q,P)$. Provided the integrability condition (116) is satisfied, then

$$H_t(Q,P) = E^Q\left[\frac{1}{2}\int_t^T \left(\widehat{\lambda}_u^2 + \psi_u^2\right)du\bigg|\widehat{\mathcal{F}}_t\right],$$

and we define $H_t(Q, P) := \infty$ otherwise. In particular, therefore, recalling that $\widehat{\lambda}_t \equiv \widehat{\lambda}(t, S_t)$ is a smooth and Lipschitz function of time and current stock price, and that the Q-dynamics of $\widehat{\lambda}_t$ do not depend on ψ_t for any $Q \in \mathcal{M}$, the minimal conditional entropy $(H_t(Q^E, P))_{0 \leq t \leq T}$ will be a deterministic function of time and stock price, given by $H_t(Q^E, P) \equiv H^E(t, S_t)$ for a $C^{1,2}([0,T] \times \mathbb{R}^+)$ function H^E defined by

$$H^E(t, s) := E^{Q^E}\left[\frac{1}{2}\int_t^T \widehat{\lambda}^2(u, S_u)du \,\bigg|\, S_t = s\right]. \tag{119}$$

5.2.4 The primal problem

We use an exponential utility function, $U(x) = -\exp(-\alpha x)$, $x \in \mathbb{R}$, $\alpha > 0$. The primal value function $u \equiv u^{(n)}$ is defined as the maximum expected utility of wealth at T from trading S and receiving n units of the claim on Y, when starting at time $t \in [0, T]$:

$$u^{(n)}(t, x, s, y) := \sup_{\pi \in \mathcal{A}} E[U(X_T + nh(Y_T))|X_t = x, S_t = s, Y_t = y], \tag{120}$$

where $\mathcal{A}$ denotes the set of admissible trading strategies. The dynamics of the state variables X, S, Y are given by (114) and (106,107). For starting time 0 we write $u^{(n)}(x) \equiv u^{(n)}(0, x, \cdot, \cdot)$.

The set of admissible strategies is defined as follows. Denote by $\Delta := \pi/S$ be the adapted S-integrable process for the number of shares held. The space of permitted strategies is

$$\mathcal{A} = \{\Delta : (\Delta \cdot S) \text{ is a } (Q, \widehat{\mathbb{F}})\text{-martingale for all } Q \in \mathcal{M}_f\},$$

where $(\Delta \cdot S)_t = \int_0^t \Delta_u dS_u$ is the gain from trading over $[0, t], t \in [0, T]$.

Denote the optimal trading strategy by $\pi^* \equiv \pi^{*,n}$, and the optimal wealth process by $X^* \equiv X^{*,n}$. The utility-based price $p^{(n)}$ and optimal hedge for a position in n claims are defined along the lines of Definitions 1 and 2. The indifference price per claim at $t \in [0, T]$, given $X_t = x, S_t = s, Y_t = y$, is $p^{(n)}$ given by

$$u^{(n)}(t, x - np^{(n)}(t, x, s, y), s, y) = u^{(0)}(t, x, s).$$

The optimal hedging strategy is to hold $(\Delta_t^H)_{0 \leq t \leq T}$ shares of stock at time t, where $\Delta_t^H S_t =: \pi_t^H S_t$, and $\pi^H := (\pi_t^H)_{0 \leq t \leq T}$, is defined by

$$\pi_t^H := \pi_t^{*,n} - \pi_t^{*,0}, \quad 0 \leq t \leq T. \tag{121}$$

It is well known that with exponential utility the indifference price is independent of the initial cash wealth x, so we shall write $p^{(n)}(t, x, s, y) \equiv p^{(n)}(t, s, y)$ from now on.

For small positions in the claim (or, equivalently, for small risk aversion), we shall later approximate the indifference price by the marginal utility-based price introduced by Davis [4]. This is the indifference price for infinitesimal diversions of funds into the purchase or sale of claims, and is equivalent (as is well-known, see for example Monoyios [25]) to the limit of the indifference price as $n \to 0$.

Definition 3 (Marginal price)**.** The marginal utility-based price of the claim at $t \in [0, T]$ is $\widehat{p}(t, s, y)$ defined by

$$\widehat{p}(t, s, y) := \lim_{n \to 0} p^{(n)}(t, s, y).$$

It is well known that with exponential utility the marginal price is also equivalent to the limit of the indifference price as risk aversion goes to zero. Under appropriate conditions (satisfied in this model) it is given by the expectation of the payoff under the optimal measure of the dual problem without the claim. For exponential utility this measure is the minimal entropy measure Q^E and, as we have already seen, in our model $Q^E = Q^M$, giving the representation $\widehat{p}(t, s, y) = E^{Q^M}[h(Y_T)|S_t = s, Y_t = y]$, as we shall see in the next section.

5.2.5 Dual problem and optimal hedge

We attack the primal utility maximisation problem (120) using classical duality results. For a problem with the random terminal endowment of a European claim, and with exponential utility, as in this paper, Delbaen et al [6] establish the required duality relations between the primal and dual problems in a semimartingale setting. We shall use these results below to establish a simple algebraic relation (Lemma 3) between the primal value function and the indifference price, which we shall then exploit to derive the representation for the optimal hedging strategy.

The dual problem with starting time 0 has value function defined by

$$\tilde{u}^{(n)}(\eta) := \inf_{Q \in \mathcal{M}} E\left[\widetilde{U}(\eta Z_T) + \eta Z_T n h(Y_T)\right],$$

where Z is the density process in (115) and $\widetilde{U}$ is the convex conjugate of the utility function. For exponential utility $\widetilde{U}$ is given by

$$\widetilde{U}(\eta) = \frac{\eta}{\alpha}\left[\log\left(\frac{\eta}{\alpha}\right) - 1\right].$$

Hence the dual value function has the well-known entropic representation

$$\tilde{u}^{(n)}(\eta) = \widetilde{U}(\eta) + \frac{\eta}{\alpha}\inf_{Q \in \mathcal{M}}\left[\mathcal{H}(Q, P) + \alpha n E^Q h(Y_T)\right].$$

Denoting the dual minimiser that attains the above infimum by $Q^{*,n}$, we observe that $Q^{*,n} \in \mathcal{M}_f$.

For a starting time $t \in [0, T]$ the dual value function is defined by

$$\tilde{u}^{(n)}(t, \eta, s, y) := \inf_{Q \in \mathcal{M}} E\left[\widetilde{U}\left(\eta\frac{Z_T}{Z_t}\right) + \eta\frac{Z_T}{Z_t}n h(Y_T)\,\middle|\, S_t = s, Y_t = y\right], \tag{122}$$

and we write $\tilde{u}^{(n)}(\eta) \equiv \tilde{u}^{(n)}(0, \eta, \cdot, \cdot)$.

Lemma 3. *The primal value function and indifference price are related by*

$$u^{(n)}(t, x, s, y) = u^{(0)}(t, x, s)\exp\left(-\alpha n p^{(n)}(t, s, y)\right), \tag{123}$$

where the value function without the claim is given by

$$u^{(0)}(t, x, s) = -\exp\left(-\alpha x - H^E(t, s)\right), \tag{124}$$

and $H^E(t, s)$ is the conditional minimal entropy function defined in (119).

Proof. For brevity, we give the proof for $t = 0$. The proof for a general starting time follows similar lines, and we make some comments on how to adapt the following argument for that case at the end of the proof.

The fundamental duality linking the primal and dual problems in Delbaen et al [6] implies that the value functions $u^{(n)}(x)$ and $\tilde{u}^{(n)}(\eta)$ are conjugate:

$$\tilde{u}^{(n)}(\eta) = \sup_{x \in \mathbb{R}}[u^{(n)}(x) - x\eta], \qquad u^{(n)}(x) = \inf_{\eta > 0}[\tilde{u}^{(n)}(\eta) + x\eta].$$

The value of η attaining the above infimum is η^*, given by $\tilde{u}^{(n)}_\eta(\eta^*) = -x$, so that

$$u^{(n)}(x) = \tilde{u}^{(n)}(\eta^*) + x\eta^*,$$

which translates to

$$u^{(n)}(x) = -\exp\left(-\alpha x - \inf_{Q \in \mathcal{M}}\left[\mathcal{H}(Q, P) + \alpha n E^Q h(Y_T)\right]\right). \tag{125}$$

So, in particular,
$$u^{(0)}(x) = -\exp\left[-\alpha x - \mathcal{H}(Q^E, P)\right], \tag{126}$$

where Q^E is the minimal entropy measure: $Q^E = Q^{*,0}$

Combining the dual representations (125) and (126) for the primal problems with and without the claim, with the definition of the indifference price, gives the dual representation for the utility-based price in the form

$$p^{(n)} = \frac{1}{\alpha n}\left[\inf_{Q\in\mathcal{M}}\left[\mathcal{H}(Q, P) + \alpha n E^Q h(Y_T)\right] - \mathcal{H}(Q^E, P)\right], \tag{127}$$

which is the representation found in Delbaen et al [6], modified slightly as we have a random endowment of n claims ([6] considered the case $n = -1$).

In particular, for $n \to 0$ or $\alpha \to 0$, we obtain the marginal price of Davis [4]:

$$\widehat{p} := \lim_{n\to 0} p^{(n)} = E^{Q^E} h(Y_T) = E^{Q^M} h(Y_T), \tag{128}$$

the last inequality following from the equality of Q^M and Q^E, as implied by (117).

From (125)–(127), the relation between the primal value functions and indifference price then follows immediately, as

$$\begin{aligned}
u^{(n)}(x) &= -\exp\left(-\alpha x - \mathcal{H}(Q^E, P) - \alpha n p^{(n)}\right) \\
&= u^{(0)}(x)\exp\left(-\alpha n p^{(n)}\right).
\end{aligned}$$

Similarly, a corresponding relation for a starting time $t \in [0, T]$ may also be derived. This is achieved using the definition (122) of the dual value function for an initial time $t \in [0, T]$, the conjugacy of $u^{(n)}(t, x, s, y)$ and $\tilde{u}^{(n)}(t, \eta, s, y)$ and the definitions (118) and (119) of the conditional entropy and conditional minimal entropy.

$\square$

Using Lemma 3 we obtain the following representation for the optimal hedging strategy associated with the indifference price. In what follows we assume that the indifference price is a suitably smooth function of (t, s, y), so that (given Lemma 3) we may assume the primal value function is smooth enough to be a classical solution of the associated Hamilton-Jacobi-Bellman (HJB) equation. This smoothness property is confirmed in [24].

Theorem 13. *The optimal hedge for a position in n claims is to hold Δ_t^H units of S at $t \in [0, T]$, where*
$$\Delta_t^H = -n\left(p_s^{(n)}(t, S_t, Y_t) + \rho\frac{\beta}{\sigma}\frac{Y_t}{S_t}p_y^{(n)}(t, S_t, Y_t)\right).$$

Remark 3. We note the extra term in the hedging formula compared with the corresponding full information result (103). The drift parameter uncertainty results in additional risk, manifested as dependence of the indifference price on the stock price, and hence the derivative with respect to the stock price appears in the theorem.

Proof. The HJB equation associated with the primal the value function is
$$u_t^{(n)} + \max_\pi \mathcal{A}_{X,S,Y} u^{(n)} = 0,$$

where $\mathcal{A}_{X,S,Y}$ is the generator of (X, S, Y) under P. Performing the maximisation over π yields the optimal Markov control as $\pi_t^{*,n} = \pi^{*,n}(t, X_t^{*,n}, S_t, Y_t)$, where

$$\pi^{*,n}(t, x, s, y) = -\left(\frac{\widehat{\lambda}u_x^{(n)} + \sigma s u_{xs}^{(n)} + \rho\beta y u_{xy}^{(n)}}{\sigma u_{xx}^{(n)}}\right),$$

and where the arguments of the functions on the right-hand-side are omitted for brevity. For the case $n = 0$ there is no dependence on y in the value function $u^{(0)}$, and we have $\pi_t^{*,0} = \pi^{*,0}(t, X_t^{*,0}, S_t)$, where

$$\pi^{*,0}(t, x, s) = -\left(\frac{\widehat{\lambda} u_x^{(0)} + \sigma s u_{xs}^{(0)}}{\sigma u_{xx}^{(0)}}\right).$$

Applying the definition (121) of the optimal hedging strategy along with the representations (123) and (124) from Lemma 3 for the value functions, gives the result.

$\square$

5.2.6 Stochastic control representation of the indifference price

The dual representation (127) of $p^{(n)}$ gives the price of the claim at time 0 as the value function of a control problem:

$$p^{(n)} = \inf_\psi E^Q\left[\frac{1}{2\alpha n}\int_0^T \psi_t^2 dt + h(Y_T)\right],$$

to be minimised over control processes $(\psi_t)_{0 \leq t \leq T}$, such that $Q \in \mathcal{M}_f$, and with dynamics for S, Y given by

$$dS_t = \sigma S_t d\widehat{B}_t^Q,$$
$$dY_t = \beta Y_t[(\widehat{\theta}(t, Y_t) - \rho\widehat{\lambda}(t, S_t) - \sqrt{1-\rho^2}\psi_t)dt + d\widehat{W}_t^Q].$$

For a starting time $t \in [0, T]$ we have

$$p^{(n)}(t, s, y) = \inf_\psi E^Q\left[\frac{1}{2\alpha n}\int_t^T \psi_u^2 du + h(Y_T)\,\middle|\, S_t = s, Y_t = y\right].$$

The HJB dynamic programming PDE associated with $p^{(n)}(t, s, y)$ is

$$p_t^{(n)} + \mathcal{A}_{S,Y}^{Q^M} p^{(n)} + \inf_\psi\left[\frac{1}{2\alpha n}\psi^2 - \beta\sqrt{1-\rho^2}\psi y p_y^{(n)}\right] = 0, \quad p(T, s, y) = h(y),$$

where $\mathcal{A}_{S,Y}^{Q^M}$ is generator of (S, Y) under minimal measure:

$$\mathcal{A}_{S,Y}^{Q^M} f(t, s, y) = \beta(\widehat{\theta}(t, y) - \rho\widehat{\lambda}(t, s))y f_y + \frac{1}{2}s^s f_{ss} + \frac{1}{2}\beta^2 y^2 f_{yy} + \rho\sigma\beta sy f_{sy}.$$

The optimal Markov control is $\psi_t^{*,n} \equiv \psi^{*,n}(t, S_t, Y_t)$, where

$$\psi^{*,n}(t, s, y) = \alpha n\sqrt{1-\rho^2}\beta y p_y^{(n)}(t, s, y),$$

and note that $\psi^{*,0} = 0$. Substituting back into the HJB equation, we find that $p^{(n)}$ solves the semi-linear PDE

$$p_t^{(n)} + \mathcal{A}_{S,Y}^{Q^M} p^{(n)} - \frac{1}{2}\alpha n(1-\rho^2)\beta^2 y^2 \left(p_y^{(n)}\right)^2 = 0, \quad p^{(n)}(T, s, y) = h(y).$$

We note that for $n = 0$ this becomes a linear PDE for the marginal price $\widehat{p}$, so that by the Feynman-Kac Theorem we have

$$\widehat{p}(t, s, y) = E_{t,s,y}^{Q^M} h(Y_T), \tag{129}$$

consistent with the general result (128). We shall see that in this case the marginal price is given by a BS-type formula.

5.2.7 Analytic approximation for the indifference price

To obtain analytic results we approximate the indifference price by the marginal price in (129). The marginal price (and hence the associated trading strategy) can be computed in analytic form since, under Q^M, $\log Y_T$ is Gaussian. We have the following result.

Proposition 3. *Under Q^M, conditional on $S_t = s, Y_t = y$, $\log Y_T \sim \mathrm{N}(m, \Sigma^2)$, where $m \equiv m(t, s, y)$ and $\Sigma^2 \equiv \Sigma^2(t)$ are given by*

$$
\begin{aligned}
m(t, s, y) &= \log y + \beta \left(\widehat{\theta}(t, y) - \rho \widehat{\lambda}(t, s) - \frac{1}{2}\beta \right)(T - t) \\
\Sigma^2(t) &= \left[1 + (1 - \rho^2)\mathrm{v}_t(T - t) \right]\beta^2(T - t)
\end{aligned}
$$

Proof. This is established by computing the SDEs for Y and for $\widehat{\theta}_t - \rho\widehat{\lambda}_t$ under Q^M. Indeed, applying the Itô formula to $\log Y_t$ under Q^M, we obtain, for $t < T$,

$$
\log Y_T = \log Y_t + \beta \int_t^T \left(\widehat{\theta}_u - \rho\widehat{\lambda}_u \right) du - \frac{1}{2}\beta^2(T - t) + \beta \int_t^T d\widehat{W}_u^{Q^M}, \tag{130}
$$

where $\widehat{W}^{Q^M}$ is a Brownian motion under Q^M. The dynamics of $\widehat{\theta}_t - \rho\widehat{\lambda}_t$ under Q^M are

$$
d(\widehat{\theta}_t - \rho\widehat{\lambda}_t) = \sqrt{1 - \rho^2}\mathrm{v}_t d\widehat{B}_t^{\perp, Q^M},
$$

where $\widehat{B}^{\perp, Q^M}$ is a Q^M-Brownian motion perpendicular to that driving the stock, related to $\widehat{W}^{Q^M}$ by $\widehat{W}^{Q^M} = \rho\widehat{B}^{Q^M} + \sqrt{1 - \rho^2}\widehat{B}^{\perp, Q^M}$, and where $\widehat{B}^{Q^M}$ is the Brownian motion driving S. Hence, for $u > t$, after changing the order of integration in a double integral, we obtain

$$
\int_t^T \left(\widehat{\theta}_u - \rho\widehat{\lambda}_u \right) du = \left(\widehat{\theta}_t - \rho\widehat{\lambda}_t \right)(T - t) + \sqrt{1 - \rho^2} \int_t^T \mathrm{v}_u(T - u) d\widehat{B}_u^{\perp, Q^M}.
$$

This can be inserted into (130) to yield the desired result.

$\square$

We are thus able to obtain BS-style formulae for the price and hedge. For a put option of strike K we easily obtain the following explicit formulae for the marginal price and the associated optimal hedging strategy, where Φ denotes the standard cumulative normal distribution function.

Corollary 1. *With m and Σ as in Proposition 3, define $b \equiv b(t, s, y)$ by*

$$
m = \log y + b - \frac{1}{2}\Sigma^2.
$$

Then the marginal price at time $t \in [0, T]$ of a put option with payoff $(K - Y_T)^+$ is $\widehat{p}(t, S_t, Y_t)$, where

$$
\begin{aligned}
\widehat{p}(t, s, y) &= K\Phi(-d_1 + \Sigma) - ye^b\Phi(-d_1), \\
d_1 &= \frac{1}{\Sigma}\left[\log\left(\frac{y}{K}\right) + b + \frac{1}{2}\Sigma^2 \right].
\end{aligned}
$$

The optimal hedging strategy given by Theorem 13 with $\widehat{p}$ as an approximation to the indifference price is $\widehat{\Delta}_t \equiv \widehat{\Delta}(t, S_t, Y_t)$, where

$$
\widehat{\Delta}(t, s, y) = n\rho\frac{\beta}{\sigma}\frac{y}{s}e^b\Phi(-d_1).
$$

In [24] these results are used to conduct a simulation study of the effectiveness of the optimal hedge under partial information (that is, with Bayesian learning about the drift parameters of the assets), compared with the BS-style hedge and the optimal hedge without learning. The results show that optimal hedging combined with a filtering algorithm to deal with drift parameter uncertainty can indeed give improved hedging performance over methods which take S as a perfect proxy for Y, and over methods which do not incorporate learning via filtering.

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