CAT 2019

Problem Sheet 1

Simplicial complexes and fundamentals of homology theory.

*Depending on your background, some of these will be either too easy or too hard. You should, however, engage with each problem (not only the ones you hand in!) enough to understand its statement completely. The more testing ones are marked with an asterisk *.*

(1) Let K be a simplicial complex. Assume (σ, τ) is a pair of its simplices such that σ is the only codimension one coface of τ. Prove that $L := K - \{\sigma, \tau\}$ is also a simplicial complex and that it is homotopy equivalent to K.

(2) Let K be a simplicial complex. The **cone** on K, written $\mathrm{Cone}(K)$ is a new simplicial complex defined as follows: the vertices of $\mathrm{Cone}(K)$ are all the vertices of K plus an additional vertex v_∞. And the k-dimensional simplices of $\mathrm{Cone}(K)$ consist of all the k-dimensional simplices of K plus all simplices of the form $v_\infty \cup \tau$ where τ ranges over the $(k-1)$ dimensional simplices of K. Show that $\mathrm{Cone}(K)$ is always contractible.

(3) * Consider the circle S^1 and cover it by three open arcs such that any pair intersects in a subarc. Construct its Čech complex and verify the Nerve Theorem in this context directly. Give an example of a cover by open arcs of the circle such that the corresponding Čech complex does not have the homotopy type of the circle.

(4) Let $S \subset \mathbb{R}^d$ be finite and consider real weights w_s for each $s \in S$. The weighted Voronoi cells are defined by

$$V(s) := \{x \in \mathbb{R}^d \text{ so that } \|x - s\| - w_s \leq \|x - v\| - w_v \text{ for } v \neq s \text{ in } S\}$$

Let $s_- = (-1, 0), s_+ = (1, 0) \in \mathbb{R}^2$ and sketch the bisectors for

- $w_{s_-} = 0, w_{s_+} = 0$
- $w_{s_-} = 0, w_{s_+} = 1$
- $w_{s_-} = 0, w_{s_+} = 2$
- $w_{s_-} = 0, w_{s_+} = -1$.

(5) Show that $\mathrm{Alpha}_\epsilon(S) \subset \check{\mathrm{C}}\mathrm{ech}_\epsilon(S)$ for all finite $S \subset \mathbb{R}^d$ and any $\epsilon > 0$. Show that $\mathrm{Alpha}_\epsilon(S)$ has the homotopy type of $\bigcup_{s \in S} B_\epsilon(s) \cap V(s)$.

(6) * Find out about 'witness complexes'.

(7) Let $f : C_* \to D_*$ be a map of chain complexes. Prove that the kernel, image and cokernel of f are chain complexes in a natural way.

(8) Prove that for a finite simplicial complex K, the Euler characteristic can be computed in terms of its homology with field coefficients in a field $\mathbb{F}$ as

$$\chi(K) = \sum_{i \geq 0} (-1)^i \dim H_i(K; \mathbb{F})$$

(9) A simplicial complex K is connected if there is a path of 1-dimensional simplices between any two vertices. Show that such a complex has $\dim H_0(K; \mathbb{F}) = 1$.

(10) Prove that a map of simplicial complexes induces a map of homology groups.

(11) Compute the homology of a sphere modelled as the boundary of a standard 3-simplex.

(12) Compute the relative homology $H_*(D^n, \partial D^n)$ and show that it is isomorphic to the reduced homology of the sphere $\tilde{H}_*(S^n)$.

(13) Find a simplicial complex such that its geometric realisation is a cylinder $S^1 \times [0,1]$. Find its homology.

(14) Do the same as in the above question for the Klein bottle.

(15) * Compute the homology of the image of the Klein bottle immersed (in the standard way) in $\mathbb{R}^3$. Compare it with the answer of the previous question.

(16) Let X and Y be simplicial complexes. Identify a vertex v from X with a vertex w from Y to form a new simplicial complex $X \vee Y$. Prove $H_p(X \vee Y) = H_p(X) \oplus H_p(Y)$ for all $p > 0$.