CAT 2019

Problem Sheet 3

Persistence, Sheaves and Discrete Morse theory

(1) Let $[a, b)$ and $[c, d)$ be two intervals (i.e., bounded connected subsets of the real line) so that $b > c$. What is the interleaving distance between the two corresponding interval modules $\mathbb{I}_\bullet^{[a,b)}$ and $\mathbb{I}_\bullet^{[c,d)}$?

(2) Let $\phi : B_0 \to B_1$ and $\psi : B_1 \to B_2$ be matchings of barcodes. How would you define their composite matching $\psi \circ \phi : B_0 \to B_2$? Show that distortion is a sublinear function of this composition, i.e., that $\mathrm{dist}(\psi \circ \phi) \leq \mathrm{dist}(\phi) + \mathrm{dist}(\psi)$.

(3) Draw the **Petersen graph**, and construct an acyclic partial matching on it so that all but one of its vertices are paired. What can you conclude about the first Betti number of the Petersen graph?

(4) Given two finite sets P, Q in Euclidean space, $\mathbb{R}^n$, the **Hausdorff distance** between P and Q is defined to be the smallest $\epsilon \geq 0$ so that
- every point of P is within ϵ of some point of Q, and
- every point of Q is within ϵ of some point of P.

Show that the Hausdorff distance satisfies all the requirements of a metric on the set of finite subsets of $\mathbb{R}^n$.

(5) Show that if the Hausdorff distance between two finite sets P and Q is ϵ, then the persistence modules of the Čech filtrations around P and Q are ϵ-interleaved.

(6) Let $\mathcal{F}$ be a sheaf over a simplicial complex K. A **global section** of $\mathcal{F}$ is an assignment of vectors to simplices of K, subject to the constraints imposed by the stalks and restriction maps. More precisely, every global section is a map of the form

$$s : K \to \coprod_{\sigma \in K} \mathcal{F}(\sigma),$$

subject to two constraints:
(a) each simplex σ is sent to some vector $s(\sigma)$ in its own stalk, and
(b) given a face relation $\sigma \leq \tau$, the restriction map $\mathcal{F}(\sigma \leq \tau) : \mathcal{F}(\sigma) \to \mathcal{F}(\tau)$ sends $s(\sigma)$ to $s(\tau)$.

Show that the set of all global sections of $\mathcal{F}$, usually denoted $\Gamma(\mathcal{F})$, has a natural vector space structure. Then show that the dimension of this vector space is exactly the dimension of the zeroth sheaf cohomology $H^0(K; \mathcal{F})$.

(7) Let $K_0 \subset K_1$ be a pair of simplicial complexes, treated as a two-step filtration of K_1. How would find the dimension of the i-th relative homology group $H_i(K_1, K_0)$ from the i-th persistent homology barcode of this filtration?

(8) Let K be a simplicial complex with the special property that one of its vertices v is contained in *every* simplex of dimension > 0. What can you say about the homology of K?

(9) Consider the filtered simplicial complex shown below (the numbers next to simplices indicate the index at which they are born). Draw an acyclic partial matching compatible with this filtration, describe its critical cells and gradient paths, and use it to simplify the construction of the barcodes in dimensions 0 and 1 of this filtered complex.