CAT 2019

Problem Sheet 1

Simplicial Complexes and Homotopy

(1) Show that for any fixed pair of topological spaces X and Y, *homotopy* is an equivalence relation on the set of all continuous functions $X \to Y$.

(2) Show that *homotopy equivalence* is an equivalence relation on the set of all topological spaces.

(3) What does it mean for two topological spaces to be homeomorphic (you can use Google if you don't know this already)? Show that homeomorphic topological spaces are automatically homotopy equivalent.

(4) Show that the one-point space $\bullet$ and the two-point space $\bullet\ \bullet$ are *not* homotopy equivalent.

(5) Let M be a finite metric subspace of an ambient metric space (Z, d). Show, for each $\epsilon > 0$, that the associated Čech complex $\check{\mathbf{C}}_\epsilon(M)$ is a subcomplex of the Vietoris-Rips complex $\mathbf{VR}_{2\epsilon}(M)$. Then, show that – no matter what Z you had chosen – this $\mathbf{VR}_{2\epsilon}(M)$ is itself a subcomplex of $\check{\mathbf{C}}_{2\epsilon}(M)$.

(6) (*Bonus! No need to solve this or hand it in, but think about how you might try to approach it*). Let M be a finite subset of points in Euclidean space $\mathbb{R}^n$ (with its standard metric). As a function of n, can you find the *smallest* δ so that $\mathbf{VR}_\epsilon(M)$ is always a subcomplex of $\check{\mathbf{C}}_\delta(M)$? [Here the Čech complex has been constructed with respect to the ambient Euclidean space $\mathbb{R}^n$]