MiniProject Guidance for C3-9

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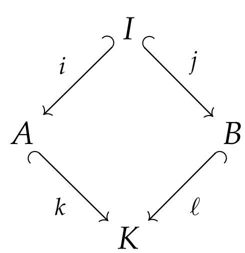
1. Diamonds are Forever

The main task is to describe how you might use discrete Morse theory to make (computational aspects of) your life easier when constructing the Mayer-Vietoris long exact sequence in cosheaf homology. Your starting point is a decomposition $K = A \cup B$ of a simplicial complex K into subcomplexes A and B whose intersection $A \cap B$ will be called I. There are four simplicial maps running around here, all inclusions, so let's give them names:



Let's agree to call this diagram the **Mayer-Vietoris diamond**.

If we work with coefficients over a field $\mathbb{F}$, then the ordinary Mayer-Vietoris short exact sequence associated to our diamond has the form

$$0 \to \mathrm{C}_\bullet(I;\mathbb{F}) \xrightarrow{\alpha_\bullet} \mathrm{C}_\bullet(A;\mathbb{F}) \oplus \mathrm{C}_\bullet(B;\mathbb{F}) \xrightarrow{\beta_\bullet} \mathrm{C}_\bullet(K;\mathbb{F}), \tag{1}$$

where for instance $\mathrm{C}_d(A;\mathbb{F})$ is the vector space generated by linear combinations of d-dimensional simplices lying in the subcomplex I. The two chain maps α and β have the following action for every γ in $\mathrm{C}_d(I;\mathbb{F})$ and every pair (μ,ν) in $\mathrm{C}_d(A;\mathbb{F}) \oplus \mathrm{C}_d(B;\mathbb{F})$:

$$\alpha_d(\gamma) = \big(\mathrm{C}i(\gamma), \mathrm{C}j(\gamma)\big) \quad \text{and} \quad \beta_d(\mu,\nu) = \mathrm{C}k(\mu) - \mathrm{C}\ell(\nu).$$

Here for instance $\mathrm{C}i$ is the chain map induced by the inclusion i, etc. You should be able to convince yourself that this sequence is indeed exact: α is injective, β is surjective, and the kernel of β exactly agrees with the image of α.

Task 1: If someone hands you a cosheaf $\mathscr{F}$ on K, can you find a sequence similar to (1) where these ordinary chain groups are replaced by the cosheaf chain groups $\mathrm{C}_\bullet(K;\mathscr{F})$, etc.? Since $\mathscr{F}$ is defined on all of K, you will have to use it to generate cosheaves on the subcomplexes I, A and B somehow that still give you a short exact sequence.

If you can solve this task successfully (i.e., if you can show that indeed your recipe produces a short exact sequence), then you will have produced the Mayer Vietoris short exact sequence for any cosheaf $\mathscr{F}$ on K with respect to our diamond.

2. Snakes on a Plane

Once you have a short exact sequence for a cosheaf $\mathscr{F}$ with respect to the M-V diamond, it is time to construct the long exact sequence on cosheaf homology. Here is a basic result in homological algebra, called the **snake lemma** or **zigzag lemma**.

LEMMA 2.1. *Given any short exact sequence of chain complexes of $\mathbb{F}$-vector spaces, say*

$$0 \longrightarrow C_\bullet \xrightarrow{p_\bullet} D_\bullet \xrightarrow{q_\bullet} E_\bullet \longrightarrow 0,$$

there is a family of linear maps $\Delta_d : H_d(E) \to H_{d-1}(C)$, called **connecting homomorphisms***, which fit into a long exact sequence*

$$\cdots \longrightarrow H_d(C) \xrightarrow{H_d p} H_d(D) \xrightarrow{H_d q} H_d(E) \xrightarrow{\Delta_d} H_{d-1}(C) \longrightarrow \cdots,$$

where $H_d p$ is the map on homology groups induced by the chain map $p_\bullet$, etc.

Giving a full proof of this lemma takes a lot of work (fortunately, you are *not* being asked to do that work). But it will help if you know just how one should construct $\Delta_\bullet$ for a given short exact sequence. You can even find the recipe on the Wikipedia page for `zigzag lemma`, not to mention the textbooks of Hatcher and Munkres, with the former being freely available online. All of the computational burden here is linear algebraic: you have to compute kernels, cokernels or images of linear maps — either p or q or the boundary operators of $C_\bullet$, $D_\bullet$ and $E_\bullet$. So, if these vector spaces can be arranged to have smaller dimensions while preserving the homology groups, then the linear maps would have tinier matrix representations, and all of this linear algebra can be done much faster. This is where discrete Morse theory might help. But first,...

PROPOSITION 2.2. *Consider a map of short exact sequences — writing the two sequences horizontally, such a map consists of a triple of vertical chain maps $(u_\bullet, v_\bullet, w_\bullet)$ which make the following diagram commute:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_\bullet & \xrightarrow{p_\bullet} & D_\bullet & \xrightarrow{q_\bullet} & E_\bullet & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle u_\bullet} & & \downarrow{\scriptstyle v_\bullet} & & \downarrow{\scriptstyle w_\bullet} & & \\
0 & \longrightarrow & C'_\bullet & \xrightarrow[p'_\bullet]{} & D'_\bullet & \xrightarrow[q'_\bullet]{} & E'_\bullet & \longrightarrow & 0
\end{array}$$

This map $(u_\bullet, v_\bullet, w_\bullet)$ induces a map of the corresponding long exact sequences:

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & H_d(C) & \xrightarrow{H_d p} & H_d(D) & \xrightarrow{H_d q} & H_d(E) & \xrightarrow{\Delta_d} & H_{d-1}(C) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle H_d u} & & \downarrow{\scriptstyle H_d v} & & \downarrow{\scriptstyle H_d w} & & \downarrow{\scriptstyle H_{d-1} p} & & \\
\cdots & \longrightarrow & H_d(C') & \xrightarrow[H_d p']{} & H_d(D') & \xrightarrow[H_d q']{} & H_d(E') & \xrightarrow[\Delta'_d]{} & H_{d-1}(C') & \longrightarrow & \cdots
\end{array}$$

And moreover, if $u_\bullet$, $v_\bullet$ and $w_\bullet$ are quasi-isomorphisms, i.e., if all three induce isomorphisms on homology, then the two long exact sequences are isomorphic.

This result forms a cornerstone of homology theory. If you want to learn more, search online for the **naturality of the connecting homomorphism**. I hope you can now see the point of the exercise. We will use discrete Morse theory on the original short exact sequence to produce a much smaller one so that there is a triple $(u_\bullet, v_\bullet, w_\bullet)$ of level-wise quasi-isomorphisms from the big one to the small one.

Task 2: In lecture we have already seen how to make an acyclic partial matching Σ on K compatible with a cosheaf $\mathscr{F}$. Given such a matching, what further conditions must be imposed to guarantee compatibility with the Mayer-Vietoris Diamond? In other words, we want the Morse chain complex $M^I_\bullet$ [associated to the restriction of Σ on I] to include into the Morse chain complexes $M^A_\bullet$ and $M^B_\bullet$, each of which in turn includes into $M^K_\bullet$ so that the diamond of Morse chain complexes commutes.

If this seems difficult, try to remember how we extended discrete Morse theory to the setting of filtered simplicial complexes $K_0 \subset K_1 \subset \cdots$ by restricting the Σ-pairs to lie in the same consecutive difference $K_i - K_{i-1}$ as each other. Is there a natural way to do something similar for the diamond? Perhaps some inspiration can be found in the 2013 paper *Morse theory for filtrations and efficient computation of persistent homology* by K Mischaikow et al., which introduced the extension of discrete Morse theory for computing persistent homology.

3. For a Few Dollars Morse

One of the glaring omissions from our lectures is that I never showed you the quasi-isomorphisms which establish that the homology of a simplicial complex K agrees with that of the Morse chain complex $M_\bullet$ associated to an acyclic partial matching Σ on K. Let's rectify this defect; you can find a description of these chain maps in the 2016 paper *Discrete Morse theory for computing cellular sheaf cohomology* by J Curry et al. This was written for sheaves and cohomology, so you'll have to turn some arrows around to get the desired chain maps for cosheaves and homology. Now we reach the end of our journey.

Task 3: Can you use what you have learned about the discrete Morse-theoretic quasi isomorphisms to verify that you are in the setting of Prop 2.2? Namely, you want to show that not only do you have some chain maps $u_\bullet, v_\bullet, w_\bullet$ which induce isomorphisms on cosheaf homology, but that these maps also fit together into a commutative diagram.

If you successfully accomplish this task, then Prop 2.2 will guarantee that you have a Morse Mayer-Vietoris long exact sequence for a cosheaf $\mathscr{F}$ on K, which contains isomorphic information as the original M-V long exact sequence, but is far cheaper to construct. All the best!

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