Conormal Spaces and Whitney Stratifications

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ABSTRACT. We describe a new algorithm for computing Whitney stratifications of complex projective varieties. The main ingredients are (a) an algebraic criterion, due to Lê and Teissier, which reformulates Whitney regularity in terms of conormal spaces and maps, and (b) a new interpretation of this conormal criterion via ideal saturations, which can be practically implemented on a computer. We show that this algorithm improves upon the existing state of the art by several orders of magnitude, even for relatively small input varieties. En route, we introduce related algorithms for efficiently stratifying affine varieties, flags on a given variety, and algebraic maps.

Introduction

The quest to define and study singular spaces counts among the most spectacular success stories of twentieth century mathematics. Much of the underlying motivation arose from algebraic geometry, where the spaces of interest – namely, the vanishing loci of polynomials – contain singular points even in the simplest of cases. Without the benefit of hindsight, it remains a Herculean task to construct good models of singular spaces that are simultaneously broad enough to include all analytic varieties and narrow enough to exclude various pathological spaces which arise as zero sets of arbitrary smooth functions. The standard solution to this conundrum, which we describe below, was first proposed by Whitney [37] and subsequently refined by Thom [31, 32], Mather [26], Goresky-MacPherson [16, 17, 18], Lê-Teissier [24], Fulton [12], Cappell-Shaneson [5], Weinberger [36] and others.

Whitney's Condition (B). As a natural starting point, one can at least require each candidate space X under consideration to embed in some Euclidean space $\mathbb{R}^n$ and to admit a partition into smooth submanifolds, say

$$X = \coprod_i M_i,$$

with $\dim M_i = i$. An entirely reasonable first attempt at constructing such M_i from X might proceed as follows. For each dimension $0 \leq i \leq n$ and subset $Y \subset \mathbb{R}^n$, let $\mathscr{E}_i(Y)$ denote the set of points p in Y which admit an open neighbourhood $U_p \subset Y$ homeomorphic to $\mathbb{R}^i$. Then, recursively define

$$M_n := \mathscr{E}_n(X), \text{ and}$$
$$M_i := \mathscr{E}_i(X - M_{>i}) \text{ for } 0 \leq i < n,$$

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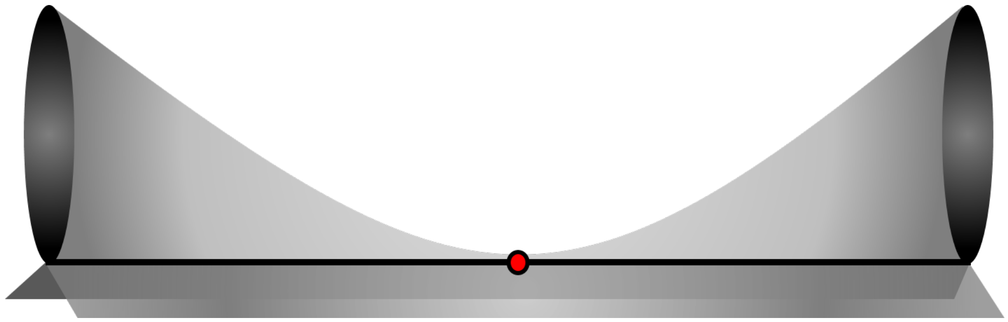
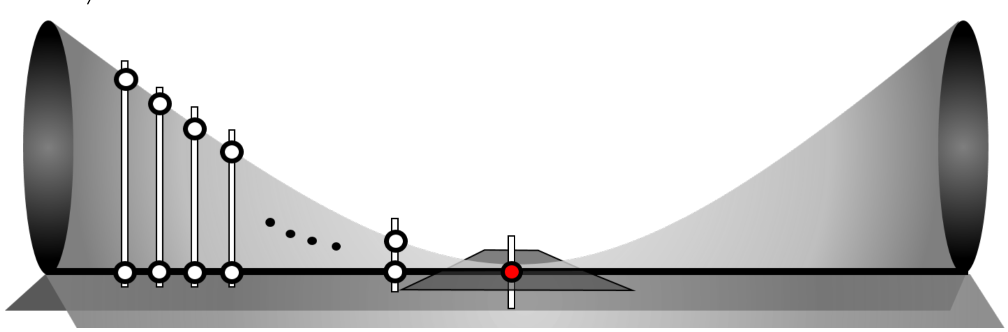
where $M_{>i}$ denotes the union $\bigcup_{j>i} M_j$. Unfortunately, this recursive strategy does not produce a desirable partition. Perhaps the simplest way to see the underlying problem is to try constructing these M_i by hand when $X \subset \mathbb{R}^3$ is the singular surface depicted below.



Since M_3 is empty, one identifies $M_2 \subset X$ as the set of points with two-dimensional Euclidean neighbourhoods; and upon removing these, only the horizontal line remains. This line must therefore equal M_1, and we obtain a partition of X into one and two-dimensional smooth manifolds. The issue here is that the highlighted point (which we will call p) has a singularity type which is different from all other points lying on M_1 — a small neighbourhood in X around p is not homeomorphic to a small neighbourhood around any other point lying on M_1. More precisely, let G be the group of homeomorphisms $f : X \to X$ so that f is isotopic to the identity and its restriction to each M_i is a diffeomorphism. It turns out that G acts transitively on the connected components of $M_1 - \{p\}$ while fixing p itself. Thus, our recursive strategy must be amended so that the M_i are *G-equisingular* in this sense, which would automatically separate p from M_1 as desired.

Whitney's ingenious approach from [37] was to consider the behaviour of limiting tangent spaces as one approaches a point in some $Y := M_i$ in two different ways: one in a tangential direction along Y itself, and another in a normal direction along some other $X := M_j$ for $j > i$. Let $\{x_i\}$ and $\{y_i\}$ be sequences of points in X and Y respectively which both converge to the same y in Y. Write T_i for the tangent space of X at x_i and ℓ_i for the secant line joining each x_i to the corresponding y_i in the ambient $\mathbb{R}^n$. The pair (X, Y) is said to satisfy Whitney's **Condition (B)** if the limiting tangent space $T = \lim T_i$ contains the limiting secant line $\ell = \lim \ell_i$ whenever both limits exist. In our example, one can find sequences of points in $X := M_2$ and $Y := M_1$, both limiting to the problematic point p, for which $\ell \not\subset T$:

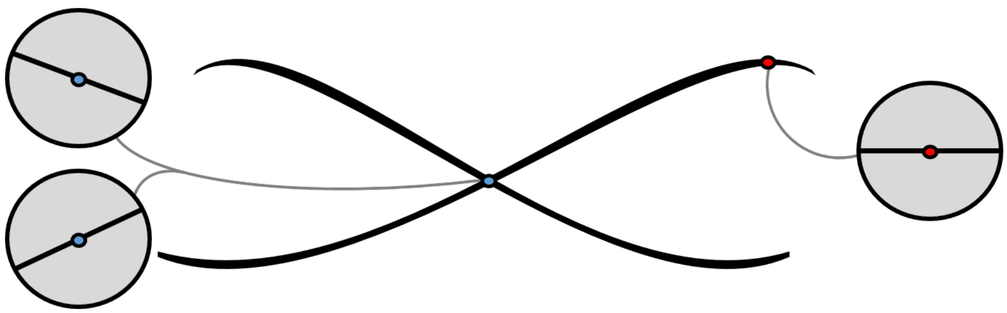


It is a foundational result in stratification theory that equisingularity is satisfied by any decomposition $X = \coprod_i M_i$ for which all pairs (M_j, M_i) satisfy Whitney's Condition (B) — see [26] or [18, Part I Ch 1.5]. Since their very inception, such *Whitney stratifications* have been used to define and compute myriad important algebraic-topological invariants of singular spaces and related structures, even in cases where the invariants do not ultimately depend on the chosen stratification. Prominent examples include intersection homology groups [16, 17], stratified vector fields [4], characteristic varieties [13], Euler obstructions [15, 30] and Chern classes [25], to name but a few.

Conormal Spaces. Given a k-dimensional projective variety $X \subset \mathbb{P}^n$, let X_{reg} be the smooth locus of X and note that there is a well-defined tangent space $T_x X_{\text{reg}}$ at each point x in X_{reg}. This tangent space naturally resides in the Grassmannian $\mathbf{Gr}(n+1,k)$ of k-dimensional subspaces of $\mathbb{C}^{n+1}$; let us consider the map

$$\tau : X_{\text{reg}} \to X \times \mathbf{Gr}(n+1,k)$$

that sends each x to the pair $(x, T_x X_{\text{reg}})$. Thus, τ's image is the graph of the Gauss map of X; taking the closure of this image creates a new (usually singular) space $\mathbf{N}(X)$, called the **Nash blowup** of X (see [37, Sec 16] or [25]). The fiber over each point $x \in X$ of the evident projection map $\mathbf{N}(X) \twoheadrightarrow X$ catalogues all the limiting tangent spaces at x:



From the perspective of analysing limiting tangent spaces, the Nash blowup is an optimal object. However, the geometry of the Grassmannian is rather intricate, and this makes it difficult to explicitly compute defining equations of $\mathbf{N}(X)$ from those of X. One remedy is to systematically replace $\mathbf{Gr}(n+1,k)$ with the dual projective space $(\mathbb{P}^n)^*$ in the construction described above. With this modification in place, we consider the set of all hyperplanes in $\mathbb{P}^n$ – or equivalently, points in the dual projective space $(\mathbb{P}^n)^*$ – that contain the tangent space at each x in X_{reg}. Passing to the closure in $X \times (\mathbb{P}^n)^*$ produces the **conormal space** $\mathbf{Con}(X)$ of X, and the natural projection $\kappa_X : \mathbf{Con}(X) \twoheadrightarrow X$ is called the **conormal map**. The conormal space retains essential information about limiting tangents; and crucially, $\mathbf{Con}(X)$ is also a projective variety whose defining equations can be easily extracted from those of X.

This Paper. Here we introduce an algorithm for building Whitney stratifications of complex projective varieties. The main reason for restricting our focus to such varieties rather than the far more general class of real semialgebraic sets (which are equally easy to represent on a computer) is that the ingredients required to implement our algorithm are only available in the complex algebraic setting. In any event, the immediate obstacle is that it appears hopeless to try verifying Condition (B) directly for a pair of smooth algebraic sets, since this would require computation of limiting tangent planes and secant lines over arbitrary pairs of infinite sequences. Attempting to bypass this problem by constructing the Nash blowup also appears to be a daunting task. Thus, we turn to the conormal space $\mathbf{Con}(X)$ of the given input variety X.

The good news comes in the form of a result by Lê and Teissier, which provides a complete characterisation of Condition (B) for projective varieties in terms of their conormal spaces [24, Proposition 1.3.8]. This criterion asserts that for any projective variety $X \subset \mathbb{P}^n$ and smooth algebraic subset $Y \subset X$, the pair (X_{reg}, Y) satisfies Condition (B) if and only if the ideal sheaf of $\mathbf{Con}(X) \cap \mathbf{Con}(Y)$ lies in the integral closure of the ideal sheaf of $\kappa_X^{-1}(Y)$. There are now two caveats to consider — first, as remarked above, we are not aware of any analogous criterion for pairs of smooth real (semi)algebraic sets. Second, and more serious from our perspective, is the fact that even the most basic algorithmic tasks involving integral closures are computationally prohibitive [34, Sec 9.3]. Our main result circumvents the latter issue by making use of ideal saturations. (Note in its statement below that I_Z indicates the defining homogeneous ideal of a given projective variety Z.)

THEOREM. *Let $X \subset \mathbb{P}^n$ be a pure-dimensional projective variety and $Y \subset X$ a nonempty irreducible subvariety of its singular locus X_{sing}. Let J be defined as the ideal saturation*

$$J := I_{\kappa_X^{-1}(Y)} : \left(I_{\mathbf{Con}(X)\cap\mathbf{Con}(Y)}\right)^{\infty},$$

and write $\mathbf{V}(J)$ for the corresponding projective variety. Then the difference

$$Y' := Y_{\text{reg}} - \overline{\kappa_X(\mathbf{V}(J))}$$

is dense in Y, and moreover, the pair (X_{reg}, Y') satisfies Condition (B).

Computing ideal saturations (such as J from the statement above) reduces to a standard Gröbner basis calculation [7, Chapter 4.4, Theorem 14], so this result directly leads to our recursive algorithm for stratifying complex projective varieties. Before describing the details, we highlight three relevant features. First, given a k-dimensional input variety $X \subset \mathbb{P}^n$, the output Whitney stratification is produced in the form of (defining equations for) a nested sequence $X_\bullet$ of subvarieties

$$X_0 \subset X_1 \subset \cdots \subset X_k = X,$$

so that the desired manifold partition $X = \coprod_i M_i$ is given by $M_i := X_i - X_{i-1}$. Second, this algorithm can easily be used to produce Whitney stratifications of affine complex varieties as well — first pass to the projective closure X, build its Whitney stratification $X_\bullet$, then dehomogenize the resulting X_i's. And third, given any flag $\mathbf{F}_\bullet$ of projective subvarieties:

$$\mathbf{F}_0 X \subset \mathbf{F}_1 X \subset \cdots \subset \mathbf{F}_\ell X = X,$$

it is quite straightforward to adapt our algorithm so that its output is subordinate to this flag. In other words, we can guarantee that each connected component of a given M_i lies in a single successive difference $\mathbf{F}_j X - \mathbf{F}_{j-1} X$. As a consequence, we are also able to stratify generic morphisms of projective varieties by availing of their Thom-Boardman flags [31, 3].

Related Work. The current state of the art in this area appears to be the recent work of Đinh and Jelonek [23]. To the best of our knowledge, given a complex algebraic variety X embedded in n-dimensional affine or projective space, all prior stratification methods (such as [28, 29] for instance) require quantifier elimination in approximately $4n$ real variables. This is generally accomplished using some version of the cylindrical algebraic decomposition algorithm of Collins [1, 6, 2], which is known to be extremely difficult in practice. For stratifications of real varieties given by unions of transversely-intersecting *smooth* subvarieties, the theoretical complexity of quantifier elimination can be somewhat improved [35]. As far as we are aware, no implementation of any such quantifier elimination based Whitney stratification algorithms has ever been produced.

The work of [23] improves upon such cylindrical algebraic decomposition (CAD) approaches by requiring only Gröbner basis (GB) computations in approximately $4n$ complex variables. The advantage enjoyed by GB methods over their CAD counterparts in practice has been well-documented [38, 10]. In fact, it is remarked in [10, III.D] that

> *"Although like CAD the calculation of GB is doubly exponential in the worst case, GB computation is now mostly trivial for any problem on which CAD construction is tractable."*

In fairness, it should be noted that cylindrical algebraic decompositions apply to a much wider class of singular spaces (i.e., real semialgebraic sets), where Gröbner basis methods are entirely unavailable.

Our algorithm further reduces the Whitney stratification problem to Gröbner basis type computations in approximately $2n$ complex variables, and additionally, is able to preserve the sparsity structure of the input in ways that make a significant difference to real-world performance. By contrast, the algorithm of [23] requires various choices of generic linear forms, which end up removing a lot of sparsity from the systems being considered. As an added bonus our algorithm is completely deterministic, while the algorithm of [23] is probabilistic. We have implemented both algorithms, and provide a performance comparison in the final Section of this paper.

Outline. In Sections 1 and 2, we briefly review conormal spaces and Whitney stratifications respectively, with a view towards describing the Lê-Teissier criterion for Whitney's Condition (B). Section 3, which is focused on the ideal saturation approach to this criterion, forms the technical heart of this paper and contains a proof of our main result. We describe our recursive algorithm for constructing Whitney stratifications of complex projective varieties in Section 4 and verify its correctness. In Section 5 we modify this algorithm to produce flag-subordinate stratifications, which are then used to stratify projective morphisms in Section 6. Finally, Section 7 provides empirical evidence that the algorithm described here readily outperforms the existing state of the art.

Acknowledgements. VN is grateful to Mark Goresky and Robert MacPherson for several valuable discussions on the local geometry of singular spaces; his work was supported by EPSRC grant EP/R018472/1 and by DSTL grant D015 funded through the Alan Turing Institute.

1. Conormal Spaces

For each pair of positive integers n and k with $n+1 \geq k$, let $\mathbf{Gr}(n+1,k)$ denote the complex Grassmannian whose points correspond to all the k-dimensional linear subspaces of $(n+1)$-dimensional affine space $\mathbb{C}^{n+1}$. The usual projective space $\mathbb{P}^n$ equals $\mathbf{Gr}(n+1,1)$ while its dual $(\mathbb{P}^n)^*$ is $\mathbf{Gr}(n+1,n)$. Let $X \subset \mathbb{P}^n$ be a connected k-dimensional complex analytic space whose regular and singular loci will be written X_{reg} and $X_{\text{sing}} := (X - X_{\text{reg}})$ respectively. Recall that the tangent space to the smooth manifold X_{reg} at a given point x is a subspace T_xX_{reg} in $\mathbf{Gr}(n+1,k)$.

DEFINITION 1.1. The **conormal space** of X is the subset of $\mathbb{P}^n \times (\mathbb{P}^n)^*$ determined by the closure

$$\mathbf{Con}(X) = \overline{\{(x, \xi) \mid x \in X_{\text{reg}} \text{ and } T_x X_{\text{reg}} \subset \xi\}}.$$

Thus, a point (x, ξ) in $\mathbb{P}^n \times (\mathbb{P}^n)^*$ lies in $\mathbf{Con}(X)$ if and only if there exists a sequence of points $\{x_i\} \subset X_{\text{reg}}$ which converge to x in $\mathbb{P}^n$ and a sequence of hyperplanes $\{\xi_i\} \subset (\mathbb{P}^n)^*$ which converge to ξ in $(\mathbb{P}^n)^*$ so that $T_{x_i} X_{\text{reg}} \subset \xi_i$ holds for all $i \gg 1$.

The map $\kappa_X : \mathbf{Con}(X) \to X$ induced by the evident projection $(x, \xi) \mapsto x$ is called the **conormal map** of X; the fiber $\kappa_X^{-1}(x)$ of κ_X over a point x in X is the set of all hyperplanes in $(\mathbb{P}^n)^*$ which contain a limiting tangent space at x. Conormal spaces and maps are well-studied classical objects of substantial interest in complex geometry, with deep connections to polar varieties, Nash blowups, microlocal analysis and beyond [11]. Here we will be interested exclusively in conormal maps of complex projective subvarieties $X \subset \mathbb{P}^n$ — in this special case, $\mathbf{Con}(X)$ is an n-dimensional complex subvariety of $X \times (\mathbb{P}^n)^*$, the conormal map κ_X is algebraic, and the dimension of each fiber $\kappa_X^{-1}(x)$ is no larger than n (see [11, Proposition 2.9] and references therein).

PROPOSITION 1.2. *Let $X \subset \mathbb{P}^n$ be a pure dimensional complex variety. If $X' \subset X$ is any Zariski-dense subset, then* $\mathbf{Con}(X) = \mathbf{Con}(X')$.

PROOF. Since X' is dense in the closed connected algebraic subspace $X \subset \mathbb{P}^n$, for any point $(x, \xi) \in \mathbf{Con}(X)$ there exists a sequence of points $\{x_i\}$ in X'_{reg} converging to x and an induced sequence of hyperplanes ξ_i containing $T_{x_i} X'_{\text{reg}}$ which converge to ξ. Since $\mathbf{Con}(X)$ is closed by definition, we have $\mathbf{Con}(X) = \mathbf{Con}(X')$. □

2. Whitney Stratifications

Let $X, Y \subset \mathbb{C}^{n+1}$ be smooth complex manifolds. A point p in Y is said to satisfy Whitney's **Condition (B)** with respect to X if the following property [37, Sec 19] holds:

> for any sequences of points $\{x_i\} \subset X$ and $\{y_i\} \subset Y$ both converging to p, if the secant lines $\ell_i = [x_i, y_i]$ converge to some limiting line ℓ in $\mathbb{P}^n$ and if the tangent spaces $T_{x_i} X$ converge to some limiting plane T in the Grassmannian $\mathbf{Gr}(n+1, \dim X)$, then $\ell \subset T$.

More generally, we say that the pair (X, Y) satisfies Condition (B) if the above property holds for every point p in Y. Note that (X, Y) vacuously satisfies Condition (B) if the closures $\overline{X}$ and $\overline{Y}$ do not intersect in $\mathbb{C}^{n+1}$. The result below is also elementary, and we have only highlighted it here since we appeal to it rather frequently.

PROPOSITION 2.1. *Assume that a pair (X, Y) of smooth complex manifolds satisfies Condition (B). If $X' \subset X$ is a dense submanifold of X and $Y' \subset Y$ an arbitrary submanifold of Y, then (X', Y') also satisfies Condition (B).*

PROOF. Since $X' \subset X$ and $Y' \subset Y$, every sequence $(\{x_i\}, \{y_i\}) \subset X' \times Y'$ is automatically a sequence in $X \times Y$. And since X' is dense in X, we have an equality $T_{x_i} X' = T_{x_i} X$ of tangent planes in the appropriate Grassmannian. Thus, (X', Y') satisfies Condition (B) because (X, Y) does. □

Condition (B) serves as a regularity axiom which can be used to induce a particularly well-behaved and useful class of decompositions of analytic spaces into submanifolds.

DEFINITION 2.2. A (k-dimensional) **Whitney stratification** of a complex analytic subspace $W \subset \mathbb{C}^{n+1}$ is a filtration $W_\bullet$ by closed subsets

$$\varnothing = W_{-1} \subset W_0 \subset W_1 \subset \cdots \subset W_{k-1} \subset W_k = W,$$

where each difference $\Delta_i := W_i - W_{i-1}$ is a complex analytic i-dimensional manifold subject to the following conditions. The connected components of Δ_i, called the i-dimensional **strata**, must obey the following axioms:

(1) **local finiteness:** every point p in W admits an open neighbourhood which intersects only finitely many strata;
(2) **frontier:** for each stratum $S \subset X$, the difference $(\overline{S} - S)$ is a finite union of lower-dimensional strata; and finally,
(3) **condition (B):** each pair of strata (regardless of dimension) satisfies Condition (B).

Whitney showed in [37, Section 19] that every variety X admits a Whitney stratification $X_\bullet$ for which each constituent $X_i \subset X$ is a subvariety. His original definition from that paper contains an additional Condition (A), which is now known by Mather's work to be superfluous [26, Proposition 2.4]. In fact, even the frontier axiom follows from Condition (B) for locally finite stratifications [26, Corollary 10.5], so Condition (B) will remain our sole focus in this paper. The inherent difficulty here is that verifying Condition (B) involves testing the behavior of limiting tangent spaces and secant lines over arbitrary families of infinite sequences. Fortunately, there is a beautiful alternate characterization in terms of conormal maps due to Lê and Teissier [24, Proposition 1.3.8], which we describe below. The following notation has been employed in its statement: for each (open or closed) subscheme Z of $\mathbb{P}^n$, the defining sheaf of ideals is written $\mathscr{I}[Z]$.

PROPOSITION 2.3. *Let $X \subset \mathbb{P}^n$ be a projective variety and $Y \subset X$ a smooth algebraic subspace. The pair (X_{reg}, Y) satisfies Whitney's Condition (B) if and only if we have a containment*

$$\mathscr{I}[\mathbf{Con}(X) \cap \mathbf{Con}(Y)] \subset \overline{\mathscr{I}}[\kappa_X^{-1}(Y)]. \tag{1}$$

Here $\kappa_X : \mathbf{Con}(X) \to X$ is the conormal map of X while $\overline{\mathscr{I}}[\kappa^{-1}(Y)]$ stands for the integral closure of the ideal sheaf $\mathscr{I}[\kappa^{-1}(Y)]$.

We recall for the reader's convenience that the integral closure of an ideal I in a commutative ring R is the ideal $\overline{I}$ consisting of all r in R which happen to be roots of monic polynomials with coefficients in I. In general, if we are only given access to a set of generating elements for I, then various algorithmic operations involving $\overline{I}$ become computationally prohibitive even in the relatively benign case $R = \mathbb{C}[x_0, \ldots, x_n]$. These hard tasks include, for instance, extracting a list of defining polynomials for $\overline{I}$ and testing whether a given $r \in R$ lies inside $\overline{I}$ (see [33, Sections 6.6 and 6.7] or [34, Section 9.3]). Thus, its considerable aesthetic appeal notwithstanding, Proposition 2.3 does not furnish an efficient algorithmic mechanism for verifying Condition (B). We are therefore compelled to employ a Corollary of this Proposition, where the integral closure has been replaced by a far more tractable object.

REMARK 2.4. Before proceeding to the promised Corollary in the next section, it may be helpful to keep the following facts in mind regarding the ideals which have appeared in Proposition 2.3. Since Y is a (smooth) subspace of X by assumption, at each point y of Y the tangent space T_yY is a $(\dim Y)$-dimensional subspace of some (in fact, every) limiting tangent space $T = \lim_i(T_{x_i}X_{\text{reg}})$ associated to a given sequence $\{x_i\} \subset X_{\text{reg}}$ converging to y. Thus, any hyperplane containing T automatically contains T_yY and we have a containment of fibers $\kappa_X^{-1}(y) \subset \kappa_Y^{-1}(y)$ whenever y lies in $Y \subset X$. Consequently, the intersection $\mathbf{Con}(X) \cap \mathbf{Con}(Y)$ lies in the inverse image $\kappa_X^{-1}(Y)$, which forces the contravariant containment

$$\mathscr{I}[\kappa_X^{-1}(Y)] \subset \mathscr{I}[\mathbf{Con}(X) \cap \mathbf{Con}(Y)] \tag{2}$$

of the associated ideal sheaves. It follows from Proposition 2.3 that the pair (X_{reg}, Y) satisfies Condition (B) whenever $\mathscr{I}[\mathbf{Con}(X) \cap \mathbf{Con}(Y)]$ is sandwiched between $\mathscr{I}[\kappa_X^{-1}(Y)]$ and its integral closure.

3. Saturations

Our quest to render Proposition 2.3 algorithmically effective begins with a reminder that if I and J are any two homogeneous ideals of a commutative ring R, then the **saturation** of I with respect to J, written $I : J^\infty$, is defined to be

$$I : J^\infty = \left\{ r \in R \mid \text{there is some } N \geq 0 \text{ satisfying } rJ^N \subset I \right\}.$$

This saturation [7, Chapter 4, Definition 8] is itself a homogeneous ideal of $\mathbb{C}[x_0, \ldots, x_n]$, and satisfies the following crucial property — writing $\mathscr{V}(I) \subset \mathbb{P}^n$ for the subscheme defined by the ideal $I \lhd \mathbb{C}[x_0, \ldots, x_n]$ and so forth, we have

$$\mathscr{V}(I : J^\infty) = \overline{\mathscr{V}(I) - \mathscr{V}(J)}, \tag{3}$$

see [7, Chapter 4, Theorem 10] for details. We will reserve the notation $\mathbf{V}(I)$ for the projective variety (the zero set rather than the scheme) corresponding to an ideal I; equivalently $\mathbf{V}(I)$ is the reduced scheme associated to the radical $\sqrt{I}$, i.e., $\mathbf{V}(I) = \mathscr{V}(I)_{\text{red}}$. Also note that all varieties will be assumed to be reduced, but not necessarily irreducible.

We now return to the setting of Proposition 2.3, where $X \subset \mathbb{P}^n$ is a projective variety with $Y \subset X$ a smooth algebraic subspace, and $\kappa_X : \mathbf{Con}(X) \to X$ is the conormal map of X. Throughout the remainder of this section, we will write coordinates of $\mathbb{P}^n$ as $\{x_0, \ldots, x_n\}$ and denote this choice as $\mathbb{P}^n_x$. Similarly, the choice of coordinates for $(\mathbb{P}^n)^*$ will be $\{\xi_0, \cdots, \xi_n\}$ and we denote this by writing $(\mathbb{P}^n)^*_\xi$. We denote the coordinate ring of the product $\mathbb{P}^n_x \times (\mathbb{P}^n)^*_\xi$ by

$$\mathbb{C}[x, \xi] := \mathbb{C}[x_0, \ldots, x_n, \xi_0, \ldots, \xi_n]$$

and examine some of its relevant ideals. When $Y \subset X$ is a subvariety defined by some ideal

$$I_Y = \langle f_1, \ldots, f_r \rangle \lhd \mathbb{C}[x],$$

then I_Y is canonically identified with an ideal $\langle f_1, \ldots, f_r \rangle$ of $\mathbb{C}[x, \xi]$ because the f_i are also polynomials in $\mathbb{C}[x, \xi]$; in a mild abuse of notation, we will also refer to this ideal in $\mathbb{C}[x, \xi]$ as I_Y. Note that both $\kappa_X^{-1}(Y)$ and $\mathbf{Con}(X) \cap \mathbf{Con}(Y)$ are also defined by ideals of $\mathbb{C}[x, \xi]$ — the next result provides a convenient relation between these three ideals.

PROPOSITION 3.1. *Let $X \subset \mathbb{P}^n_x$ be a projective variety with conormal map $\kappa_X : \mathbf{Con}(X) \to X$. For any subvariety $Y = \mathbf{V}(f_1, \ldots, f_r) \subset X$, we have*

$$I_{\kappa_X^{-1}(Y)} = I_{\mathbf{Con}(X)} + I_Y$$

in the polynomial ring $\mathbb{C}[x,\xi]$.

PROOF. The polynomials $\{f_1,\ldots,f_r\}$ in $\mathbb{C}[x]$ are also polynomials in the ring $\mathbb{C}[x,\xi]$; set $K = \langle f_1,\ldots,f_r\rangle \subset \mathbb{C}[x,\xi]$. Let $\pi : \mathbb{P}^n_x \times (\mathbb{P}^n)^*_\xi \to \mathbb{P}^n_x$ be the standard coordinate projection onto the first factor. We have an equality of schemes

$$\kappa_X^{-1}(Y) = \mathbf{Con}(X) \cap \pi^{-1}(Y),$$

and it therefore suffices to determine the ideal which generates $\pi^{-1}(Y)$. Since Y is a variety and hence reduced, $\pi^{-1}(Y)$ is also reduced (although $\mathbf{Con}(X)\cap\pi^{-1}(Y)$ may not be). Hence we only need to show that $\mathbf{V}(K)$ and $\pi^{-1}(Y)$ are equal as sets. Now for any (x,ξ) in $\mathbf{Con}(X)$, we have:

$$\begin{aligned}
(x,\xi)\in\pi^{-1}(Y) &\iff x \in Y && \text{by Definition of } \pi\\
&\iff f_\ell(x) = 0 \text{ for } 1 \le \ell \le r && \text{since } Y = \mathbf{V}(f_1,\ldots,f_r)\\
&\iff f_\ell(x,\xi) = 0 \text{ for } 1 \le \ell \le r && \text{by Definition of } K\\
&\iff (x,\xi) \in \mathbf{V}(K)
\end{aligned}$$

Consequently, have an equality

$$\kappa_X^{-1}(Y) = \mathbf{Con}(X) \cap \mathbf{V}(K)$$

of schemes; the desired conclusion now follows since the scheme on the right hand side is defined by $I_{\mathbf{Con}(X)} + K$. □

Our next goal, given a candidate stratum-closure Y inside a projective variety X, is to identify all those points of Y (if any) which fail to satisfy Condition (B) with respect to X_{reg}. The next result constructs an ideal J in $\mathbb{C}[x,\xi]$ and shows that that these offending points must lie within $\overline{\kappa_X(\mathbf{V}(J))}$; and moreover, removing these points leaves us with a Zariski dense (and hence nonempty) subset Y' of Y. Here we restrict to the case where the variety X is *pure-dimensional*, meaning that all of its irreducible components must have the same dimension.

LEMMA 3.2. *Let $X \subset \mathbb{P}^n_x$ be a pure dimensional projective variety and Y a nonempty irreducible subvariety of its singular locus X_{Sing}. If $J \lhd \mathbb{C}[x,\xi]$ is the saturation*

$$J = I_{\kappa_X^{-1}(Y)} : \left(I_{\mathbf{Con}(X)\cap\mathbf{Con}(Y)}\right)^\infty,$$

and $\mathbf{V}(J) \subset \mathbb{P}^n_x \times (\mathbb{P}^n)^*_\xi$ the corresponding variety, then, the set $Y' := Y_{\text{reg}} - \overline{\kappa_X(\mathbf{V}(J))}$ is Zariski dense in Y.

PROOF. We seek to show that the Zariski closure of Y' equals Y; and since Y is irreducible, it is enough to establish that $Y - \overline{\kappa_X(\mathbf{V}(J))}$ is nonempty. To this end, let $i > j$ be the dimensions of X and Y respectively, and impose a Whitney stratification on X:

$$\varnothing = X_{-1} \subset X_0 \subset \cdots \subset X_{i-1} \subset X_i = X,$$

so that the j-stratum $Z = X_j - X_{j-1}$ intersects Y in a dense subset $Z \cap Y$. This density has three immediate but important consequences:

(i) by Proposition 1.2, we know that $\mathbf{Con}(Z\cap Y) = \mathbf{Con}(Y)$; also,
(ii) since $Y \neq \varnothing$ by assumption, $\mathbf{Con}(Z\cap Y)\cap\mathbf{Con}(X) = \mathbf{Con}(Y)\cap\mathbf{Con}(X)$ is also non-empty; and finally,

(iii) since κ_X is a projection map, we have $\dim(\kappa_X^{-1}(Z \cap Y)) = \dim(\kappa_X^{-1}(Y))$.

Moreover, since $Z \subset X$ is a Whitney stratum, the pair $(X_{\text{reg}}, Y \cap Z)$ satisfies Condition (B); so by (1) and (2) we have containments

$$\mathscr{I}[\kappa_X^{-1}(Y)] \subset \mathscr{I}[\kappa_X^{-1}(Y \cap Z)] \subset \mathscr{I}[\mathbf{Con}(X) \cap \mathbf{Con}(Y)] \subset \overline{\mathscr{I}}[\kappa_X^{-1}(Y \cap Z)]. \tag{4}$$

of ideal sheaves. Let's write $\mathscr{S}$ and $\mathscr{S}^+$ to indicate the schemes associated to $\mathscr{I}[\kappa_X^{-1}(Y)]$ and $\overline{\mathscr{I}}[\kappa_X^{-1}(Y \cap Z)]$ respectively. As noted in consequence (iii) above, $\dim(\kappa_X^{-1}(Z \cap Y))$ equals $\dim(\kappa_X^{-1}(Y))$. Since an ideal sheaf and its integral closure have the same support (see [22, Remark 1.1.3] for instance), the containments in (4) guarantee

$$\dim(\mathscr{S}) = \dim(\kappa_X^{-1}(Y)) = \dim(\kappa_X^{-1}(Z \cap Y)) = \dim(\mathbf{Con}(X) \cap \mathbf{Con}(Y)) = \dim(\mathscr{S}^+).$$

Since by (4) the scheme $\mathscr{S}$ contains $\mathbf{Con}(X) \cap \mathbf{Con}(Y)$ and these schemes have the same dimension, they must agree in at least one irreducible component of maximal dimension; choose some (x, ξ) in this component, so (x, ξ) does not lie in $\mathbf{V}(J)$. Finally, since J is defined as a saturation, we know from (3) that

$$\mathbf{V}(J) = \overline{\kappa_X^{-1}(Y) - (\mathbf{Con}(X) \cap \mathbf{Con}(Y))},$$

whence x lies in $Y - \overline{\kappa_X(\mathbf{V}(J))}$ as desired. □

We now arrive at our main result, which establishes that the variety Y' constructed in Lemma 3.2 satisfies Condition (B) with respect to X_{reg}.

THEOREM 3.3. *Let $Y \subset X \subset \mathbb{P}^n_x$ and $J \lhd \mathbb{C}[x, \xi]$ be defined as in the statement of Lemma 3.2; writing Y' for the difference $Y_{\text{reg}} - \overline{\kappa_X(\mathbf{V}(J))}$, the pair (X_{reg}, Y') satisfies Condition (B).*

PROOF. By (3) and the definition of Y', we have the containment

$$\kappa_X^{-1}(Y') \subseteq \overline{\mathscr{V}(I_Y + I_{\mathbf{Con}(X)}) - \mathscr{V}(J)}$$

as subschemes of $\mathbb{P}^n_x \times (\mathbb{P}^n)^*_\xi$. By Lemma 3.2 we know that Y' is dense in Y, and therefore $\mathbf{Con}(Y') = \mathbf{Con}(Y)$ by Proposition 1.2. We will show that $\kappa_X^{-1}(Y')$ and $\mathbf{Con}(X) \cap \mathbf{Con}(Y)$ are equal. By (2) and Proposition 1.2, we immediately have the containment

$$\mathbf{Con}(X) \cap \mathbf{Con}(Y) = \mathbf{Con}(X) \cap \mathbf{Con}(Y') \subseteq \kappa_X^{-1}(Y').$$

For the reverse inclusion, note by (3) that $\overline{\mathscr{V}(I_Y + I_{\mathbf{Con}(X)}) - \mathscr{V}(J)}$ equals

$$\overline{\mathscr{V}(I_Y + I_{\mathbf{Con}(X)}) - (\overline{\mathscr{V}(I_Y + I_{\mathbf{Con}(X)}) - (\mathbf{Con}(X) \cap \mathbf{Con}(Y))})},$$

which is evidently a subset of $\mathbf{Con}(X) \cap \mathbf{Con}(Y)$. Therefore, we have

$$\kappa^{-1}(Y') = \mathbf{Con}(X) \cap \mathbf{Con}(Y) = \mathbf{Con}(X) \cap \mathbf{Con}(Y')$$

as schemes. Hence the conclusion follows by Proposition 2.3 along with the fact that an ideal is always contained in it's integral closure. □

In particular all points in Y_{reg} at which Condition (B) fails with respect to X_{reg} are forced to lie in the closure $\overline{\kappa_X(\mathbf{V}(J))}$. Algebraically, one computes this closure via elimination, i.e., by using the fact that $\overline{\kappa_X(\mathbf{V}(J))} = \mathbf{V}(J \cap \mathbb{C}[x])$.

4. Stratifying Projective Varieties

In this section we describe a recursive algorithm which uses Theorem 3.3 to compute Whitney stratifications of pure-dimensional projective varieties. Since each irreducible component of an arbitrary projective variety can be stratified separately, this is by no means a severe restriction. And since various intermediate varieties which get constructed in our algorithm won't satisfy this purity criterion, it will be convenient to let $\text{Pure}_d(Z)$ denote the set of all the pure d-dimensional irreducible components of a given projective variety Z. These components can be algorithmically extracted by computing the *minimal associated primes* of the defining ideal I_Z, and the main cost is a Gröbner basis computation — see [8].

4.1. Computing Conormal Ideals. In light of Theorem 3.3, we will often be required to compute (the equations which define) the conormal space $\mathbf{Con}(Z)$ of a given projective variety $Z \subset \mathbb{P}^n$. The ideal $I_{\mathbf{Con}(Z)} \lhd \mathbb{C}[x, \xi]$ is extracted in practice as follows. First, we let $I_Z = \langle f_1, \ldots, f_r \rangle$ be any defining ideal of Z, and let c be its codimension $n - \dim(Z)$. One computes the Jacobian ideal $\mathbf{Jac}_Z$ of Z which is generated by all the $c \times c$ minors of the matrix of partial derivatives

$$\begin{bmatrix} \partial f_1/\partial x_0 & \cdots & \partial f_1/\partial x_n \\ \vdots & \ddots & \vdots \\ \partial f_r/\partial x_0 & \cdots & \partial f_r/\partial x_n \end{bmatrix},$$

see [27, pg. 27–28]. The singular set $Z_{\text{sing}} \subset Z$ is (by definition) the zero locus of the Jacobian ideal, namely:

$$Z_{\text{sing}} = \mathbf{V}(I_Z + \mathbf{Jac}_Z).$$

Now if we let $\mathbf{Jac}_Z^{\xi}$ be the ideal generated by the $(c+1) \times (c+1)$ minors of the ξ_i-augmented Jacobian matrix:

$$\begin{bmatrix} \xi_0 & \cdots & \xi_n \\ \partial f_1/\partial x_0 & \cdots & \partial f_1/\partial x_n \\ \vdots & \ddots & \vdots \\ \partial f_r/\partial x_0 & \cdots & \partial f_r/\partial x_n \end{bmatrix},$$

then $I_{\mathbf{Con}(Z)}$ is given by the saturation $(I_Z + \mathbf{Jac}_Z^{\xi}) : (\mathbf{Jac}_Z)^{\infty}$, see [9, Eq. (5.1)] or [21, §2] for instance. Algorithms for computing ideal sums and saturations are standard fare across computational algebraic geometry, and may be found in [7] for instance.

4.2. The Saturation Subroutine. For any pair of projective varieties $X, Y \subset \mathbb{P}^n$ with $Y \subset X_{\text{sing}}$, the following subroutine, called **Decompose**, implements the saturation-based constructions of Lemma 3.2. Although we present the inputs and outputs of all our algorithms as projective varieties, in practice these must be represented on a machine by some choice of generating polynomials of their defining ideals.

	Decompose(Y, X)
	Input: Projective varieties $Y \subset X$ in $\mathbb{P}^n$, with $d := \dim Y$. **Output:** A list of subvarieties $Y_\bullet$ of Y.
01	**Set** $Y_\bullet := (Y_d, Y_{d-1}, \ldots, Y_0) := (Y, \varnothing, \ldots, \varnothing)$
02	**For each** irreducible component Z of Y
03	**Set** $J := (I_{\mathbf{Con}(X)} + I_Z) : (I_{\mathbf{Con}(X)} + I_{\mathbf{Con}(Z)})^\infty \subset \mathbb{C}[x, \xi]$
04	**Set** $K := J \cap \mathbb{C}[x]$
05	**Set** $W := Z \cap \mathbf{V}(K)$
06	**For each** irreducible component V of W
07	**Add** V to $Y_{\geq \dim(V)}$
08	**Return** $Y_\bullet$

The notation in Line 07 is meant to indicate that V is added to Y_i for all $i \geq \dim V$. This subroutine terminates because the **For** loops in lines 02 and 06 are indexed over irreducible components of projective varieties, of which there can only be finitely many. The following result is a consequence of Lemma 3.2 and Theorem 3.3.

PROPOSITION 4.1. *Let $Y \subset X$ be a pair of projective varieties in $\mathbb{P}^n$ so that $Y \subset X_{\text{sing}}$ and $\dim Y = d$. If* **Decompose** *is called with input (Y, X), then:*

(1) for all $i \in \{0, \ldots, d-1\}$, its output varieties satisfy $Y_i \subset Y_{i+1}$; also,
(2) Y_{d-1} is a (possibly empty) subvariety of Y with $\dim Y_{d-1} < \dim Y$; and finally,
(3) all points of Y_{reg} where Condition (B) fails with respect to X_{reg} lie in Y_{d-1}, so the pair $(X_{\text{reg}}, Y_{\text{reg}} - Y_{d-1})$ satisfies Condition (B).

PROOF. The first assertion holds because of Line 07: whenever an irreducible Z is added to $Y_{\dim Z}$, it is also added to all the subsequent Y_i with $i > \dim Z$. For each irreducible $Z \subset Y$, the variety W from Line 05 is precisely $\overline{\kappa_X(\mathbf{V}(J))}$, where

$$J = I_{\kappa_X^{-1}(Z)} : (I_{\mathbf{Con}(X) \cap \mathbf{Con}(Z)})^\infty.$$

(As before, κ_X is the conormal map of X.) Using the fact that Y equals the union $\bigcup_Z Z$ as Z ranges over its irreducible components, the second assertion follows from Lemma 3.2. Finally, by Theorem 3.3, this variety W contains all those points of Z_{reg} where Condition (B) fails to hold with respect to X_{reg}. Thus, the third assertion follows from the fact that $Y_{\text{reg}} \subseteq \bigcup_Z Z_{\text{reg}}$. ☐

It might be helpful to note that the output $Y_\bullet$ of **Decompose** described above need not constitute a Whitney stratification of the input variety Y — Proposition 4.1 does not guarantee that Condition (B) holds among successive differences of the form $Y_i - Y_{i-1}$.

4.3. The Main Algorithm. Let $X \subset \mathbb{P}^n$ be a pure k-dimensional complex projective variety defined by a radical homogeneous ideal I_X. The algorithm **WhitStrat**, described below, takes in X as input and returns a nested sequence of its subvarieties $X_\bullet$. An essential part of this algorithm is a **Merge** statement, which accepts two nested sequences of varieties $V_\bullet$ and $W_\bullet$ of lengths k and d, with $k > d$. To **Merge** $V_\bullet$ with $W_\bullet$, one updates the longer sequence $V_\bullet$ via the following rule:

$$V_i \leftarrow V_i \cup W_{\max(i,d)}.$$

The property $V_i \subset V_{i+1}$ continues to hold after such an operation. The following algorithm **WhitStrat** uses **Decompose** in order to construct Whitney stratifications of pure-dimensional projective varieties.

	WhitStrat(X)
	Input: A pure k-dimensional variety $X \subset \mathbb{P}^n$. **Output:** A list of subvarieties $X_\bullet$ of X.
01	**Set** $X_\bullet := (X_k, X_{k-1}, \ldots, X_0) := (X, \varnothing, \ldots, \varnothing)$
02	**Compute** X_{Sing} and $\mu := \dim(X_{\text{Sing}})$
03	**For each** irreducible component Z of X_{sing}
04	**Add** Z to $X_{\geq \dim Z}$
05	**For each** d in $(\mu, \mu-1, \ldots, 1, 0)$
06	**Merge** $X_\bullet$ with **Decompose**$(\text{Pure}_d(X_d), X)$
07	**Merge** $X_\bullet$ with **WhitStrat**$(\text{Pure}_d(X_d))$
08	**Return** $X_\bullet$

To verify that this algorithm terminates, we note that the **For** loop on Line 03 runs once per irreducible component of X_{sing}, of which there can only be finitely many. And the **For** loop on Line 05 will terminate provided that the recursive call on Line 07 terminates; but the dimension of the variety X_d is bounded above by $\mu < k$, i.e., it is strictly less than the dimension of the input variety. Thus, the recursion terminates after finitely many steps and produces a nested sequence $X_\bullet$ of subvarieties of X. Our goal in the next section is to establish that $X_\bullet$ constitutes a valid Whitney stratification of X.

4.4. Correctness. Let $X_\bullet(d)$ denote the nested sequence of varieties $X_\bullet$ as they stand at the *end* of the d-th iteration of **For** loop (in Line 05 of **WhitStrat**) where $d = (\mu, \mu - 1, \ldots, 1, 0)$. These $X_i(d)$ fit into a $(\mu+1) \times (k+1)$ grid of projective varieties and inclusion maps:

$$\begin{array}{ccccccccc}
X_k(\mu) & \hookleftarrow & X_{k-1}(\mu) & \hookleftarrow & \cdots & \hookleftarrow & X_1(\mu) & \hookleftarrow & X_0(\mu) \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
X_k(\mu-1) & \hookleftarrow & X_{k-1}(\mu-1) & \hookleftarrow & \cdots & \hookleftarrow & X_1(\mu-1) & \hookleftarrow & X_0(\mu-1) \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \ddots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
X_k(1) & \hookleftarrow & X_{k-1}(1) & \hookleftarrow & \cdots & \hookleftarrow & X_1(1) & \hookleftarrow & X_0(1) \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
X_k(0) & \hookleftarrow & X_{k-1}(0) & \hookleftarrow & \cdots & \hookleftarrow & X_1(0) & \hookleftarrow & X_0(0)
\end{array} \tag{5}$$

The vertical inclusion maps arise from the fact that each $X_\bullet(d)$ is obtained from the previous $X_\bullet(d+1)$ by performing two **Merge** operations (in Lines 06 and 07 respectively). Note that each iteration of the **For** loop in Line 05 moves us from one row to the next, until at last the bottom row (corresponding to index $d = 0$) contains the output.

REMARK 4.2. Before entering the **For** loop of Line 05, the sequence $X_\bullet$ contains the input variety X in the top dimension (i.e., $X_k = X$) and irreducible components of X_{sing}

in lower dimensions. It follows that the columns of our grid (5) come in three flavours, depending on the index i:

(1) the left-most column (with index $i = k$) identically equals X, i.e., $X_k(d) = X$ regardless of the row index d in $\{0, \ldots, k\}$;
(2) the next few columns (with index $k > i \geq \mu$) are also constant — independent of the row index d, each $X_i(d)$ in this range equals X_{sing}; and finally,
(3) the i-th column for $\mu > i \geq 0$ stabilizes below its $(i+1)$-indexed entry:

$$X_i(d) = X_i(i+1) \quad \text{for all} \quad 0 \leq d \leq i < \mu. \tag{6}$$

These three assertions follow from the observation that during the d-indexed iteration of the **For** loop in Line 05, the two **Merge** operations (from Lines 06 and 07) which produce the row $X_\bullet(d)$ are only allowed to merge subvarieties of $X_d(d+1)$ to the preceding row $X_\bullet(d+1)$.

Define the successive differences across the rows of the grid (5), i.e.,

$$S_i(d) := X_i(d) - X_{i-1}(d). \tag{7}$$

It follows from Lines 02-04 of **WhitStrat** that we have the containment

$$X_i(d)_{\text{Sing}} \subset X_{i-1}(d) \text{ for all } d \leq i,$$

whence $S_i(d)$ is a smooth manifold whenever $d \leq i$. Therefore, $S_i(0)$ is smooth for all i. In the remainder of this Section, we will now show that $S_i(0)$ constitutes the i-strata of a Whitney stratification $X_\bullet(0)$ of the input variety X. Here is a first step in this direction.

PROPOSITION 4.3. *The pair $(S_k(i), S_d(i))$ satisfies Condition (B) for all $0 \leq i \leq d \leq k$. In other words, $X_{d-1}(i)$ contains all points of $X_d(i)_{\text{reg}}$ where Condition (B) fails with respect to $S_k(i)$.*

PROOF. By Remark 4.2, we may safely restrict to the case $0 \leq i \leq d \leq \mu$ since we have

$$X_\mu(d) = X_{\mu+1}(d) = \cdots = X_{k-1}(d)$$

for all d by Remark 4.2(2). Let $V_\bullet$ denote the output of **Decompose** obtained (in Line 06) during the d-th iteration of the **For** loop in Line 05. It follows from Proposition 4.1(3) that the pair

$$P := \left(X_{\text{reg}}, X_d(d+1)_{\text{reg}} - V_d\right)$$

satisfies Condition (B). By Proposition 4.1(2), we know that V_d is a subvariety of $X_d(d+1)$ of dimension strictly smaller than d. Since $V_\bullet$ is merged with $X_\bullet(d+1)$ in Line 06 en route to producing $X_\bullet(d)$, we know that V_d must in fact be a subvariety of $X_{d-1}(d)$. Now note that

$$\begin{aligned} S_k(d) &= X_k(d) - X_{k-1}(d) && \text{by (7)}, \\ &= X - X_{k-1}(d) && \text{by Remark 4.2(1).} \end{aligned}$$

Since $\dim X_{k-1}(d) < k$, we know that $S_k(d)$ is dense in X_{reg}. Moreover, since V_d is entirely contained in $X_{d-1}(d)$, we may apply Proposition 2.1 to the pair P above and conclude that the new pair

$$P' := \left(S_k(d), X_d(d+1)_{\text{reg}} - X_{d-1}(d)\right)$$

also satisfies Condition (B). Now let $W_\bullet$ be the output of the recursive call to **WhitStrat** in Line 07 during the d-th iteration of the **For** loop in Line 05. From Lines 02-04, we deduce that $X_d(d+1)_{\text{sing}} \subset W_{d-1}$, and after the **Merge** operation of Line 07 we are also guaranteed $W_{d-1} \subset X_{d-1}(d)$. Putting these containments together gives

$$X_d(d+1)_{\text{sing}} \subset W_{d-1} \subset X_{d-1}(d).$$

By (6), we have $X_d(d+1) = X_d(d)$, whence $X_d(d)_{\text{sing}} \subset X_{d-1}(d)$. Therefore, the difference $S_d(d) := X_d(d) - X_{d-1}(d)$ in fact equals $X_d(d+1)_{\text{reg}} - X_{d-1}(d)$. Using this in the pair P' guarantees that the pair $(S_k(d), S_d(d))$ satisfies Condition (B). Finally, the conclusion for the pair $(S_k(i), S_d(i))$ follows since subsequent iterations of the the **For** loop (corresponding to lower d values) do not alter $X_{\geq d}$. □

Next, we will establish that Condition (B) is satisfied by arbitrary pairs of successive differences in (5) for sufficiently small row index.

PROPOSITION 4.4. *The pair $(S_j(i), S_i(i))$ satisfies Condition (B) for all $0 \leq i < j \leq k$.*

PROOF. From Remark 4.2, we have $S_j(i) = S_j(j)$ since $i < j$. Let $W_\bullet$ be the output of the recursive call in Line 07 of **WhitStrat** during the $d = j$ iteration of the **For** loop in Line 05, so we have $W_j = \text{Pure}_j(X_j(j))$. Recall by (7) that $S_j(i) = X_j(i) - X_j(i-1)$; now any irreducible component $Y \subset X_j(i)$ with $\dim Y < j$ must also lie in $X_{j-1}(i)$, whence

$$\begin{aligned} S_j(i) &= \text{Pure}_j(X_j(j)) - X_{j-1}(i) \\ &= W_j - X_{j-1}(i). \end{aligned}$$

Now W_{j-1} lies in $X_{j-1}(j)$ because of the **Merge** operation in Line 07 of **WhitStrat**, and in turn $X_{j-1}(j)$ is a subvariety of $X_{j-1}(i)$ as described in (5). Since both varieties $W_{j-1} \subset X_{j-1}(i)$ have dimension strictly smaller than j, we have that $S_j(i)$ is dense in $W_j - W_{j-1}$. Thus, it suffices to show that all points in $W_j \cap X_i(i)$ where Condition (B) fails with respect to $S_j(i)$ lie within $W_j \cap X_{i-1}(i)$. To confirm this, note that by construction $X_i(i)$ is a union of the form $W_i \cup Z$, where the subvariety $Z \subset X_i(i)$ has $\dim(Z) \leq i$. First we consider the case where Z is empty; in this case, we know from Proposition 4.4 that the pair

$$(W_j - W_{j-1}, W_i - W_{i-1})$$

satisfies Condition (B), so the desired result follows immediately from Proposition 2.1. On the other hand, if Z is nonempty, we can assume without loss of generality that Z is not contained in W_i. Now any point of $W_j \cap (X_i(i) - W_i) = W_j \cap (Z - W_i)$ where Condition (B) fails with respect to W_j must lie in W_{j-1} by Proposition 4.3. Thus, no such point lies in $S_j(i)$, and it remains to show that all points in $W_i \cap X_i(i) = W_i$ where Condition (B) fails with respect to $S_j(i)$ are contained in $X_{i-1}(i)$. But since $W_{i-1} \subset X_{i-1}(i)$ due to the **Merge** operation, this follows immediately from Proposition 4.3. □

We can now confirm that the output of **WhitStrat** constitutes a valid Whitney stratification of X.

THEOREM 4.5. *When called on a pure k-dimensional complex projective variety $X \subset \mathbb{P}^n$, the output $X_\bullet$ of* **WhitStrat** *forms a Whitney stratification of X.*

PROOF. As remarked after (7), each $S_i(d)$ is smooth for $d \leq i$. The conclusion follows immediately from Proposition 4.4 since for any pair $(S_j(i), S_i(i))$ with $0 \leq i \leq j \leq k$ we have $S_i(i) = S_i(0)$ and $S_j(i) = S_j(0)$ by Remark 4.2. Thus, Condition (B) holds for every pair $(S_j(0), S_i(0))$ with $i \leq j \leq k$. □

This algorithm can also be used to stratify affine complex varieties via the following dictionary: for each affine complex variety $X \subset \mathbb{C}^n$, we write $PX \subset \mathbb{P}^n$ for its *projective closure* [27, Proposition 2.8], which is constructed as follows. Let $\{f_1, \ldots, f_r\}$ be a Gröbner basis for the defining ideal $I_X \lhd \mathbb{C}[x_1, \ldots, x_n]$. For each i in $\{1, \ldots, r\}$, write F_i for the homogenisation of f_i in $\mathbb{C}[x_0, x_1, \ldots, x_n]$. Then, the projective closure $PX \subset \mathbb{P}^n$ is the complex projective variety given by $\mathbf{V}(F_1, \ldots, F_r)$. Conversely, one can recover X from PX by *dehomogenizing*, i.e., by setting $x_0 = 1$ in each defining polynomial F_i.

COROLLARY 4.6. *Let $X \subset \mathbb{C}^n$ be a pure k-dimensional affine complex variety and let $PX \subset \mathbb{P}^n$ be its projective closure. If $PX_\bullet$ is the output of* **WhitStrat**(PX)*, then a valid Whitney stratification of X is given by $X_\bullet$, where each X_i is the dehomogenization of PX_i.*

PROOF. Since $PX_\bullet$ defines a Whitney stratification of PX, it follows from Proposition 2.1 that intersecting each PX_i with a dense subset $D \subset PX$ constitutes a Whitney stratification of D. The conclusion follows by considering $D = PX - \mathbf{V}(x_0)$. ☐

It should be noted that the stratifications produced by **WhitStrat** may not be minimal; and moreover, the stratification of X described in the preceding Corollary may not be minimal even if the output stratification of PX is minimal.

5. Flag-Subordinate Stratifications

By a *flag* $\mathbf{F}_\bullet$ on a variety X we mean any finite nested set of subvarieties of the form

$$\varnothing = \mathbf{F}_{-1}X \subset \mathbf{F}_0X \subset \mathbf{F}_1X \subset \cdots \subset \mathbf{F}_{\ell-1}X \subset \mathbf{F}_\ell X = X.$$

If X is projective, we implicitly require each $\mathbf{F}_iX$ to also be projective. We call ℓ the *length* of the flag $\mathbf{F}_\bullet$. Aside from these containments, there are no restrictions on the dimensions of the individual $\mathbf{F}_iX$; and in particular, we do not require successive differences $\mathbf{F}_iX - \mathbf{F}_{i-1}X$ to be smooth manifolds, let alone satisfy Condition (B).

DEFINITION 5.1. Let $X \subset \mathbb{P}^n$ be a projective variety and $\mathbf{F}_\bullet$ a flag on X of length ℓ. A Whitney stratification $X_\bullet$ of X is **subordinate** to $\mathbf{F}_\bullet$ if for each stratum $S \subset X$ of $X_\bullet$ there exists some $j = j(S)$ in $\{0, \ldots, \ell\}$ satisfying $S \subset (\mathbf{F}_jX - \mathbf{F}_{j-1}X)$.

It is crucial to note that the number $j(S)$ from the preceding Definition need not equal $\dim S$, and that one does not require $j(S) = j(S')$ whenever $\dim S = \dim S'$.

Fix a pure-dimensional complex projective variety $X \subset \mathbb{P}^n$ as well as a flag $\mathbf{F}_\bullet$ on X of length $\ell < \infty$. The following subroutine accepts as input any (not necessarily pure dimensional) subvariety $W \subset X$ along with the flag $\mathbf{F}_\bullet$, and constructs the *induced flag* $\mathbf{F}'_\bullet$ on W defined by

$$\mathbf{F}'_jW := W \cap \mathbf{F}_jX$$

for all j in $\{0, 1, \ldots, \ell\}$.

	InducedFlag$(W, \mathbf{F}_\bullet)$
	Input: A subvariety $W \subset X$ and a flag $\mathbf{F}_\bullet$ on X of length ℓ. **Output:** A flag $\mathbf{F}'_\bullet$ on W of length ℓ.
01	**Set** $\mathbf{F}'_\bullet W := (\mathbf{F}'_\ell W, \ldots, \mathbf{F}'_0 W) := (\varnothing, \ldots, \varnothing)$
02	**For each** irreducible component V of W
03	**Add** V to $\mathbf{F}'_i W$ **for all** $\mathbf{F}'_i$ where $V \subset \mathbf{F}'_i$
04	**For each** j with $\dim(\mathbf{F}_j X \cap V) < \dim V$
05	**Add** $V_j := (\mathbf{F}_j X \cap V)$ to $\mathbf{F}'_i W$ **for all** $\mathbf{F}'_i$ where $V_j \subset \mathbf{F}'_i$
06	**Return** $\mathbf{F}'_\bullet W$

The strategy for producing an $\mathbf{F}_\bullet$-subordinate stratification of X is to modify the algorithms of Section 4 as follows: whenever one wishes to **Add** an irreducible i-dimensional subvariety $W \subset X$ to the output sequence $X_{\geq i}$, one **Merge**s $X_\bullet$ with **InducedFlag**$(W, \mathbf{F}_\bullet)$ instead. For completeness, we have written out the modified versions of both **Decompose** and **WhitStrat** (the originals can be found in Sections 4.2 and 4.3 respectively).

5.1. Flag-Subordinate Stratification Algorithms. Here is the flag-subordinate avatar of **Decompose**; as promised, it only differs from the original in Line 06: the statement which **Add**ed an irreducible variety to a sequence has now been replaced with a **Merge**.

	DecomposeFlag$(Y, X, \mathbf{F}_\bullet)$
	Input: Proj. varieties $Y \subset X$ with $d := \dim Y$ and a flag $\mathbf{F}_\bullet$ on X. **Output:** A list of subvarieties $Y_\bullet \subset Y$.
01	**Set** $Y_\bullet := (Y_d, Y_{d-1}, \ldots, Y_0) := (\varnothing, \ldots, \varnothing)$
02	**For each** irreducible component Z of Y
03	**Set** $J := (I_{\mathbf{Con}(X)} + I_Z) : (I_{\mathbf{Con}(X)} + I_{\mathbf{Con}(Z)})^\infty \subset \mathbb{C}[x, \xi]$
04	**Set** $K := J \cap \mathbb{C}[x]$
05	**Set** $W := Z \cap \mathbf{V}(K)$
06	**Merge** $Y_\bullet$ with **InducedFlag**$(W, \mathbf{F}_\bullet)$
07	**Return** $Y_\bullet$

And here is the variant of **WhitStrat** which produces an $\mathbf{F}_\bullet$-subordinate stratification of $X \subset \mathbb{P}^n$. Again, the only difference occurs in Line 04.

	WhitStratFlag$(X, \mathbf{F}_\bullet)$
	Input: A pure k-dimensional variety $X \subset \mathbb{P}^n$ and a flag $F_\bullet$ on X. **Output:** A list of subvarieties $X_\bullet \subset X$.
01	**Set** $X_\bullet := (X_k, X_{k-1}, \ldots, X_0) := (X, \varnothing, \ldots, \varnothing)$
02	**Compute** X_{Sing} and $\mu := \dim(X_{\text{Sing}})$
03	**Set** $X_d = X_{\text{Sing}}$ for all d in $\{\mu, \mu+1, \ldots, k-1\}$
04	**Merge** $X_\bullet$ with **InducedFlag**$(X_{\text{sing}}, \mathbf{F}_\bullet)$
05	**For each** d in $(\mu, \mu-1, \ldots, 1, 0)$
06	**Merge** $X_\bullet$ with **DecomposeFlag**$(X_d, X, \mathbf{F}_\bullet)$
07	**Merge** $X_\bullet$ with **WhitStratFlag**$(X_d, \mathbf{F}_\bullet)$
08	**Return** $X_\bullet$

5.2. Correctness. The following result confirms that **WhitStratFlag** produces valid flag-subordinate Whitney stratifications.

THEOREM 5.2. *Let $X \subset \mathbb{P}^n$ be a pure dimensional complex projective variety and $\mathbf{F}_\bullet$ a flag on X. When called with input $(X, \mathbf{F}_\bullet)$, the algorithm* **WhitStratFlag** *terminates and its output $X_\bullet$ is an $\mathbf{F}_\bullet$-subordinate Whitney stratification of X.*

PROOF. Termination follows for the same reasons as the ones used for **WhitStrat**, and the fact that $X_\bullet$ is a valid Whitney stratification follows from Theorem 4.5. Thus, it remains to show that each connected component S of $X_i - X_{i-1}$ is contained entirely in a single $\mathbf{F}_jX - \mathbf{F}_{j-1}X$. Any such S can be written as $Y - X_{i-1}$, where Y is an irreducible component of X_i. Note that, in particular, this Y will appear as a V in Line 03 of the **InducedFlag** subroutine when it is called with first input X_i. Let j be the smallest index of the flag $\mathbf{F}_\bullet$ for which $Y \subset \mathbf{F}_jX$ holds. Then $\dim(Y \cap \mathbf{F}_\ell X) < i$ for every $\ell < j$, and so $Y_\ell = Y \cap \mathbf{F}_\ell X$ is **Add**ed to $X_\bullet$ via Line 05 of the **InducedFlag** subroutine and hence Y_ℓ is contained in X_m for some $m < i$. Since $X_m \subset X_i$, it follows that

$$S \cap (\mathbf{F}_pX - \mathbf{F}_{p-1}X) = \varnothing \text{ whenever } p < j.$$

But since $S \subset \mathbf{F}_jX$ and Y is irreducible, we have $Y \subset \mathbf{F}_jX$, whence $Y \subset \mathbf{F}_pX$ for all $p \geq j$. Thus, S also has empty intersections with $\mathbf{F}_pX - \mathbf{F}_{p-1}X$ for $p > j$, and the desired result follows. □

We note in passing that the algorithms described in this Section can also be used to produce flag-subordinate stratifications of affine complex varieties. As before, we will write PX for the projective closure of each affine variety X; and given a flag $\mathbf{F}_\bullet$ on X, we write $\mathbf{PF}_\bullet$ for the flag on PX defined by

$$\mathbf{PF}_i(PX) := P(\mathbf{F}_iX).$$

The following result forms a natural flag-subordinate counterpart to Corollary 4.6.

COROLLARY 5.3. *Let $X \subset \mathbb{C}^n$ be a complex variety and let $\mathbf{F}_\bullet$ be a flag on X. Writing $PX_\bullet$ for the output of* **WhitStratFlag** *when called with input $(PX, \mathbf{PF}_\bullet)$, its dehomogenization $X_\bullet$ constitutes an $\mathbf{F}_\bullet$-subordinate Whitney stratification of $X_\bullet$.*

PROOF. This follows immediately from Corollary 4.6 and Theorem 5.2 along with the the fact that containment relations between varieties are preserved by both projective closures and dehomogenizations. □

Our motivation for computing flag-subordinate Whitney stratifications stems from the desire to algorithmically stratify algebraic maps between projective varieties.

6. Stratifying Algebraic Maps

A continuous map between topological spaces is called *proper* if pre-images of compact sets are compact. Reproduced below is the content of [4, Definition 3.5.1], which highlights a natural class of maps between Whitney stratified spaces; we recall for the reader's convenience that the tangent space at each point p on a smooth manifold S is denoted T_pS.

DEFINITION 6.1. Let X and Y be Whitney stratified spaces. A proper map $f : X \to Y$ is called a **stratified map** if for each stratum $S \subset X$ there exists a a stratum $R \subset Y$ so that

(1) the image $f(S)$ is wholly contained in R; and moreover,
(2) at each point x in S, the Jacobian $J_x(f|_S) : T_xS \to T_{f(x)}R$ is a surjection.

We refer to any pair $(X_\bullet, Y_\bullet)$ of Whitney stratifications of X and Y which satisfy the above requirements as a stratification of f. It follows from *Thom's first isotopy lemma* [26, Proposition 11.1] that if $f : X \to Y$ is a stratified map in the sense of this definition, then for each stratum $R \subset Y$ the restriction

$$f|_{f^{-1}(R)} : f^{-1}(R) \to R$$

has the structure of a fibration (with possibly singular fibers).

Our aim here is to algorithmically construct stratifications tailored to algebraic maps between projective varieties. We describe these maps in terms of the coordinate ring $\mathbb{C}[x] := \mathbb{C}[x_0, \ldots, x_n]$ of $\mathbb{P}^n$.

DEFINITION 6.2. A **projective morphism** $f : X \to \mathbb{P}^m$ consists of an $(m+1)$-tuple of homogeneous polynomials f_i in $\mathbb{C}[x]$, i.e.,

$$f(x) = (f_0(x), \ldots, f_m(x)),$$

where $d := \deg(f_i)$ is constant for all i and where $X \cap \mathbf{V}(f_0, \ldots, f_m) = \emptyset$.

Projective morphisms as defined above are always proper [20, Ch II, Thm 4.9], so at least that requirement of Definition 6.1 holds automatically. We will restrict to the case where X is pure dimensional; and for each projective morphism $f : X \to \mathbb{P}^m$ as defined above, we can always consider some pure-dimensional projective variety $Y \subset \mathbb{P}^m$ which contains the image $f(X)$, whence f constitutes an algebraic map $X \to Y$. We now seek to describe an algorithm which will produce a stratification $(X_\bullet, Y_\bullet)$ for any generic triple (X, Y, f). Both the genericity condition and the algorithm itself make essential use of the Thom-Boardman flag of f, which is described below.

6.1. The Thom-Boardman Flag. Let $f : X \to \mathbb{P}^m$ be a projective morphism and consider its Jacobian operator

$$Jf := \begin{bmatrix} \partial f_0/\partial x_0 & \cdots & \partial f_0/\partial x_n \\ \vdots & \ddots & \vdots \\ \partial f_m/\partial x_0 & \cdots & \partial f_m/\partial x_n \end{bmatrix}.$$

We denote by J_pf the $(m+1) \times (n+1)$ matrix which results from evaluating the partial derivatives of the polynomials f_i at each point p of $\mathbb{P}^n$.

Let $k \leq \min(n+1, m+1)$ be the largest possible rank of J_pf across $p \in \mathbb{P}^n$ and consider, for each i in $\{0, \ldots, k+1\}$, the homogeneous ideal $\mathbf{Jac}_i f$ of $\mathbb{C}[x]$ generated by all $i \times i$ minors of Jf. It follows by cofactor expansion that these ideals fit into a descending sequence, and so the corresponding projective varieties $\mathbf{T}_i^+\mathbb{P}^n := \mathbf{V}(\mathbf{Jac}_i f)$ form a flag of length $k+1$ on $\mathbb{P}^n$. By construction, we have

$$\mathbf{T}_i^+\mathbb{P}^n = \left\{ p \in \mathbb{P}^n \mid \text{rank } J_pf \leq i-1 \right\} \tag{8}$$

for $0 \leq i \leq k+1$. We call $\mathbf{T}_\bullet^+$ the the Thom-Boardman flag [31, 3] of f on $\mathbb{P}^n$. Our main focus here is not on $\mathbf{T}_\bullet^+$, but rather on the flag induced by $\mathbf{T}_\bullet^+$ on the domain variety $X \subset \mathbb{P}^n$. The desired genericity condition on f is described below.

DEFINITION 6.3. The projective morphism $f : X \to \mathbb{P}^m$ is **generic** if the varieties X and $\mathbf{T}_i^+\mathbb{P}^n$ intersect transversely in $\mathbb{P}^n$ for each i in $\{0, \ldots, k+1\}$.

Although we will not require this fact here, it is known that for a dense subset of algebraic morphisms, each $\mathbf{T}_i^+\mathbb{P}^n$ is a subvariety of $\mathbb{P}^n$ of dimension

$$\dim \mathbf{T}^+_{k'+1-i}\mathbb{P}^n = (n+1) - i \cdot (i + |m-n|),$$

with $k' := \min(n+1, m+1)$. A complete derivation of this dimension formula along with other properties of Thom-Boardman singularities can be found in [14, Chapter VI, Part I, §1]. Our main interest here is not in $\mathbf{T}^+_\bullet$, but rather the flag which it induces on X.

DEFINITION 6.4. The **Thom-Boardman flag of** f **on** X, denoted $\mathbf{T}_\bullet$, is defined for all $0 \le i \le k+1$ via the intersection $\mathbf{T}_iX := \mathbf{T}_i^+\mathbb{P}^n \cap X$. Equivalently, $\mathbf{T}_iX := \mathbf{V}(I_X + \mathbf{Jac}_i f)$, where $I_X \lhd \mathbb{C}[x]$ is the defining ideal of X.

Applying f to the $\mathbf{T}_\bullet$ produces a flag on $f(X)$ which extends to a flag on Y.

DEFINITION 6.5. Given the Thom-Boardman flag $\mathbf{T}_\bullet$ on X, its **image** is the flag $\mathbf{B}_\bullet$ of length $k+2$ on Y defined by setting

$$\mathbf{B}_iY := \begin{cases} f(\mathbf{T}_iX) & i \le k+1, \\ Y & i = k+2. \end{cases}$$

(Note that by assumption $\mathbf{B}_{k+1}Y := f(X)$ is a subvariety of $\mathbf{B}_{k+2}Y := Y$.)

6.2. Computing Images and Pre-Images. To compute the flag $\mathbf{B}_\bullet$ in practice, we require a mechanism for producing equations for $f(X')$ where $X' \subset X$ is a subvariety of X — note that this image $f(X')$ is Zariski closed in $\mathbb{P}^m$ because X is projective (see [27, Theorem 4.22]). This can be accomplished using elimination. To this end, consider the ideal

$$J := \langle y_0 - uf_0(x), \ldots, y_m - uf_m(x) \rangle$$

in the ring $\mathbb{C}[u, x_0, \ldots, x_n, y_0, \ldots, y_m]$ and set $J_\Gamma := J \cap \mathbb{C}[x, y]$. Let $\Gamma(X') \subset \mathbb{P}^n \times \mathbb{P}^m$ be the graph of the restricted map $f|_{X'}$, which is defined by the bi-homogeneous ideal

$$I_{\Gamma(X')} := I_{X'} + J_\Gamma.$$

Now $f(X') \subset \mathbb{P}^m$ is given by the elimination ideal

$$I_{f(X')} := I_{\Gamma(X')} \cap \mathbb{C}[y].$$

We will also require the dual operation to implement our algorithm; i.e., given some subvariety $Y' \subset Y$, we wish to algebraically compute its pre-image $f^{-1}(Y')$ within X. Let γ be the natural map $X \to \mathbb{P}^n \times \mathbb{P}^m$ sending each x to the pair $(x, f(x))$, so the image of γ coincides with the graph $\Gamma(X)$. Consider the coordinate projections π_x and π_y

$$\begin{array}{ccc} & \Gamma(X) & \\ \overset{\pi_x}{\swarrow} & & \overset{\pi_y}{\searrow} \\ \mathbb{P}^n & & \mathbb{P}^m \end{array}$$

onto the first and second factor, respectively. By construction, for each point x in X we have the equality $f(x) = \pi_y \circ \gamma(x)$; and since all spaces and maps in sight are projective, the images of closed sets remain closed. Now consider a subvariety $Y' \subset f(X)$ and note that its pre-image under f is

$$f^{-1}(Y') := \pi_x(\Gamma(X) \cap \pi_y^{-1}(Y')).$$

Treating $I_{Y'} \lhd \mathbb{C}[y]$ as an ideal in $\mathbb{C}[x,y]$, the desired pre-image $f^{-1}(Y')$ may be computed algebraically as the intersection

$$I_{f^{-1}(Y')} := \mathbb{C}[x] \cap (I_{\Gamma(X)} + I_{Y'}).$$

6.3. Algorithm. Assume that $f : X \to Y$ is a generic projective morphism in the sense of Definition 6.3, where $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$ are pure dimensional projective varieties with $f(X) \subset Y$, and let $k \leq \min(n+1, m+1)$ as in Section 6.1. The following algorithm relies on **WhitStratFlag** (from Sec 5.1) to build Whitney stratifications $X'_\bullet$ and $Y'_\bullet$ of X and Y respectively via the following basic strategy. First we construct the image $\mathbf{B}_\bullet$ of the Thom-Boardman flag $\mathbf{T}_\bullet$ as described in Definition 6.5. Next, we create a $\mathbf{B}_\bullet$-subordinate stratification $Y_\bullet$ of the codomain Y and pull it back across f to obtain a flag $\mathbf{F}_\bullet$ on the domain X. Finally, we create an $\mathbf{F}_\bullet$-subordinate stratification $X'_\bullet$ of X.

	WhitStratMap(X, Y, f)
	Input: Pure dimensional varieties X, Y and a generic morphism $f : X \to Y$. **Output:** Lists of subvarieties $X_\bullet \subset X$ and $Y_\bullet \subset Y$.
01	**Set** $\mathbf{T}_\bullet X := (\mathbf{T}_{k+1}X, \ldots, \mathbf{T}_0 X) := (X, \varnothing, \ldots, \varnothing)$
02	**Set** $\mathbf{B}_\bullet Y := (\mathbf{B}_{k+2}Y, \ldots, \mathbf{B}_0 Y) := (Y, \varnothing, \ldots, \varnothing)$
03	**For each** j in $(0, 1, \ldots, k)$
04	**Set** $\mathbf{T}_j X := \mathbf{V}(I_X + \mathbf{Jac}_j f)$
05	**Set** $\mathbf{B}_j Y := f(\mathbf{T}_j X)$
06	**Set** $\mathbf{B}_{k+1} Y := f(X)$
07	**Set** $Y'_\bullet :=$ **WhitStratFlag**$(Y, \mathbf{B}_\bullet)$
08	**For each** i in $(0, 1, \ldots, \dim Y)$
09	**Set** $\mathbf{F}_i X := f^{-1}(Y'_i)$
10	**Set** $X'_\bullet :=$ **WhitStratFlag**$(X, \mathbf{F}_\bullet)$
11	**Set** $(X_\bullet, Y_\bullet) :=$ **Refine**$(X'_\bullet, Y'_\bullet, f)$
12	**Return** $(X_\bullet, Y_\bullet)$

As we have not described the **Refine** subroutine invoked in the penultimate line, we are not yet able to check whether this algorithm terminates, and whether it returns a correct stratification of f in the sense of Definition 6.1. The next result, which involves the Whitney stratifications $X'_\bullet$ and $Y'_\bullet$ produced in Lines 10 and 07 respectively, will explain why this additional subroutine is needed.

PROPOSITION 6.6. *For each stratum S of $X'_\bullet$, there exists a unique stratum R of $Y'_\bullet$ satisfying $f(S) \subset R$. Moreover, at each point x in S, the Jacobian $J_x f|_S : T_x S \to T_{f(x)} R$ of the restricted map $f|_S : S \to R$ has full rank, i.e.,*

$$\operatorname{rank}(J_x f|_S) = \min(\dim S, \dim R)$$

PROOF. Noting that of $X'_\bullet$ is subordinate to the flag $\mathbf{F}_\bullet$ by Line 10, we know that for each stratum S of $X'_\bullet$ there is a number $i := i(S)$ satisfying $S \subset (\mathbf{F}_i X - \mathbf{F}_{i-1} X)$. Now by Line 09, for any such stratum we have $f(S) \subset (Y'_i - Y'_{i-1})$. By Definition 2.2 we know that S is connected, and so its image under the continuous map f must also be connected; thus there is a unique stratum $R \subset (Y'_i - Y'_{i-1})$ of $Y'_\bullet$ satisfying $f(S) \subset R$. Since $Y'_\bullet$ is $\mathbf{B}_\bullet$-subordinate by Line 07, there exists a number $j := j(R)$ satisfying $R \subset (\mathbf{B}_j Y - \mathbf{B}_{j-1} Y)$. By Definition 6.5, we have $f^{-1}(R) \subset (\mathbf{T}_j X - \mathbf{T}_{j-1} X)$, where $\mathbf{T}_\bullet$ is the Thom-Boardman flag of f from Definition 6.4. In particular, this gives $S \subset (\mathbf{T}_j X - \mathbf{T}_{j-1} X)$ and it follows that $X'_\bullet$ is subordinate to $\mathbf{T}_\bullet$. Consequently, for each $x \in S$ we know that rank $J_x f = (j-1)$. Cconsier the commuting diagram of vector spaces

$$\begin{array}{ccc} T_x S & \xrightarrow{J_x(f|_S)} & T_{f(x)} R \\ \big\downarrow & & \big\downarrow \\ \mathbb{C}^{n+1} & \xrightarrow[J_x f]{} & \mathbb{C}^{m+1} \end{array}$$

Here the vertical arrows depict inclusions of tangent spaces, e.g., on the left we have the natural inclusion of $T_x S$ in $T_x \mathbb{P}^n \simeq \mathbb{C}^{n+1}$. Since $S \subset (\mathbf{T}_j X - \mathbf{T}_{j-1} X)$, we know from (8) that the rank of $J_x f$ is precisely $(j-1)$ at every $x \in S$; and by genericity of f it follows that the subspaces $T_x S$ and $\ker J_x f$, whose intersection equals $\ker J_x f|_S$, meet transversely inside $\mathbb{C}^{n+1}$. Thus, we obtain

$$\begin{aligned} \dim\ker(J_x f|_S) &= \dim\ker J_x f + \dim T_x S - (n+1) && \text{by transversality} \\ &= [(n+1) - (j-1)] + \dim S - (n+1) && \text{by rank/nullity} \\ &= \dim S - (j-1). \end{aligned}$$

There are now two cases to consider — either $\dim S < (j-1)$, or $\dim S \geq (j-1)$. In the first case, $J_x f|_S$ is injective and hence already has full rank. In the latter case, we have $R \subset f(\Delta_j)$ where $\Delta_j := (\mathbf{T}_j X - \mathbf{T}_{j-1} X)$. Thus, $\dim Y \leq \dim f(\Delta_j)$; but since the rank of f on Δ_j is $(j-1)$ by (8), the implicit function theorem guarantees that $\dim f(\Delta_j) = (j-1)$. So, we have $\dim R \leq (j-1)$, and combining this inequality with our calculation of $\dim\ker(J_x f|_S)$ above gives

$$\dim\ker(J_x f|_S) \leq \dim S - \dim R.$$

Since we have assumed $\dim S \geq \dim R$, the kernel of $J_x f|_S$ can not have dimension smaller than the codimension $\dim S - \dim R$, so the above inequality is an equality in this case and $J_x f|_S$ has full rank, as desired. □

When comparing the requirements of Definition 6.1 to the properties guaranteed by the preceding result, we note that the stratifications $X'_\bullet$ and $Y'_\bullet$ are insufficient for our puposes. While every stratum S of $X'_\bullet$ does indeed have a unique stratum R of $Y'_\bullet$ containing $f(S)$, the crucial Jacobian-surjectivity requirement is not satisfied. Instead, we might have $\dim S < \dim R$ with the Jacobian $J_x f|_S$ being injective at every point x in S. The **Refine** subroutine invoked in Line 13 of **WhitStratMap**, which we describe in the next Section, has been designed to rectify this defect.

6.4. Refinement and Correctness. Before Line 13 of **WhitStrat** has been executed, we have stratifications $X'_\bullet$ and $Y'_\bullet$ of X and Y respectively. In light of Proposition 6.6, consider the set of problematic strata-pairs $\mathscr{P} = \mathscr{P}(X'_\bullet, Y'_\bullet)$ given by:

$$\mathscr{P} := \{(S, R) \mid f(S) \subset R \text{ with } \dim S < \dim R\}.$$

The purpose of the **Refine** subroutine described here is to modify the stratifications $X'_\bullet$ and $Y'_\bullet$ until this problematic set $\mathscr{P}$ becomes empty. By Proposition 6.6, each stratum S of $X'_\bullet$ can appear in a pair of $\mathscr{P}$ with at most one stratum R of $Y'_\bullet$. However, the converse need not hold — a given stratum R of $Y'_\bullet$ might be paired with several different strata of $X'_\bullet$ within $\mathscr{P}$. Recalling that $X'_\bullet$ is $\mathbf{F}_\bullet$-subordinate, for each index ℓ we will use $\mathfrak{S}_\ell = \mathfrak{S}_\ell(X'_\bullet)$ to denote the set of all strata S of $X'_\bullet$ which lie in the difference $\mathbf{F}_{\ell+1}X - \mathbf{F}_\ell X$.

	Refine$(X'_\bullet, Y'_\bullet, f)$
	Input: Stratifications $X'_\bullet, Y'_\bullet$ of pure dimensional varieties X and Y and a generic morphism $f : X \to Y$ so that Prop. 6.6 holds. **Output:** Lists of subvarieties $X_\bullet \subset X$ and $Y_\bullet \subset Y$.
01	**For each** $(S, R) \in \mathscr{P}(X'_\bullet, Y'_\bullet)$ with $\dim R$ maximal
02	**Set** $Y^+_\bullet := Y'_\bullet$
03	**Set** $d := \dim \overline{f(S)}$
04	**Add** $\overline{f(S)}$ to $Y^+_{\geq d}$
05	**Merge** $Y^+_\bullet$ with **WhitStrat**$(\text{Pure}_d(Y'_d))$
06	**For each** $\ell = (d, d-1, \ldots, 1, 0)$
07	**For each** irreducible $W \subset \overline{Y^+_\ell - Y'_\ell}$ and $S' \in \mathfrak{S}_\ell(X'_\bullet)$
08	**If** $Z \cap S' \neq \varnothing$ for an irreducible $Z \subset f^{-1}(W)$
09	**Set** $r := \dim Z$
10	**Add** Z to $X'_{\geq r}$
11	**Merge** $X'_\bullet$ with **WhitStrat**$(\text{Pure}_r(X'_r))$
12	**Set** $Y'_\bullet = Y^+_\bullet$
13	**Recompute** $\mathscr{P}(X'_\bullet, Y'_\bullet)$
14	**Return** $(X'_\bullet, Y'_\bullet)$

This subroutine processes the problematic strata-pairs (S, R) from $\mathscr{P}(X'_\bullet, Y'_\bullet)$ one at a time, in descending order of $\dim R$. For each such pair, Lines 02-05 further partition R by forcing the closure of the image $f(S)$ to form (one or more) new strata. As a result of subdividing R along $f(S)$, some of the strata S' of $X'_\bullet$ satisfying $f(S') \subset R$ no longer have their images contained in a single stratum. Lines 06-11 are designed to correct this problem by finding and further subdividing all such S' appropriately.

PROPOSITION 6.7. *The* **Refine** *subroutine terminates, and its output* $(X_\bullet, Y_\bullet)$ *constitutes a valid stratification (as in Definition 6.1) of the generic projective morphism* $f : X \to Y$.

PROOF. If $\mathscr{P}$ is empty, then the algorithm terminates immediately with a correct stratification, so let (S, R) be a strata-pair in $\mathscr{P}$ with R of maximal dimension; thus, we have $\dim S < \dim R$. In Lines 02-05, **WhitStratMap** constructs a new Whitney stratification $Y^+_\bullet$ of Y by subdividing the closure of R into finitely many new strata

$$\overline{R} = \coprod_i R_i$$

so that all points lying in the (closure of the) immersed image $f(S) \subset R$ lie within strata of dimension no larger than $\dim S$. Thus, for each point $x \in S$, there is a unique index $i(x)$ for which $f(x) \in R_{i(x)}$. Moreover, we have

$$\dim R_{i(x)} \leq \dim S < \dim R.$$

Since $R_{i(x)} \subset R$, we know by Proposition 6.6 that the Jacobian $J_x f : T_x S \to T_{f(x)} R_{i(x)}$ is surjective for all $x \in S$ as desired. Unfortunately, the act of partitioning R into smaller strata might violate the other property required by Definition 6.1, i.e., we may have strata S' of $X'_\bullet$ whose images $f(S')$ were entirely contained in R, but which now intersect several new R_i's. Therefore, in Lines 06-11, the algorithm paritions all such S' along their intersections with $f^{-1}(R_i)$ for all i — this creates new Whitney strata and hence refines $X'_\bullet$. Finally, in Line 12 we also update $Y'_\bullet$ to the new stratification $Y^+_\bullet$. After this update, there might be several new problematic strata pairs in $\mathscr{P}$; but the key observation here is that none of these new pairs (S^*, R^*) can have $\dim R^* > \dim R$ since all of the new strata R^* of $Y'_\bullet$ have dimension bounded above by $\dim R$. Moreover, even when $\dim R^* = \dim R$, it is impossible to have any (S^*, R^*) in $\mathscr{P}$ where $S^* \subset S$ is a newly-created stratum of $X'_\bullet$ — any such S' must have its image $f(S')$ entirely contained in a stratum of dimension $< \dim R$. Thus, **WhitStratMap** eventually terminates and outputs a valid stratification of $f : X \to Y$. □

REMARK 6.8. An analogous algorithm for stratifying proper algebraic maps between affine varieties can be produced by making a few standard modifications to **WhitStratMap**. Implicit in (the proofs of) Corollaries 4.6 and 5.3 are routines which would compute (flag-subordinate) Whitney stratifications of a complex affine algebraic variety. These routines accept as input affine equations, homogenize the associated Gröbner basis, run the projective stratification algorithms described here, and then dehomogenize the output to produce the final result. Let **WhitStratAff** and **WhitStratFlagAff** be the resulting algorithms. If $X \subset \mathbb{C}^n$ and $Y \subset \mathbb{C}^m$ are affine varieties and $f : X \to Y$ is a proper morphism, then an analogous definition of the map being generic and of a Thom-Boardman flag on X may be given. To obtain a stratification of the generic proper morphism $f : X \to Y$ in the sense of Definition 6.1, one simply replaces each occurence of **WhitStrat** and **WhitStratFlag** by **WhitStratAff** and **WhitStratFlagAff**, respectively in the **WhitStratMap** and **Refine** algorithms described above.

7. Empirical Performance

In this section we will briefly describe the real life performance of the basic algorithm **WhitStrat** from §4. Our implementation is in Macaulay2 [19], and can be found along with documentation at the link below:

`http://martin-helmer.com/Software/WhitStrat`

As remarked in the Introduction, the state of the art for Whitney stratification algorithms appears to be the recent algorithm of [23, §2]. We are not aware of any existing implementations of earlier algorithms based on cylindrical algebraic decomposition, and in any event we would expect their performance to be slower than that of [23] overall. Since the authors of [23] did not provide an implementation of their algorithm, we have implemented it ourselves in Macaulay2. This implementation can be found at:

`http://martin-helmer.com/Software/WhitStrat/DCG.m2`.

The comparison between the two implementations is documented in Table 1.

The first entry of this table is the projective analogue of the Whitney umbrella, which was one of the earliest examples of singular spaces used to illustrate the need for Condition (B). We ran the algorithm of [23] on this example for over 29 hours. In this time,

INPUT	WhitStrat	[23, §2]
$\mathbf{V}\left(x_0x_1^2 - x_1^2x_2\right) \subset \mathbb{P}^3$	0.2s	-
$\mathbf{V}\left(x_1^4x_2 - x_0^5 - x_0^4x_3 - x_0^4x_4\right) \subset \mathbb{P}^4$	0.4s	-
$\mathbf{V}\left(x_3^3 - x_1x_2^2 - x_0^2x_3 + x_0^2x_4 - x_3x_4^2\right) \subset \mathbb{P}^4$	0.5s	-
$\mathbf{V}\left(x_6^2 - x_1x_2 + x_0x_4, x_0^2 - x_0x_3 - x_5^2\right) \subset \mathbb{P}^6$	0.9s	-
$\mathbf{V}\left(x_0^2x_4 - x_1x_2^2 + x_3^3, x_0^2 - x_1x_4\right) \subset \mathbb{P}^4$	1.6s	-
$\mathbf{V}\left(x_4x_7 - x_1x_2 + x_7^2, x_0^2 - x_0x_5 - x_7^2, x_3x_7 - x_6^2\right) \subset \mathbb{P}^7$	242.5s	-

TABLE 1. Run times in Macaulay2 working over $\mathbb{Q}$ on a Intel i7-8700 CPU with 64 GB of RAM. The - denotes computations which did not finish after 8 hours.

it was unable to find strata of codimension > 1, i.e., the strata lying below the singular locus. All computations involving this algorithm used at least 27 GB of RAM during their 8 hour run. In sharp contrast, **WhitStrat** used between 0.0005 and 0.347 GB of RAM, with all but the last entry in the table requiring no more than 0.016 GB. While it is certainly possible that a more optimized version of [23] than ours could be produced, in our view it is unlikely to significantly alter the general trends indicated in Table 1.

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