Number Theory — Examples Sheet 2

Michaelmas Term 2011 *V. Neale*

1. Evaluate the following Jacobi symbols (in fact, they are Legendre symbols): $\left(\frac{20964}{1987}\right)$, $\left(\frac{741}{9283}\right)$, $\left(\frac{5}{160465489}\right)$, $\left(\frac{3083}{3911}\right)$.

2. Find all odd primes p for which 15 is a quadratic residue modulo p.

3. Prove that 3 is a quadratic non-residue modulo any Mersenne prime $2^n - 1$, with $n > 2$.

4. Let p be a prime with $p \equiv 1 (\bmod 4)$. Prove that the sum of the quadratic residues in the interval $[1, p-1]$ is equal to the sum of the quadratic non-residues in this interval. Does this hold if $p \equiv 3 (\bmod 4)$?

5. Let a be a positive integer that is not a square. Prove that there are infinitely many odd primes p such that $\left(\frac{a}{p}\right) = -1$.

6. Are the forms $3x^2 + 2xy + 23y^2$ and $2x^2 + 4xy + 5y^2$ equivalent (under the action of $SL_2(\mathbb{Z})$)? Are the forms $15x^2 - 15xy + 4y^2$ and $3x^2 + 9xy + 8y^2$ equivalent?

7. Prove that equivalence of binary quadratic forms is an equivalence relation.

8. Make a list of all reduced positive definite quadratic forms of discriminant $-d$, where $d = 8$, 11, 12, 16, 19, 23, 163.

9. Find the smallest positive integer that can be represented by the form $4x^2 + 17xy + 20y^2$.

10. Find a necessary and sufficient condition for a prime p to be represented by the form $x^2 + 3y^2$.

11. Is there a positive definite binary quadratic form that represents 2 and the primes congruent to 1 (mod 8) or 3 (mod 8), but no other primes?

12. Find a necessary and sufficient condition for a positive integer n to be properly represented by at least one of the two forms $x^2 + xy + 4y^2$ and $2x^2 + xy + 2y^2$.

    Assume that n is coprime to 15, and properly represented by at least one of the forms. Show that congruence conditions modulo 15 allow one to decide which form represents n.

Please e-mail me with comments, suggestions and queries (v.r.neale@dpmms.cam.ac.uk).