List Colouring

Given: A graph $G$ and a list $L(v)$ of available colours for each $v \in V(G)$.

Goal: To find an acceptable colouring, i.e., a colouring $f$ of the vertices such that
1 adjacent vertices receive different colours ($f$ is proper), and
2 $f(v) \in L(v)$ for all $v \in V(G)$.

Problem: Find the list chromatic number $\chi^\ell(G)$: $\chi^\ell(G)$ is the minimum $k$ such that there is an acceptable colouring whenever $|L(v)| \geq k$ for all $v \in V(G)$.

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Famous List Colouring Conjectures

The List Colouring Conjecture: $\chi_\ell = \chi$ for line graphs.

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Ohba’s Conjecture (2002)

If $|V(G)| \leq 2\chi(G) + 1$, then $\chi_\ell(G) = \chi(G)$. 
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Theorem 1 (Erdős, Rubin, Taylor, 1979).

$$\chi_\ell \left( K_{2, 2, \ldots, 2} \right) = k.$$
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Note: In Theorem 2, $|V(G)| = 2\chi(G) + 1$. 

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With Stronger Hypotheses

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Theorem (He, Li, Shen, Zheng, 2009).
If $|V(G)| \leq 2\chi(G) + 1$ and $\alpha(G) \leq 3$, then $\chi_\ell(G) = \chi(G)$.

Theorem (Kostochka, Stiebitz, Woodall, 2011).
If $|V(G)| \leq 2\chi(G) + 1$ and $\alpha(G) \leq 5$, then $\chi_\ell(G) = \chi(G)$. 

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Theorem (N., Reed, Wu, 2012). If $|V(G)| \leq 2\chi(G) + 1$, then $\chi_\ell(G) = \chi(G)$.
An Observation

Observe: Ohba’s Conjecture is true if and only if it is true for complete $k$-partite graphs.

Let $n := |V(G)|$, $k := \chi(G)$, $|L(v)| \geq k$ for all $v \in V(G)$, $C := \bigcup v \in V(G) L(v)$.

Note: $n \leq 2k + 1$. 

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Key Definitions

Def: A colour $c \in C$ is frequent if it is contained in at least $k + 1$ lists.

Def: A proper colouring $f$ of $G$ is near-acceptable if for every $v \in V(G)$ either $f(v) \in L(v)$, or $f(v)$ is frequent and $f^{-1}(f(v)) = \{v\}$.
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Lemma 2: If there are at least \( k \) frequent colours, then there is a near-acceptable colouring.

Lemma 3: If Ohba's Conjecture is false, then there is a counterexample with at least \( k \) frequent colours.
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Proof of Lemma 1

Let $f$ be a near-acceptable colouring. We construct an acceptable colouring $f'$ with the same colour classes as $f$.

Let $V_f := \{ f^{-1}(c) : c \in f(V(G)) \}$.

Let $B_f$ be the bipartite graph with bipartition $(V_f, C)$ where each $f^{-1}(c) \in V_f$ is joined to the colours of $\cap_{v \in f^{-1}(c)} L(v)$.

If there is a matching in $B_f$ that saturates $V_f$, then we are done.

Otherwise, by Hall’s Theorem, there is a set $S \subseteq V_f$ such that $|N_{B_f}(S)| < |S|$.
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Proof of Lemma 2

(For simplicity, assume $n \leq 2^k$). Let $F$ be a set of $k$ frequent colours.

**Phase 1:** For each colour $c \in C - F$, in turn, we use $c$ to colour the largest stable set which has not yet been coloured and for which $c$ is available.

For each part $P$ of $G$, let $R_P$ be the set of uncoloured vertices in $P$.

Order the parts $P_1, \ldots, P_k$ so that $|R_{P_1}| \geq \cdots \geq |R_{P_k}|$.

**Phase 2:** For each $i = 1, \ldots, k$, in turn, we try to colour all of $R_{P_i}$ with a colour in $F$ which has not yet been used and which is available for every vertex of $R_{P_i}$.

Terminate Phase 2 when we reach an $i$ for which this is not possible.

**Phase 3:** For every colour $c \in F$ not used in Phase 2, colour at most one uncoloured vertex with $c$ (ignoring the lists).
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Suppose that $R_{P_1}, \ldots, R_{P_i}$ are coloured in Phase 2, but $R_{P_{i+1}}, \ldots, R_{P_k}$ are not.
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Let $U$ be the colours in $F$ not used in Phase 2. We have

$$|F - U| = i$$
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If there are at most $|U|$ vertices that are not coloured after Phase 2, then we are done.
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Proof. It suffices to show that every vertex is coloured by our procedure.

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$$|F - U| = i$$

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If there are at most $|U|$ vertices that are not coloured after Phase 2, then we are done. Therefore,

$$\sum_{j=i+1}^{k} |R_{P_j}| \geq k - i + 1.$$
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$$2k - (k - i + 1) = k + i - 1.$$
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|R_{P_{i+1}}| \geq \left\lceil \frac{k - i + 1}{k - i} \right\rceil = 2.
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Also, by our ordering,

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and therefore,

\[ |R_{P_{i+1}}| \geq \left\lceil \frac{k - i + 1}{k - i} \right\rceil = 2. \]

By our ordering, this means that there are at least 2\(i\) vertices that are coloured in Phase 2.

(ie. each of \(R_{P_1}, \ldots, R_{P_i}\) also has size at least 2)
Proof of Lemma 2

Putting it together:

We know that at most $k + i - 1$ vertices are coloured in Phases 1 and 2.

We also know that at least $2i$ vertices are coloured in Phase 2.

So the number of vertices coloured in Phase 1 is at most $(k + i - 1) - 2i = k - i - 1$.

(*) If a colour $c \in C - F$ is used on $\ell$ vertices, then there are at most $\ell$ vertices $v \in R_{P_i + 1}$ with $c \in L(v)$.
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$$(k + i - 1) - 2i = k - i - 1.$$
Proof of Lemma 2

Putting it together:

We know that at most $k + i - 1$ vertices are coloured in Phases 1 and 2.

We also know that at least $2i$ vertices are coloured in Phase 2.

So the number of vertices coloured in Phase 1 is at most

\[(k + i - 1) - 2i = k - i - 1.\]

(* ) If a colour $c \in C - F$ is used on $\ell$ vertices, then there are at most $\ell$ vertices $v \in R_{P_{i+1}}$ with $c \in L(v)$. 

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A Proof of a Conjecture of Ohba
Proof of Lemma 2

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Summary

Main Ideas:

(1) Suppose that $f$ is a colouring in which some vertices are coloured outside of their lists. Under what conditions can $f$ be modified to produce an acceptable colouring? If not possible, then what information about the distribution of colours in the lists can you obtain from a Hall's Theorem argument?

(2) If an 'almost' acceptable colouring is sufficient, then exploit its more relaxed properties in a greedy colouring procedure.
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   - Under what conditions can $f$ be modified to produce an acceptable colouring?
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2. If an ‘almost’ acceptable colouring is sufficient, then exploit its more relaxed properties in a greedy colouring procedure.
**Def:** A colour $c \in C$ is \textit{frequent} if it is contained in at least $k + 1$ lists.

**Def:** A proper colouring $f$ of $G$ is \textit{near-acceptable} if for every $v \in V(G)$ either

- $f(v) \in L(v)$, or
- $f(v)$ is frequent and $f^{-1}(f(v)) = \{v\}$.

**Lemma 1:** If there is a near-acceptable colouring, then there is an acceptable colouring.

**Lemma 2:** If there are at least $k$ frequent colours, then there is a near-acceptable colouring.

**Lemma 3:** If Ohba’s Conjecture is false, then there is a counterexample with at least $k$ frequent colours.
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**Observe:** If $P$ is a non-singleton part of $G$, then

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Lemma 3 requires us to find a ‘special counterexample’ (with \(k\) frequent colours).

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\textbf{Observe:} If \(P\) is a non-singleton part of \(G\), then

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This means that the lists are fairly \textbf{spread apart}.
Def:
Let $B$ be the bipartite graph on $(V(G), C)$ where each vertex $v \in V(G)$ is joined to the colours of $L(v)$.

Observe:
There is no matching in $B$ which saturates $V(G)$.

Lemma (Kierstead (2000), Reed and Sudakov (2005)):
There is a matching $M$ in $B$ which saturates $C$.

Corollary:
$|C| \leq |V(G)| \leq 2k + 1$.

Proof.
Otherwise, $M$ would saturate $V(G)$.

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Suppose that $f$ is a proper colouring such that for every vertex $v \in V(G)$, either $f(v) \in L(v)$, or $f^{-1}(f(v)) = \{v\}$, and some other condition holds (to be determined). We try to find an acceptable colouring using Hall's Theorem and minimality of $G$.

Again, there is a set $S \subseteq V$ such that $|N_B(f)(S)| < |S|$. Pick $S$ such that $|S| - |N_B(f)(S)|$ is maximal.
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More Bad Colourings With Good Properties

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Lemma 1 Revisited

Definition: Say that a colour \( c \in C \) is frequent among singletons if it is contained in the lists of at least \( \gamma \) singletons.

We have proved the following:

Lemma 1': It suffices to find a proper colouring \( f \) such that for every vertex \( v \in V(G) \) either \( f(v) \in L(v) \), or \( f(v) \) is either frequent or frequent among singletons and \( f^{-1}(f(v)) = \{v\} \).
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**Lemma 1′:** It suffices to find a proper colouring \( f \) such that for every vertex \( v \in V(G) \) either
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By Lemma 1′ there is an acceptable colouring. For the other direction, we omit the details.
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**Lemma 3’:** If Ohba’s Conjecture is false, then there is a minimal counterexample with at least \( k \) colours \( c \) such that either
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**Proof.** Another trick using Hall’s Theorem (we omit the details). $\square$
Therefore, it suffices to prove:

**Lemma 3″**: If Ohba’s Conjecture is false, then there is a minimal counterexample with at least $p$ colours $c$ such that either

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Ideas:

- Use Hall’s Theorem again.
- Combine various counting arguments.
Open Problems

Question (Erdős, Rubin, Taylor):
What is $\chi^\ell(K_{m,m,...,m})$?
For $m = 2$ it is $k$.
For $m = 3$ it is $\lceil 4k - \frac{1}{3} \rceil$ (Kierstead 2000).
No other values have been calculated.

There are graphs with $|V(G)| = 2\chi(G) + 2$ and $\chi^\ell(G) > \chi(G)$.

Question:
For a function $f(k) > 2k + 1$, what is a good upper bound on $\chi^\ell$ for complete $k$-partite graphs on at most $f(k)$ vertices?
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Thanks!
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