Choosability of Graphs with Bounded Order: Ohba’s Conjecture and Beyond

Jonathan Noel¹

Joint work with

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Graphs and Colourings

Definition:
A graph is a mathematical structure made up from a collection of points (called vertices), some pairs of which are connected by lines (called edges).

Figure 1: A graph $G$. 

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![Graph Image]

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![A proper colouring of $G$ using red, green, blue and orange.](image)

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In our previous example, $\chi(G) = 3$. 
Origins of Graph Colouring

Graph colouring goes back more than 160 years.

Definition:
A graph $G$ is planar if it can be drawn on a flat surface without crossing edges.

The Four Colour Conjecture (Guthrie, 1852):
If $G$ is a planar graph, then $\chi(G) \leq 4$.
(Equivalently, every map can be coloured with 4 colours so that neighbouring countries are coloured differently.)
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A Generalization: List Colouring

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In **list colouring**, each vertex \(v\) must be mapped to a colour in its own personal list \(L(v)\) of colours.
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\{\text{pink, yellow, orange}\} & \quad \bullet \quad \{\text{red, blue, orange}\} \\
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\{\text{green, blue, orange}\} & \quad \bullet \quad \{\text{pink, yellow, red}\}
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In list colouring, each vertex \(v\) must be mapped to a colour in its own personal list \(L(v)\) of colours.
The Choice Number

Definition:
Given a graph $G$ and a list assignment $L$, we say that a colouring is acceptable if
1. it is a proper colouring and
2. every vertex $v$ is mapped to a colour in $L(v)$.

{pink, yellow, orange}
{red, blue, orange}
{red, blue, green}
{green, blue, orange}
{red, blue, orange}
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Definition:
The choice number of $G$, denoted $\text{ch}(G)$, is the minimum $k$ such that there is an acceptable colouring whenever $|L(v)| \geq k$ for every vertex $v$. 
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Claim: $\text{ch}(K_{3,3}) > 2$. 

Proof. Each side must be mapped to a set of at least 2 colours. Therefore, the image of an acceptable colouring must contain at least 4 colours, a contradiction.
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An Example: $K_{3,3}$

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Therefore, the image of an acceptable colouring must contain \textbf{at least 4 colours}, a contradiction.
Theorem: A graph $G$ is bipartite if and only if $\chi(G) \leq 2$. 

Theorem (Erdős, Rubin and Taylor 1979): For every integer $k$ there is a bipartite graph $G$ such that $\text{ch}(G) \geq k$.

Four Colour Theorem (Appel and Haken, 1977): If $G$ is a planar graph, then $\chi(G) \leq 4$.

Thomassen's Five Colour Theorem (Thomassen, 1994): If $G$ is a planar graph, then $\text{ch}(G) \leq 5$.

There are many conjectures which claim that certain types of graphs satisfy $\text{ch} = \chi$. 

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There are many conjectures which claim that certain types of graphs satisfy $\text{ch} = \chi$. 
Ohba’s Conjecture (2002)

If $G$ is a graph with at most $2\chi(G) + 1$ vertices, then $\text{ch}(G) = \chi(G)$. 

Motivating example:
Theorem (Erd˝ os, Rubin, Taylor 1979).

$\text{ch} (K^2 \ast k) = k.$

Proof Ingredients.
Induction and Hall’s Theorem.
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Main Partial Results

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Theorem (Kostochka, Stiebitz, Woodall 2011). If $G$ is a graph with at most $2\chi(G) + 1$ vertices and independence number at most 5, then $\text{ch}(G) = \chi(G)$. 
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Theorem (N., Reed, Wu 2012). Ohba’s Conjecture is true!

Proof Ingredients. Induction, Hall’s Theorem, and greedy colouring.
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Definition: Given a finite collection $S$ of sets, a **system of distinct representatives** is a set $X = \{x_A : A \in S\}$ such that

1. $x_A \in A$ for all $A \in S$, and
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**Example:** A collection of sets with a system of distinct representatives:

$$\{1, 4\}, \{2, 4\}, \{1, 5, 3\}, \{4, 1, 2\}, \{1, 2, 5\}$$
The Main Tool: Hall’s Theorem

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Observation: If there exists a system of distinct representatives, then for every $t \geq 1$ and distinct $A_1, \ldots, A_t \in S$ we have

$$\left| \bigcup_{i=1}^{t} A_i \right| \geq t.$$
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Hall’s Theorem (Hall, 1939): There exists a system of distinct representatives if and only if for every $t \geq 1$ and distinct $A_1, \ldots, A_t \in S$ we have

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Back to Choosability

Theorem (Erdős, Rubin, Taylor 1979).

\[ \text{ch (K}_2^\ast \text{k}) = k. \]

Proof.

Let \( L \) be a list assignment such that \( |L(v)| \geq k \) for each vertex. We show that there is an acceptable colouring by induction on \( k \).

The case \( k = 1 \) is trivial. So, assume that \( k \geq 2 \).

Case 1: There exists a part \( V_i = \{u, v\} \) such that \( L(u) \cap L(v) \neq \emptyset \).

Use the inductive hypothesis.

Case 2: Every part \( V_i = \{u, v\} \) satisfies \( L(u) \cap L(v) = \emptyset \).

Use Hall’s Theorem.
Theorem (Erdős, Rubin, Taylor 1979).

$$\text{ch}(K_{2\times k}) = k.$$
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\[ \text{ch}(K_{2^k}) = k. \]

Proof. Let \( L \) be a list assignment such that \( |L(v)| \geq k \) for each vertex.
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Use Hall’s Theorem. \( \square \)
Theorem (N., Reed, Wu 2012). If $G$ is a graph with at most $2\chi(G) + 1$ vertices, then $\text{ch}(G) = \chi(G)$.

As in Erdős, Rubin and Taylor's proof, we apply induction and Hall's Theorem. We use these tools to obtain information about the distribution of colours in the lists. After that, we show that an acceptable colouring can be obtained from a simple greedy colouring procedure.
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What about graphs on more than $\chi(G) + 1$ vertices? In general, we cannot say that $\chi(G) = \chi(G)$. However, perhaps we can still obtain a good bound on $\chi(G)$ in terms of $\chi(G)$ with a less restrictive bound on the number of vertices. In particular, what is the best upper bound on $\chi(G)$ for graphs with at most $3\chi(G)$ vertices?
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Graphs for which $|V(G)| \leq 3\chi(G)$

Theorem (Kierstead 2000).

Theorem (N., West, Wu, Zhu 2013). If $G$ is a graph with at most $3\chi(G)$ vertices, then $\text{ch}(G) \leq \left\lceil \frac{4\chi(G) - 1}{3} \right\rceil$. 

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Choosability of Graphs with Bounded Order: Ohba’s
Theorem (Kierstead 2000).

\[ \text{ch} (K_{3^* k}) = \left\lceil \frac{4k - 1}{3} \right\rceil. \]
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\[ \text{ch}(K_{3* k}) = \left\lceil \frac{4k - 1}{3} \right\rceil. \]

Theorem (N., West, Wu, Zhu 2013). If \( G \) is a graph with at most \( 3\chi(G) \) vertices, then \( \text{ch}(G) \leq \left\lceil \frac{4\chi(G) - 1}{3} \right\rceil. \)
The Full Theorem


For every graph $G$, $\text{ch}(G) \leq \max\{\chi(G), \lceil |V(G)| + \chi(G) - 1 \rceil\}$.

Proof Ingredients.
Induction and Hall's Theorem.

Note: This result implies Ohba's Conjecture, as well as the result on graphs for with at most $3\chi(G)$ vertices.
Theorem (N., West, Wu, Zhu 2013). For every graph $G$,

$$\text{ch}(G) \leq \max \left\{ \chi(G), \left\lceil \frac{|V(G)| + \chi(G) - 1}{3} \right\rceil \right\}.$$
The Full Theorem

Theorem (N., West, Wu, Zhu 2013). For every graph $G$,

$$ch(G) \leq \max \left\{ \chi(G), \left\lceil \frac{|V(G)| + \chi(G) - 1}{3} \right\rceil \right\}.$$

Note: This result implies Ohba’s Conjecture, as well as the result on graphs for with at most $3\chi(G)$ vertices.
Theorem (N., West, Wu, Zhu 2013). For every graph $G$,
\[
\text{ch}(G) \leq \max \left\{ \chi(G), \left\lceil \frac{|V(G)| + \chi(G) - 1}{3} \right\rceil \right\}.
\]

Proof Ingredients. Induction and Hall’s Theorem.

Note: This result implies Ohba’s Conjecture, as well as the result on graphs for with at most $3\chi(G)$ vertices.
A Theme: the choice number of $K_{m^*k}$ might give an upper bound on the choice number of all $k$-chromatic graphs on at most $mk$ vertices.

Question 1: Does this remain true for $m \geq 4$?

Question 2: What is the choice number of $K_{4^*k}$?

Lower bound: $\lceil \frac{3k^2}{2} \rceil$ due to Yang 2003.

Upper bound: $\lceil \frac{5k - 1}{3} \rceil$ due to N., West, Wu, Zhu 2013.
A Theme: the choice number of $K_{m^*k}$ might give an upper bound on the choice number of all $k$-chromatic graphs on at most $mk$ vertices.

This is true for $m = 1, 2, 3$. 

Question 1: Does this remain true for $m \geq 4$?

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Lower bound: $\left\lfloor \frac{3k}{2} \right\rfloor$ due to Yang 2003.

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A Theme: the choice number of $K_{m \ast k}$ might give an upper bound on the choice number of all $k$-chromatic graphs on at most $mk$ vertices.

This is true for $m = 1, 2, 3$.

Question 1: Does this remain true for $m \geq 4$?

Question 2: What is the choice number of $K_{4 \ast k}$?
A Theme: the choice number of $K_{m\ast k}$ might give an upper bound on the choice number of all $k$-chromatic graphs on at most $mk$ vertices.

This is true for $m = 1, 2, 3$.

Question 1: Does this remain true for $m \geq 4$?

Question 2: What is the choice number of $K_{4\ast k}$?

Lower bound: $\left\lceil \frac{3k}{2} \right\rceil$ due to Yang 2003.

Upper bound: $\left\lfloor \frac{5k-1}{3} \right\rfloor$ due to N., West, Wu, Zhu 2013.
**Question 1:** What is the choice number of $K_{4* k}$?

**Question 2:** Is it true that $K_{m* k}$ has the largest choice number among $k$-chromatic graphs on at most $m k$ vertices?

**Question 3:** Does there exist a function $f(k) = o(k^2)$ such that for every graph $G$,

$$\text{ch} (G^2) \leq f (\chi (G^2))$$

(it is known that we cannot do better than $f(k) = ck \log(k)$).