Some of the following questions are exam questions

Note that the WKB method (additional notes), is not on the exam.

Problem 1 *(M.Sc. in Mathematical Modelling and Scientific Computing, Thursday 27 April, 2000)*
The differential equation

$$y'' + \lambda y \;=\; f, \qquad 0 \le x \le 1,$$

is subject to the boundary conditions

$$y'(0) = ay(0) + by(1), \qquad y'(1) = cy(0) + dy(1),$$

where a, b, c, d and λ are real constants and f is a continuous function. What is the homogeneous adjoint problem? In terms of the solutions of this adjoint problem, state the conditions on f under which the original problem has solutions.

Show that the original problem is self-adjoint if and only if $b + c = 0$, and assume this holds for the rest of the question. Let $y_1, y_2, \ldots$ be a complete set of orthonormal eigenfunctions, with corresponding eigenvalues λ_n such that

$$y_n'' + \lambda_n y_n \;=\; 0.$$

The Green's function $G_\lambda(x, x')$ is to be the solution of

$$\frac{d^2 G_\lambda}{dx^2} + \lambda G_\lambda \;=\; \delta(x - x'),$$

on $[0, 1]$ with the same boundary conditions. For what values of λ does such a Green's function exist? When it does exist, express the solution of the original problem as an integral. Show also that

$$G_\lambda(x, x') \;=\; \sum_{n=1}^{\infty} c_n y_n(x) y_n(x'),$$

and calculate the coefficients c_n in the terms of λ and λ_n. (You may assume that G_λ has a uniformly convergent eigenfunction expansion.)

Problem 2 *(related to Problem 1)*
Assume that $b = c = 0$ and $a = d = 1$ in the previous problem, and find an explicit formula for G_λ without using the expansion $G_\lambda(x, x') = \sum_{n=1}^{\infty} c_n y_n(x) y_n(x')$.

Problem 3 *(M.Sc. in Mathematical Modelling and Scientific Computing, Thursday 27 April, 2000)*
(i) If the Fourier transform of $y(x)$ is defined as

$$\tilde{y}(k) \;=\; \int_{-\infty}^{\infty} y(x) e^{ikx} dx,$$

state the inversion integral expressing $y(x)$ in terms of $\tilde{y}(k)$. Assuming that the Fourier transforms of $xy(x), y'(x)$ and $y''(x)$ exist, express them in terms of $\tilde{y}(k)$. If $y'' + xy = 0$ and y has a Fourier transform $\tilde{y}$, show that $\tilde{y}(k) = Ae^{\frac{ik^3}{3}}$ for some constant A. Why does this involve only one arbitrary constant, whereas the differential equation is of second order?

(ii) What is meant by saying that a function $y(x,\epsilon)$ has the asymptotic expansion

$$y(x,\epsilon) \sim \sum_{n=0}^{\infty} y_n(x)\epsilon^n \qquad \text{as } \epsilon \to 0?$$

Consider the boundary value problem

$$\begin{aligned} \epsilon y'' + y' + e^{cy} &= 0, \quad \text{for } 0 \le x \le 1, \\ y(0) = a,\ y(1) &= b, \end{aligned}$$

where a, b, c and ϵ are constants with $\epsilon > 0$ and $e^{-bc} > c$. State, with a reason, whether the boundary layer as $\epsilon \to 0$ will be at $x = 0$ or at $x = 1$. Find the leading term in the asymptotic expansion of y, both inside and outside the boundary layer.

Problem 4 *(related to Problem 3)*
State a leading order composite expansion of the solution of Problem 3 (ii), and compute the second term in the expansion of $y(x,\epsilon)$.

Problem 5 *(M.Sc. in Mathematical Modelling and Scientific Computing, Thursday 27 April, 2000)*
Use the method of multiple scales to find the leading term in the solution, valid up to times of order ϵ^{-1}, of the initial value problem

$$\ddot{x} + \epsilon\dot{x} + x + \epsilon a x^3 = 0, \qquad x(0) = 1,\ \dot{x}(0) = 0,$$

where a is constant.

Problem 6 *(related to Problem 5)*
Considering the equation (without any boundary conditions)

$$\ddot{x} + b\dot{x} + x + \epsilon a x^3 = 0,$$

find the critical points (and their linearised stability) for the phase plane $(x, \dot{x})$. Give a very rough sketch of the phase plane vector field and orbits. Find all the different values of (a, b) such that there exists a periodic solution. Finally calculate the leading order term of the expansion of the periodic solution.

Problem 7 *(M.Sc. in Mathematical Modelling and Scientific Computing,*
M.Sc. in Applied and Computational Mathematics, Thursday 19 April, 2001)
Consider the non-linear oscillator equation

$$\frac{d^2x}{dt^2} + k(x^2 - x - 2)\frac{dx}{dt} + x = 1.$$

Show that in the limit $k \ll 1$, the closed trajectory has period $2\pi + \mathcal{O}(k^2)$, and that its leading-order approximation is $x_0 = 1 + 2\sqrt{2}\cos t$.
[*You may use the fact that* $\cos^2 y \sin y = \frac{1}{4}(\sin y + \sin 3y)$.]
Find $F(x)$ such that the system may be written in Liénard form

$$\frac{dx}{dt} = k(y - F(x)), \qquad k\frac{dy}{dt} = 1 - x.$$

In the limit $k \gg 1$, sketch the trajectories in the xy-plane, and show that the period of the closed trajectory is approximately $k\left(\frac{27}{4} - 2\log\frac{35}{8}\right)$.