Linear algebraic and integral equations. Fredholm Alternative. Linear differential equations. Initial and boundary value problems. Green's functions.

Notation: variables, vector components, matrix entries etc. are *real*; $\mathbf{b}$, $\mathbf{u}$ are vectors; $u'(x)$, $u''(x)$ are du/dx, d^2u/dx^2; $\alpha(x)$, $\beta(x)$ are continuous real-valued functions.

Problem 1
Suppose A is a square matrix, $n \times n$. State (without proof) the Fredholm Alternative that gives necessary and sufficient conditions under which the system $A\mathbf{u} = \mathbf{b}$ has a solution $\mathbf{u}$. What does the Fredholm Alternative say about the system $(A - \lambda I)\mathbf{u} = \mathbf{b}$? (I is the $n \times n$ identity matrix.) For what values of λ is there a unique solution ? When λ is such that there is not a unique solution, what condition(s) must $\mathbf{b}$ satisfy in order for a solution to exist ? When those conditions do hold, what is the most general solution $\mathbf{u}$?

Problem 2
Consider the differential equation $u''(x) + \alpha(x)u'(x) + \beta(x)u(x) = 0$ on $0 \le x \le 1$. For each set of primary boundary conditions (a)–(d) below, find the adjoint boundary conditions.
(a) $u'(0) = u'(1) = 0$;
(b) $u(0) = u'(1) = 0$;
(c) $u(0) = u(1)$, $u'(0) = u'(1)$;
(d) $u(0) + u(1) = 0$, $u'(1) + u(0) = u'(0)$.

Problem 3
For each of the following problems (a)–(e), use the Fredholm alternative to state the conditions under which it has a unique solution, the conditions under which it has no solution, and the conditions under which it has multiple solutions. If you know another method of solving the problem (e.g. variation of parameters, integrating factor) check that it gives the same conclusion.
(a) $A\mathbf{u} = \mathbf{b}$, A an oblong matrix; $\mathbf{u}$ to be found.
(b) $u''(x) - u(x) = b(x)$ on $0 < x < 1$, $u(0) = u(1) = 0$; $u(x)$ to be found.
(c) $u''(x) + u(x) = b(x)$ on $0 < x < L$, $u(0) = u(L) = 0$; $u(x)$ to be found. ($L > 0$ is a given constant, possibly $n\pi$.)
(d) $u''(x) + \alpha(x)u'(x) + \beta(x)u(x) = b(x)$ on $0 < x < 1$, $u(0) = u'(0) = 0$; $u(x)$ to be found.
(e) $u'(x) + \alpha(x)u(x) = b(x)$ on $0 < x < 1$, $u(0) = u(1)$; $u(x)$ to be found.

Problem 4
Suppose u_1 and u_2 each satisfy $u''(x) + \alpha(x)u'(x) + \beta(x)u(x) = 0$, with $u_1(0) = 0$, $u_2(1) = 0$. The Wronskian of u_1 and u_2 is

$$W(y) = u_1(y)u_2'(y) - u_2(y)u_1'(y);$$

under what condition on u_1 and u_2 is $W(y)$ non-zero ? Suppose that condition holds, and define $K(x, y)$ by

$$K(x,y) \quad = \quad \begin{cases} u_1(x)u_2(y)/W(y) & \text{if } 0 \le x \le y \le 1, \\ u_1(y)u_2(x)/W(y) & \text{if } 0 \le y \le x \le 1. \end{cases}$$

Show that a solution of $u''(x) + \alpha(x)u'(x) + \beta(x)u(x) = b(x)$ with $u(0) = u(1) = 0$ is

$$u(x) = \int_0^1 K(x,y)b(y)dy.$$

Is this the *only* solution of that problem ? If $\alpha(x) = 0$ and $\beta(x) = n^2\pi^2$, what goes wrong with the approach followed in this question ?