Distributions. Green's functions. (Stakgold, Green's Functions and Boundary Value Problems, Wiley 1998)

Notation: Again, all functions, matrices etc. are to be considered real.

Problem 1
Define a sequence of functions $\delta_n(x)$ by

$$\delta_n(x) = \sqrt{\frac{n}{\pi}}\exp(-nx^2).$$

Sketch $\delta_n(x)$ for $n = 1, 10, 100$, show that $\delta_n(x) \to 0$ as $n \to \infty$ for all $x \neq 0$, and that $\int_{-\infty}^{\infty} \delta_n(x)dx = 1$ for each n. [You should know, and may use, the result that $\int_{-\infty}^{\infty} \exp(-x^2)dx = \sqrt{\pi}$.] Thus it is natural to interpret the "limit" of the δ_n as the δ-function or δ-distribution. Confirm this by showing that if $\phi(x)$ is a test function (i.e. $\phi \in C_0^\infty(\mathbb{R})$, which means that ϕ is differentiable arbitrarily often and $\phi = 0$ outside some bounded interval) then

$$\int_{-\infty}^{\infty} \delta_n(x)\phi(x)dx \to \phi(0) \quad \text{as } n \to \infty.$$

[Hint: Change variable to $y = x\sqrt{n}$ and then let $n \to \infty$.] Show also that

$$\int_{-\infty}^{\infty} \delta_n'(x)\phi(x)dx \to -\phi'(0), \text{ and } \int_{-\infty}^{\infty} \delta_n(kx)\phi(x)dx \to \frac{\phi(0)}{|k|},$$

where k is a non-zero constant. These correspond to the facts that $\delta_n'(x)$ is a sequence of functions that can be regarded as converging to the δ' distribution, and that $\delta(kx) = \delta(x)/|k|$.

Problem 2
Let H be the Heaviside step function

$$H(x) = \begin{cases} 1 & \text{if x} > 0, \\ 0 & \text{if x} < 0. \end{cases}$$

Show that $H'(x) = \delta(x)$ as distributions. Then if $f(x) = \max(x, 0)$ show that $f''(x) = \delta(x)$.

Problem 3
Find the Green's function $G(x, \xi)$ for the problem

$$u'' - u = f \qquad \text{on } 0 < x < 1,$$

with $u(0) = 0$, $u(1) = 0$. Formally verify that $G'' - G = \delta(x - \xi), G(0, \xi) = 0 = G(1, \xi)$, and give a sketch of $G(x, \xi)$ for fixed $\xi = \frac{2}{3}$.

Problem 4
Suppose L is a second order linear differential operator with linear homogeneous boundary conditions at $x = 0$ and $x = 1$ and

$$L[u] = f(x) \qquad \text{for } 0 < x < 1.$$

Show that if $L^*[K(x, \xi)] = \delta(x - \xi)$ for $0 < x, \xi < 1$, where K satisfies the adjoint boundary conditions, then

$$u(\xi) = \int_0^1 f(x)K(x, \xi)dx.$$

Problem 5
Some examples of adjoints : check that $\langle v, L[u]\rangle = \langle L^*[v], u\rangle$ in each case.
(a) The adjoint of a matrix : if A is an $n \times m$ matrix, u is an m-vector and v an n-vector, show that

$$\langle v, Au\rangle = \langle A^t v, u\rangle$$

where $\langle .,.\rangle$ denotes the usual scalar product of two vectors and A^t denotes the transpose of A.
(b) L and L^* are the Fredholm Integral Operators

$$L[u](x) = \int_0^1 K(x,\xi)u(\xi)d\xi, \quad L^*[v](\xi) = \int_0^1 K(x,\xi)v(x)dx.$$

(c) L and L^* are the Volterra Integral Operators

$$L[u](x) = \int_0^x K(x,\xi)u(\xi)d\xi, \quad L^*[v](\xi) = \int_\xi^1 K(x,\xi)v(x)dx.$$

(d) L and L^* are both given by the Sturm-Liouville operator (these are self-adjoint problems):

$$L[u] = L^*[u] = \frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u(x)$$

together with the boundary conditions that either u and v both vanish at 0 and 1, or u' and v' both do.

Problem 6 (optional question if you have time)
In 2 dimensions with $r^2 = x^2 + y^2$ consider the problem of finding a Green's function $G = G(r)$ for Laplace's equation, so we want

$$\nabla^2 G(r) \equiv \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) G(r) = \delta(x)\delta(y). \qquad (*)$$

Find the solution by regarding the distribution $\delta(x)\delta(y)$ as a generalized limit as $a \to 0$ of the function

$$f(x,y) = \begin{cases} 1/(\pi a^2) & \text{if r < a,} \\ 0 & \text{if r > a.} \end{cases}$$

[Why do we write $1/(\pi a^2)$ here ?] Solve $\nabla^2 G = f$ with $G = G(r)$ by $G = Ar^2 + B$ in $r < a$, $G = C + D\log r$ in $r > a$, with $G(r)$ and $G'(r)$ continuous at $r = a$. Then let $a \to 0$. Now show that your answer satisfies $(*)$ in the sense of distribution theory. You will need to show that

$$\int\int \phi(x,y)\nabla^2 G(r)\, dx\, dy = \phi(0,0),$$

for all test functions ϕ. Remember that the definition of $\nabla^2 G(r)$ is given in terms of its action on a test function to be consistent with integration by parts for regular functions. The 2 dimensional equivalent of integration by parts is Greens theorem:

$$\int\int \phi\nabla^2 G\, dx\, dy = \int\int G\nabla^2\phi\, dx\, dy.$$

Now cut out a small hole around the origin and reverse the integration by parts procedure. [See J I Richards and H K Youn, Theory of Distributions, CUP, 1991]