Nonlinear Systems: Phase Planes and Nonlinear Oscillators.

Much of the theory of these can be found in *Nonlinear Ordinary Differential Equations*, D. W. Jordan and P. Smith, OUP.

Problem 1
Water flows into a rectangular tank at a variable rate. The tank has a perforated bottom, and water flows out at a rate proportional to the square root of the depth. So in scaled variables, as long as the tank does not overflow, we have

$$dy/dt = g(t) - \sqrt{y}, \quad (g(t) \geq 0, y \geq 0).$$

(i) Show that the right hand side is not Lipschitz in y.

(ii) Show that if $g = 0$ and $y(0) = y_0 > 0$ then the tank empties in a *finite* time $2\sqrt{y_0}$. Deduce that the trajectory $y(t)$ in $0 \leq t \leq t_0$ is *not* uniquely determined by $y(t_0)$ and the function g.

(iii) Show that the trajectory in $t \geq t_0$ *is* uniquely determined by $y(t_0)$ and the function g.

(iv) Suppose $g(t) \leq g_{\max}$ for all t. Show that a tank of depth $g_{\max}^2$ will never overflow.
(i.e. show that if $y(0) \leq g_{\max}^2$ then $y(t) \leq g_{\max}^2$ for all $t \geq 0$.)

[If stuck on (iii), suppose y_1 and y_2 are distinct solutions, say $y_1(t_p) > y_2(t_p)$ for some $t_p > t_0$. Let t_e be the last time before t_p at which $y_1(t_e) = y_2(t_e)$. Then on $[t_e, t_p]$ we have $y_1 \geq y_2$ so $\dot{y}_1 \leq \dot{y}_2$ and this leads to a contradiction.]

Problem 2
Consider the differential equations in the plane,

$$dx/dt = x - y - (x^2 + \frac{3}{2}y^2)x, \quad dy/dt = x + y - (x^2 + \frac{1}{2}y^2)y.$$

Show that $(0,0)$ is the only equilibrium point. The matrix A of the system linearized about $(0,0)$ is

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

with eigenvalues $1 \pm i$. Since the eigenvalues have positive real part, trajectories starting near the origin initially move exponentially away. Since the eigenvalues have non-zero imaginary part, the trajectories are *spiralling* away. To determine what trajectories eventually do,

(i) show that for some $R_1 > 0$ all trajectories cross the circle $x^2 + y^2 = R_1^2$ with *positive* radial velocity;

(ii) show that for some $R_2 > R_1$ all trajectories cross the circle $x^2 + y^2 = R_2^2$ with *negative* radial velocity.

What can be deduced from the Poincaré-Bendixson Theorem about the solution in the annulus $R_1^2 \leq x^2+y^2 \leq R_2^2$?
[Best values are $R_1^2 = 16/17, R_2^2 = 2$. If stuck, see Jordan and Smith, Ch.11]

Problem 3
The simplest one-dimensional partial differential equation incorporating both nonlinear and diffusive effects is Burgers' equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = k\frac{\partial^2 u}{\partial x^2},$$

where $k > 0$ is constant. If we attempt to find "travelling wave" solutions of this, by $u = U(x - ct)$ for some constant c, show that U obeys a second order nonlinear autonomous differential equation. Write this in the form $dU/dx = V, \quad dV/dx = V(U - c)/k$, and find the equilibrium points in the (U, V) plane. Sketch the trajectories and interpret what they would mean for possible travelling wave solutions of Burgers' equation. [In fact you can calculate the form of such a wave as an explicit $U(x)$.]

Problem 4
The most famous one-dimensional partial differential equation incorporating both nonlinear and dispersive effects is the Korteweg de Vries equation, one form of which is

$$\frac{\partial u}{\partial t} + \frac{\partial (u^2)}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0,$$

where $u(x)$, $\frac{\partial u}{\partial x}$ and $\frac{\partial^2 u}{\partial x^2}$ should tend to 0 as $x \to \pm\infty$. Similarly to the previous question, find travelling wave solutions of this, by setting $u = U(x - ct)$ for some constant c. Show that U obeys the second order nonlinear autonomous differential equation

$$U'' = cU - U^2.$$

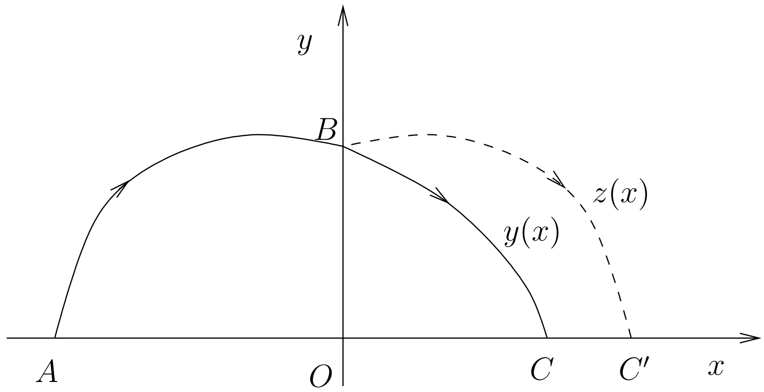
Again writing $V = U'$ and, considering the (U, V) phase plane, sketch the trajectories and interpret what they mean for possible travelling wave solutions of the equation.
[Again, $U(x)$ for such a wave can be calculated explicitly.]

Problem 5
Calculate the period of the closed trajectory of

$$\frac{d^2 x}{dt^2} + k(x^2 - 4)\frac{dx}{dt} + x = 1$$

both in the limit $k \ll 1$ and in the limit $k \gg 1$.
[$2\pi + O(k^2)$ for small k; $\sim k(12 - 3\log 5)$ for large.]

Problem 6
Consider the system $\ddot{x}+f(x)\dot{x}+g(x)=0$, with g odd and $g(x)>0$ for all $x>0$. Considering the (x,y) phase plane with $y=\dot{x}$, show that $(0,0)$ is the only equilibrium point. If the decomposition $f(x)=f_e(x)+f_o(x)$ of f into even and odd parts has $f_e(x)>0$ for all x, prove that there are *no* closed trajectories, as follows:



Any closed trajectory encircles the origin. Consider a trajectory arc ABC from $A=(-x_0,0)$ to $B=(0,y_0)$ to $C=(x_1,0)$, where $x_0>0$, $y_0>0$, $x_1>0$. Let $y(x)$ be the function whose graph is the trajectory arc BC. Let BC' be the reflection of the trajectory arc AB in the y axis, and let $z(x)$ be the function thus described. Show that $(d/dx)(z-y)=2f_e(x)+g(x)(z-y)/(yz)$, and deduce that $z-y$ remains positive, hence the picture really is as drawn above, and so C is nearer to the origin than A was, $x_1<x_0$. Repeating the argument, the trajectory from C will hit the negative x axis again somewhere even closer to the origin, so the trajectory is in fact not closed, but spiralling inwards.
[This ingenious argument is due to Gabriele Villari, Florence.]