Singular Perturbations. Matched Asymptotic Expansions.

Examples of asymptotic expansions.

Problem 1
Let $F(x)$ be defined for $x > 0$ by

$$F(x) = \int_0^\infty \frac{e^{-t}}{t+x}\, dt.$$

Show by integration by parts that

$$F(x) = \sum_{n=1}^{N} \frac{(-1)^{n-1}(n-1)!}{x^n} + (-1)^N N! \int_0^\infty \frac{e^{-t}}{(t+x)^{N+1}}\, dt.$$

Show that the absolute value of this last integral is at most $1/x^{N+1}$, and deduce that $F(x)$ has the asymptotic expansion

$$F(x) \sim \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{x^n}, \quad (\text{as } x \to \infty).$$

If x is allowed to be complex, show that the same method proves this asymptotic expansion in the sector $|\arg(x)| \le \pi - \delta$, for fixed $\delta > 0$.

Problem 2
Consider the nonlinear boundary value problem

$$u''(x) - \epsilon \exp(u(x)\ln(1-\epsilon)) \;=\; 0 \quad \text{for } -1 < x < 1, \qquad u(-1) = u(1) = 0.$$

Sketch the phase plane. Are there different solutions to this problem ? Find the first three terms of an expansion of u for small ϵ.
[We choose the exotic nonlinearity because the equation can be explicitly integrated by using $\int \frac{1}{\sqrt{a+be^{cx}}}dx = -\frac{2}{c\sqrt{a}}\operatorname{arctanh}\left(\frac{\sqrt{a+be^{cx}}}{\sqrt{a}}\right)$. The phase plane and the exact solutions can be found by using MAPLE (an interesting and useful exercise). More conventional problems would be $u'' - \epsilon u^2 = 0$, $u(0) = u(1) = 1$, or $u'' - \epsilon e^{\epsilon u} = 0$, $u(0) = u(1) = 0$.]

Singular Perturbation problems to solve by Matched Asymptotic Expansions.

Problem 3
Consider the singular perturbation problem

$$\epsilon y''(x) - y'(x) - Ny(x) = 0, \quad \text{for } 0 \le x \le 1; \quad y(0) - \epsilon y'(0) = 1, \quad y'(1) = 0,$$

with constants N and ϵ such that $0 < \epsilon \ll 1$. You may assume that there is a boundary layer only near $x = 1$. Determine the first 2 terms in the asymptotic expansion of the solution in powers of ϵ, i.e. find functions $y_{m0}(x)$, $y_{m1}(x)$ such that outside the boundary layer

$$y(x) = y_{m0}(x) + \epsilon y_{m1}(x) + \mathcal{O}(\epsilon^2), \quad 1 - x \gg \epsilon,$$

and functions $y_{r0}(x_r)$, $y_{r1}(x_r)$ such that for $x_r = (x-1)/\epsilon$ in the boundary layer we have

$$y(x) = y_r(x_r) = y_{r0}(x_r) + \epsilon y_{r1}(x_r) + \mathcal{O}(\epsilon^2), \quad 1 - x = \mathcal{O}(\epsilon).$$

[If you want to check your answer, solve the equation exactly.]

Problem 4
Determine one-term expansions for the solution of the nonlinear problem

$$\epsilon y'' + (2x+1)y' + y^2 = 0, \quad \text{for } 0 \leq x \leq 1; \quad y(0) = a, \quad y(1) = b.$$

(In other words, if there may be boundary layers on the left and right, determine the functions y_{l0}, y_{m0}, y_{r0} that are the *leading* terms in the asymptotic expansions of y in the left, middle and right regions.) Repeat this for the problem

$$\epsilon y'' - (2x+1)y' + y^2 = 0, \quad \text{for } 0 \leq x \leq 1; \quad y(0) = a, \quad y(1) = b.$$

What restrictions on a and b are required for your solutions to be valid ?