Matched Asymptotic Expansions.

Problem 1
The vibration of a piano wire is approximately governed by

$$\epsilon\frac{d^4y}{dx^4}-\frac{d^2y}{dx^2}=\omega^2y \quad \text{for } 0\le x\le 1; \quad y(0)=y'(0)=y(1)=y'(1)=0,$$

where $0<\epsilon\ll 1$. The natural frequencies ω are the values $\omega=\omega_n$ for which this equation has a non-zero solution $y=y_n(x)$. Determine the first term involving ϵ in the asymptotic expansions of the frequencies ω_n and corresponding mode shapes $y_n(x)$. [Note that the problem for $\epsilon=0$ with the boundary conditions on y' dropped has natural frequencies $\omega_n=n\pi$ for $n\ge 1$. In fact $\epsilon=B/(TL^2)$ in terms of the bending stiffness B, tension T and length L. To check your answer for the frequency, see Rayleigh, *Theory of Sound*, §190.]

The Method of Multiple Scales.

Problem 2
Apply the method of multiple scales to Duffing's equation in the form

$$\ddot{x}+x+\epsilon x^3=0,$$

where $0<\epsilon\ll 1$: take the "slow" time to be $T=\epsilon t$. Show that if there is a periodic solution with $x=a\cos(\omega t)+\mathcal{O}(\epsilon)$, then $\omega=1+3\epsilon a^2/8+\mathcal{O}(\epsilon^2)$.

Problem 3
In the differential equation

$$\ddot{x}+\dot{x}=\epsilon\cos(\omega t)E(x),$$

$E(x)$ is a prescribed smooth bounded function, and $0<\epsilon\ll 1$. Introducing a slow time $T=\epsilon^2 t$, and assuming an expansion

$$x(t,\epsilon)\sim x_0(T)+\epsilon x_1(t,T)+\epsilon^2 x_2(t,T)+\dots,$$

show that $x_0(T)$ obeys

$$\frac{dx_0}{dT}=-\frac{E(x_0)E'(x_0)}{2(1+\omega^2)},$$

and deduce that x_0 tends to a point where $|E|$ is a minimum. If you had not been told the correct scaling to use, how could you have deduced it ?

Problem 4
Consider the Mathieu equation in the form

$$\ddot{x}+(\alpha+\beta\cos t)x=0$$

with α and β constants. If $\alpha=1+c\beta^2$ with $c>0$ fixed and $0<\beta\ll 1$, use the method of multiple scales with a slow time $T=\beta^2 t$ to deduce that the solutions become unbounded if $-1<12c<5$.