Matched Asymptotic Expansions.

Problem 1
The vibration of a piano wire is approximately governed by

$$\epsilon\frac{d^4y}{dx^4} - \frac{d^2y}{dx^2} = \omega^2 y \quad \text{for } 0 \le x \le 1; \quad y(0) = y'(0) = y(1) = y'(1) = 0,$$

where $0 < \epsilon \ll 1$. The natural frequencies ω are the values $\omega = \omega_n$ for which this equation has a non-zero solution $y = y_n(x)$. Determine the first term involving ϵ in the asymptotic expansions of the frequencies ω_n and corresponding mode shapes $y_n(x)$. [Note that the problem for $\epsilon = 0$ with the boundary conditions on y' dropped has natural frequencies $\omega_n = n\pi$ for $n \ge 1$. In fact $\epsilon = B/(TL^2)$ in terms of the bending stiffness B, tension T and length L. To check your answer for the frequency, see Rayleigh, *Theory of Sound*, §190.]

The Method of Multiple Scales.

Problem 2
Apply the method of multiple scales to Duffing's equation in the form

$$\ddot{x} + x + \epsilon x^3 = 0,$$

where $0 < \epsilon \ll 1$: take the "slow" time to be $T = \epsilon t$. Show that if there is a periodic solution with $x = a\cos(\omega t) + \mathcal{O}(\epsilon)$, then $\omega = 1 + 3\epsilon a^2/8 + \mathcal{O}(\epsilon^2)$.

Problem 3
In the differential equation

$$\ddot{x} + \dot{x} = \epsilon\cos(\omega t)E(x),$$

$E(x)$ is a prescribed smooth bounded function, and $0 < \epsilon \ll 1$. Introducing a slow time $T = \epsilon^2 t$, and assuming an expansion

$$x(t,\epsilon) \sim x_0(T) + \epsilon x_1(t,T) + \epsilon^2 x_2(t,T) + \dots,$$

show that $x_0(T)$ obeys

$$\frac{dx_0}{dT} = -\frac{E(x_0)E'(x_0)}{2(1+\omega^2)},$$

and deduce that x_0 tends to a point where $|E|$ is a minimum. If you had not been told the correct scaling to use, how could you have deduced it ?

Problem 4
Consider the Mathieu equation in the form

$$\ddot{x} + (\alpha + \beta\cos t)x = 0$$

with α and β constants. If $\alpha = 1 + c\beta^2$ with $c > 0$ fixed and $0 < \beta \ll 1$, use the method of multiple scales with a slow time $T = \beta^2 t$ to deduce that the solutions become unbounded if $-1 < 12c < 5$.