The following questions are modified exam questions

Problem 1
The differential equation

$$u'' + u' = f, \qquad 0 \le x \le 1,$$

is subject to the boundary conditions

$$\begin{aligned} u'(0) &= 0, \\ u'(1) &= a u(0), \end{aligned}$$

where a is a real constant, f is a real continuous function, and $' = \frac{d}{dx}$. What is the homogeneous adjoint problem? The Green's function $G(x, y)$ for the original problem satisfies

$$\frac{d^2G}{dx^2} + \frac{dG}{dx} = \delta(x - y)$$

on $[0, 1]$ with the same boundary conditions, where δ is the Dirac δ-function. Find the Green's function, and express the solution of the original problem in terms of an integral. Sketch (roughly) $G(x, y)$ for $y \in [0, 1]$ and $0 \le x \le 1$. Suppose

$$h(x) = \begin{cases} 0 & \text{for} \quad x < s, \\ 1 - e^{s-x} & \text{for} \quad x > s. \end{cases}$$

Find $\frac{dh}{dx}$, $\frac{d^2h}{dx^2}$ and show that

$$\frac{d^2h}{dx^2} + \frac{dh}{dx} = \delta(x - s)$$

in the sense of distributions.

Problem 2
Consider the nonlinear oscillator equation

$$\frac{d^2x}{dt^2} + k(x^2 - 1)\frac{dx}{dt} + x = \lambda,$$

where λ and k are real constants. Show that in the limit $k \ll 1$ there is a closed trajectory with period $2\pi + \mathcal{O}(k^2)$ providing $|\lambda| < 1$, and that its leading-order approximation is $x_0 = \lambda + 2(1 - \lambda^2)^{\frac{1}{2}} \cos t$. [You may use the fact that $\cos^2 y \sin y = \frac{1}{4}(\sin y + \sin 3y)$.]
Find $F(x)$ such that the system may be written in Liénard form

$$\begin{aligned} \frac{dx}{dt} &= k(y - F(x)), \\ k\frac{dy}{dt} &= \lambda - x. \end{aligned}$$

Sketch the trajectories in the (x, y)-plane for the cases $\lambda = 0, k = 3$ and $\lambda = 2, k = 3$. Find the critical point and show that its stability changes between these cases, and state what this implies about the stability of the closed trajectory (assuming it exists)?

Problem 3
Let y be the solution of

$$\epsilon y'' + xy' + xy^2 = 0, \qquad y(0) = 0, \quad y(1) = \frac{1}{2},$$

where $y' = \frac{dy}{dx}$ and $0 < \epsilon \ll 1$.

(i) Show that there can be no (nontrivial, i.e. constant) boundary layer at $x = 1$.

(ii) Find the leading-order outer solution valid for $x = \mathcal{O}(1)$ (use $y(1) = \frac{1}{2}$).

(iii) Show that the scaling of the boundary layer at $x = 0$ is $x = \epsilon^{\frac{1}{2}}\hat{x}$.

(iv) By solving the leading-order boundary layer equation and matching with the outer solution, show that the leading-order boundary layer solution is

$$A\int_0^{\hat{x}} e^{-\frac{u^2}{2}}\,du, \qquad \text{where } A^{-1} = \int_0^{\infty} e^{-\frac{u^2}{2}}\,du = \sqrt{\frac{\pi}{2}}.$$

(v) State a leading order composite expansion for $y(x, \epsilon)$.

(vi) Now suppose that the boundary condition at $x = 1$ is changed to $y(1) = 1$. What happens to the outer solution y as $x \to 0$? Show that the boundary layer scaling is now $x = \epsilon^{\frac{1}{2}}\bar{x}, y = \epsilon^{-\frac{1}{2}}\bar{y}$, and find the new boundary layer equation.