Optimal Control: the Pontryagin Maximum Principle.

Problem 1

A process obeys $\ddot{y} + \dot{y} = u$, and $y(t)$ and $\dot{y}(t)$ are to be brought from specified initial values to zero in minimum time by choice of input $u(t)$ subject to $|u| \leq 1$. Show that "bang-bang" control is optimal (i.e. always use an extreme value of u) and that there is never more than one switch of control on any optimal trajectory. Sketch the $(y, \dot{y})$ phase plane showing the optimal trajectories.

Problem 2

A process obeys the first order differential equation

$$\dot{x} = -x + u$$

where u is an unrestricted input. It is desired to minimize, for given $x(0)$, the integral

$$I = \int_0^T (x^2 + u^2)\, dt.$$

Write down the Hamiltonian and show that the optimal value of u is $-\frac{1}{2}p$, where p is the adjoint variable, and write down the differential equation for p. Show that the differential equations are satisfied by an optimal feedback-control law in the form

$$u(t) = -K(t)x(t)$$

where the time-varying gain satisfies the Riccati equation

$$\dot{K} = -1 + 2K + K^2.$$

Specify a boundary value for K and hence calculate K explicitly. What happens in the limiting case $T \to \infty$?

Problem 3

A colony of insects consists of workers and queens, of numbers $w(t)$ and $q(t)$ at time t. If a time-dependent proportion $u(t)$ of the colony's effort is put into producing workers $(0 \leq u(t) \leq 1)$, then w and q obey the differential equations

$$\dot{w} = auw - w, \quad \dot{q} = c(1-u)w,$$

where a and c are positive constants with $a > 1$. The function u is to be chosen so as to maximize the number of queens at the end of the season. Show that the optimal policy is to produce only workers up to some moment, and only queens thereafter. Show that the optimal switching time is $\log(a/(a-1))$ before the end of the season.

Problem 4

(a) Find, using variational techniques, the curve which gives the shortest distance between the points (a, b) and (c, d).

(b) A light ray passes through a medium consisting of the uniform infinite slabs $-a \leq x \leq 0$ (air, with speed of travel c_1) and $0 \leq x \leq b$ (water, with speed of travel c_2), where $-\infty < y < \infty$. Find the path with shortest travel time from $(-a, 0)$ to $(b, 1)$. Hence deduce Snell's law of refraction for light passing from air to water, namely

$$\frac{\sin\alpha}{c_1} = \frac{\sin\beta}{c_2},$$

where α, β are the incident, transmitted angles of the light path with the normal to the plane $x = 0$.

Problem 5
For D a simply connected and bounded domain in $\mathbb{R}^2$, find the function $u(x, y)$ with mean zero (i.e. $\int_D u(x, y)dxdy = 0$), which minimises

$$\int_D \frac{1}{2}(\nabla u)^2 + \lambda e^u dxdy,$$

for constant $\lambda \geq 0$. What happens if we drop the condition that u has mean zero?