Eigenfunction expansions, Fourier transforms.

Problem 1
Show that in each of the cases below, L and its boundary conditions form a self-adjoint system. For each, calculate all the eigenvalues and eigenfunctions of $L[u] = \lambda u$. (Remember that when λ has a 2-dimensional eigenspace you must produce 2 orthogonal eigenfunctions.)
(a) $L[u] = -u''$ on $0 \leq x \leq \pi$, with $u(0) = u(\pi) = 0$. (The stretched string)
(b) $L[u] = -u''$ on $0 \leq x \leq \pi$, with $u(0) + u'(0) = 0$, $u(\pi) + u'(\pi) = 0$.
(c) $L[u] = -u''$ on $0 \leq x \leq 2\pi$, with $u(0) = u(2\pi)$, $u'(0) = u'(2\pi)$. (The periodic stretched string)
(d) $L[u] = d^4u/dx^4$ on $0 \leq x \leq 1$, with $u(0) = u'(0) = u''(1) = u'''(1) = 0$. (The clamped-free bending beam)

Problem 2
Using the answers to question 1,
(a) State sufficient conditions on $f(x)$ for there to be a uniformly convergent expansion

$$f(x) = \sum_{1}^{\infty} a_n \sin nx, \quad (0 \leq x \leq \pi).$$

Express a_n as an integral in terms of f.
(b) State sufficient conditions on $f(x)$ for there to be a uniformly convergent expansion

$$f(x) = \sum_{1}^{\infty} b_n (\sin nx - n \cos nx), \quad (0 \leq x \leq \pi).$$

Express b_n as an integral in terms of f.
(c) State sufficient conditions on $f(x)$ for there to be a uniformly convergent expansion

$$f(x) = \sum_{-\infty}^{\infty} c_n \exp(inx), \quad (0 \leq x \leq 2\pi).$$

Express c_n as an integral in terms of f.
[If 1(c) does not appear directly to help with this, notice that for $n \neq 0$ the pairs of functions $\{\cos nx, \sin nx\}$ and $\{\exp(inx), \exp(-inx)\}$ are orthogonal bases for the *same* complex vector space.]
(d) State sufficient conditions on $f(x)$ for there to be a uniformly convergent expansion

$$f(x) = \sum_{1}^{\infty} d_n u_n(x), \quad (0 \leq x \leq 1),$$

where $u_n(x)$ is the nth clamped-free bending beam eigenfunction of part (d) in question 1. Express d_n as an integral in terms of f.
[Note that in all of these cases, f still has sensible expansions under *weaker* conditions, e.g. if f is continuous then the expansion converges pointwise except on a null set, and if $f \in L^2$ then the expansion converges in mean square.]

Problem 3
Consider a self-adjoint system $L[u] = f(x)$ on $0 \le x \le 1$, with linear homogeneous boundary conditions at $x = 0, 1$. Let λ_n be the eigenvalues of $L[u] = \lambda u$ as usual, and let u_n be real orthonormal eigenfunctions, a basis for each eigenspace. Suppose that $\lambda = 0$ is *not* an eigenvalue of the system.
Show that if u solves the inhomogeneous system and if f has an expansion $f(x) = \sum a_n u_n(x)$ then $\langle u_n, u\rangle = a_n/\lambda_n$.
Hence show that if the solution u has an eigenfunction expansion then it is

$$u(x) = \sum_1^\infty \frac{a_n u_n(x)}{\lambda_n}.$$

Comparing this with the Green's function representation, deduce that

$$G(x, y) = \sum_1^\infty \frac{u_n(x)u_n(y)}{\lambda_n}.$$

[Assume that the order of integration and summation can be interchanged where necessary. In fact this series for G is uniformly convergent in cases where *all* the λ_n have the same sign (Mercer's Theorem) and it always converges in mean square, and it is rather difficult to construct examples where it does not hold pointwise.]
Applying L to this, deduce that at least formally we have

$$\sum_1^\infty u_n(x)u_n(y) = \delta(x - y).$$

Show that this is true as equality of distributions on $0 < x < 1$ (using the fact that a test function $\phi \in C_0^\infty(0, 1)$ will have a uniformly convergent eigenfunction expansion).
Also deduce from the above that

$$\int_0^1 G(x, x)dx = \sum_1^\infty \frac{1}{\lambda_n}.$$

Revision of some basic results on Fourier Transforms.

If you need somewhere to look to revise this material, see Carrier, Krook and Pearson, *Functions of a Complex Variable*, Chapter 7.

Problem 4
Defining

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x)\exp(ikx)\,dx,$$

express the Fourier transforms of $xf(x)$ and $f'(x)$ in terms of $\tilde{f}(k)$, assuming convergence of all necessary integrals.

Notation: in the sequel, the convention for Fourier transforms is to have the forward and inverse transforms given by

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x) \exp(ikx)\, dx, \quad f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(k) \exp(-ikx)\, dk.$$

Note that different books use different conventions about whether the $-$ sign is in the exponent in the forwards or inverse transform, and about whether the factor $1/2\pi$ is put in the forward transform, or the inverse, or $1/\sqrt{2\pi}$ in each.

Problem 5
Assuming the inversion result that

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(k) \exp(-ikx)\, dk,$$

and assuming that $g(x)$, $\tilde{g}(k)$ are a similar Fourier transform pair, show that

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{\tilde{f}(k)}\, \tilde{g}(k)\, dk = \int_{-\infty}^{\infty} \overline{f(x)}\, g(x)\, dx$$

and that therefore

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |\tilde{f}(k)|^2\, dk = \int_{-\infty}^{\infty} |f(x)|^2\, dx.$$

These are both called Parseval's formula.
[Again just do the formal manipulations, reverse the order of double integrals without worrying about convergence etc. It can be shown that the Fourier transform does extend to a scaled isometry of $L^2(\mathbb{R})$ with these properties.]

Problem 6
Show (just manipulating the formulas as before) that if h is the *convolution* of f and g,

$$h(x) = \int_{-\infty}^{\infty} f(y) g(x-y)\, dy,$$

then $\tilde{h}(k) = \tilde{f}(k)\tilde{g}(k)$.
[In fact, if f, $g \in L^1(\mathbb{R})$ then $h(x)$ is defined except on a null set, $h \in L^1(\mathbb{R})$ and the transform result holds for all real k.]

Problem 7
If $f(-x) = f(x)$ for all x then we say $f(x)$ is an even function of x, or to be brief, f is even. Similarly if $f(-x) = -f(x)$ then f is odd; if $f(-x) = \overline{f(x)}$ then f is Hermitian; if $f(-x) = -\overline{f(x)}$ then f is skew-Hermitian.
Show that if f is even, odd, real-valued, imaginary-valued, Hermitian or skew-Hermitian, then $\tilde{f}$ is even, odd, Hermitian, skew-Hermitian, real-valued or imaginary-valued respectively.

Problem 8
(a) Calculate the Fourier transform of $f_\epsilon(x) = \exp(-\epsilon|x|)$ for $\epsilon > 0$.
(b) Show that $2\epsilon/(\epsilon^2 + k^2) \to 2\pi\delta(k)$ (in the sense of distributions) as $\epsilon \to 0+$ (i.e. as ϵ tends to 0 from above).
[Recall that to do this you have to let ϕ be a test function and show

$$\int_{-\infty}^{\infty} \frac{2\epsilon}{\epsilon^2 + k^2}\phi(k)\,dk \to 2\pi\phi(0)$$

as $\epsilon \to 0+$. To do this, put $k = \epsilon y$ in the integral and then let $\epsilon \to 0+$.]
Combining (a) and (b) show that the natural definition of the Fourier transform of $f(x) = 1 = \lim_{\epsilon\to 0+}(f_\epsilon(x))$ is the distribution $\tilde{f}(k) = 2\pi\delta(k)$. Check that the inversion formula holds formally in this case.

Problem 9
Considering similarly the functions

$$H_\epsilon(x) = \begin{cases} \exp(-\epsilon x) & \text{if } x > 0 \\ 0 & \text{if } x < 0, \end{cases}$$

show that the Heaviside step function $H(x) = \lim_{\epsilon\to 0+} H_\epsilon(x)$ has Fourier transform $\tilde{H}(k) = \pi\delta(k) + i/k$. Check that the inversion formula holds formally using the Cauchy principal value in this case.
[It may be useful to know that $\int_{-R}^{R} \sin(kx)/k\,dk \to \pi\mathrm{sign}(x)$ as $R \to \infty$.]

Solving some partial differential equations by using the Fourier transform to convert them into ordinary differential equations.

Problem 10
Suppose $u(x,t)$ satisfies the diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad (x \in \mathbb{R})$$

for $t > 0$, with initial values $u(x,0) = u_0(x)$. Assuming that the solution u has a Fourier transform

$$\tilde{u}(k,t) = \int_{-\infty}^{\infty} u(x,t)\exp(ikx)dx$$

and assuming that the formal manipulations carry through successfully, show that $\tilde{u}$ obeys $\partial\tilde{u}/\partial t = -k^2\tilde{u}$ and $\tilde{u}(k,0) = \tilde{u}_0(k)$. Deduce that $\tilde{u}(k,t) = \tilde{u}_0(k)\exp(-k^2 t)$, and hence deduce that $u(x,t)$ is the convolution of $u_0(x)$ with the function $E_t(x)$ whose Fourier transform is $\tilde{E}_t(k) = \exp(-k^2 t)$. Calculate $E_t(x)$ by inverting the transform, and hence show the solution is

$$u(x,t) = \int_{-\infty}^{\infty} u_0(x-y)\frac{\exp(-y^2/(4t))}{2\sqrt{\pi t}}\,dy.$$

(Existence of the Fourier transform precludes any "blasts of heat from infinity" (thinking of u as temperature).)

Problem 11
Suppose $u(x,t)$ satisfies the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

for $t > 0$, with zero initial displacement $u(x,0) = 0$, and prescribed initial velocity $partialu/ \partial t(x,0) = v_0(x)$. Proceeding as in the previous question, show that $\tilde{u}(k,t) = \tilde{v}_0(k)\sin(kt)/k$. Show that the Fourier transform of the function that is $\frac{1}{2}$ on $[-t,t]$ and 0 elsewhere is $\sin(kt)/k$. Deduce that

$$u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} v_0(y)\,dy.$$

(You should know this from d'Alembert's solution of the 1D wave equation.)

Problem 12
Suppose $u(x,y)$ satisfies Laplace's equation in 2 dimensions

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

in $y > 0$, with $u(x,0) = u_0(x)$ (a bounded continuous function), and $u(x,y)$ bounded in the whole region $y > 0$. Define

$$\tilde{u}(k,y) = \int_{-\infty}^{\infty} u(x,y)\exp(ikx)\,dx,$$

and show that $\partial^2\tilde{u}/\partial y^2 = k^2\tilde{u}$. Deduce that $\tilde{u}(k,y) = \tilde{u}_0(k)\exp(-|k|y)$ (without worrying too much about the details of why you reject the other solution). Hence show that

$$u(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{u_0(\xi)}{(x-\xi)^2 + y^2}\,d\xi.$$

(Note that some bound on $u(x,y)$ is needed for uniqueness, otherwise Ay can be added to any solution u.)

Problem 13 (for fun)
The "uncertainty principle" says that the more concentrated f is as a function of space x, the more spread out $\tilde{f}$ must be as a function of wavenumber k. In detail, let us define the standard deviation in x to be

$$\Delta x = \min_a \left(\int_{-\infty}^{\infty} (x-a)^2 |f(x)|^2\,dx \Big/ \int_{-\infty}^{\infty} |f(x)|^2\,dx \right)^{1/2},$$

so the minimum occurs when a is the mean, $a = \int_{-\infty}^{\infty} x|f|^2 / \int_{-\infty}^{\infty} |f|^2$. The standard deviation in k is similarly

$$\Delta k = \min_b \left(\int_{-\infty}^{\infty} (k-b)^2 |\tilde{f}(k)|^2\,dk \Big/ \int_{-\infty}^{\infty} |\tilde{f}(k)|^2\,dk \right)^{1/2}.$$

Then the uncertainty principle is

$$\Delta x\,\Delta k \geq \frac{1}{2}.$$

The proof depends on Weyl's inequality (see e.g. Hardy, Littlewood and Pólya, *Inequalities*, Theorem 226),

$$\left(\frac{1}{2}\int_{-\infty}^{\infty}|g(x)|^2\,dx\right) \le \left(\int_{-\infty}^{\infty}x^2|g(x)|^2\,dx\right)^{1/2}\left(\int_{-\infty}^{\infty}|g'(x)|^2\,dx\right)^{1/2},$$

which assumes g is differentiable and both integrals on the right converge. Show from this using Parseval's formula that

$$\frac{1}{2} \le \left(\frac{\int_{-\infty}^{\infty}x^2|g(x)|^2\,dx}{\int_{-\infty}^{\infty}|g(x)^2|\,dx}\right)^{1/2}\left(\frac{\int_{-\infty}^{\infty}k^2|\tilde{g}(k)|^2\,dk}{\int_{-\infty}^{\infty}|\tilde{g}(k)|^2\,dk}\right)^{1/2}.$$

Then apply this to $g(x) = f(x+a)\exp(ibx)$ to deduce the result $\Delta x\,\Delta k \ge \frac{1}{2}$ originally stated.
If you know Quantum Mechanics, show that this is equivalent to Heisenberg's Uncertainty Principle.
[You need not prove Weyl's inequality. If you ever need to, one method is first to consider real-valued g. Use integration by parts to show that $\int_a^b g^2 = [xg^2]_a^b - 2\int_a^b xgg'$. Let $b = b_n \to \infty$ through a sequence of values such that $b_n g(b_n)^2 \to 0$. (If no such sequence exists then $x|g(x)|^2 \ge \epsilon > 0$ for large x contradicting convergence of $\int_{-\infty}^{\infty}x^2|g|^2$.) Take $a = a_n$ for a similar sequence $a_n \to -\infty$. Hence show $\frac{1}{2}\int_{-\infty}^{\infty}g^2 = -\int_{-\infty}^{\infty}xgg'$. Use Schwarz' inequality $(\int_{-\infty}^{\infty}FG)^2 \le \int_{-\infty}^{\infty}F^2\int_{-\infty}^{\infty}G^2$ with $F = xg,\ G = g'$ to deduce the result for real-valued g. Apply this to $|g|$ if g is complex-valued.]

MERRY CHRISTMAS AND A HAPPY NEW YEAR