On the Bachelier–DrawDown equation and Azéma-Yor martingales

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1 Introduction

In his paper "Théorie des probabilités continues" published in 1906 [3], the French mathematician Louis Bachelier was the first to consider and study stochastic differential equations. He focused his attention on some particular types of SDE’s most of which are now very classical. However, he also considered and ”solved” an SDE of a special type, that has not been studied since 1906, and which we accordingly call the Bachelier equation.

In [2] Azéma and Yor introduced a family of simple local martingales, associated with Brownian motion or more generally with a continuous martingale, which they exploited to solve the Skorokhod embedding problem. These processes, called Azéma-Yor processes, extended to a semimartingale context as defined in (2) below are simply functions of the underlying

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semimartingale $X$ and its running maximum. They proved to be very useful especially in investigating laws of the maximum of stopped martingales (cf. Azéma and Yor [1], Oblój and Yor [21]). In this paper we explore new properties and applications of these processes.

We start by showing that Azéma-Yor processes allow to solve explicitly the Bachelier equation, which we also identify with the Draw-Down equation. In the proof we uncover a duality relation which induces a group structure on the (sub)set of Azéma-Yor martingales. Then in Section 4 we study in detail Azéma-Yor processes defined from a non-negative local martingale $N_t$ with continuous supremum process and with $N_t \to 0$ as $t \to \infty$. We show how one can identify the process from its terminal value and in particular how to build an Azéma-Yor martingale with a prescribed terminal law. This allows us to re-discover the Azéma-Yor [2] solution to the Skorokhod embedding problem. Finally, in the last section, we prove optimal properties of Azéma-Yor martingales for the concave ordering of terminal variables.

2 The set of Azéma-Yor processes

Throughout, all processes are assumed to be taken right-continuous with left-hand limits (càdlàg) and defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ satisfying the usual hypothesis. All functions are assumed to be Borel measurable. Given a process $(X_t)$ we denote its running supremum $\overline{X}_t = \sup_{s \leq t} X_s$. In that follows, we are essentially concerned with semimartingales with continuous running supremum, that we called max-continuous semimartingales. Observe that under this assumption, the process $\overline{X}_t = \sup_{s \leq t} X_s$ only increases when $\overline{X}_t = X_t$ or equivalently

$$\int_0^T (\overline{X}_t - X_t) d\overline{X}_t = 0 . \quad (1)$$

We let $T_b(X) = \inf\{t \geq 0 : X_t = b\}$ be the first hitting time of $b$ by process $(X_t)$. Note that by max-continuity $T_b(X) = \inf\{t \geq 0 : X_t \geq x\}$.

2.1 Definition and Properties

There are two different ways to introduce the Azéma-Yor processes, and their equivalence has been proven by several authors (see the comments after Definition 1).
Definition 1. Let \((X_t)\) be a max-continuous semimartingale starting from \(X_0 = a\), and \(\overline{X}_t\) its (continuous) running supremum. With any locally bounded Borel function \(u\) we associate the primitive function \(U(x) = a^* + \int_a^x u(s)ds\) defined on \([a, +\infty)\). The \((U, X)\)-Azéma-Yor process is defined by one of these two equations,

\[
M_t^U(X) := U(\overline{X}_t) - u(\overline{X}_t)(\overline{X}_t - X_t)
\]

or

\[
= a^* + \int_0^t u(\overline{X}_s)dX_s.
\]

In consequence, \(M^U(X)\) is a semimartingale and it is a local martingale when \(X\) is a local martingale.

Remarks. Note that when \(u\) is defined only on some interval \([a, b)\) we can still define \(M_t^U(X)\) for \(t \leq T_b(X)\) and (2) holds with \(t \wedge T_b(X)\) instead of \(t\).

Azéma and Yor [2] were the first to introduce these processes (using the first equation) when \((X_t)\) is a continuous local martingale. The equivalence between both equations is easy to establish when \(u\) is smooth enough to apply Itô’s formula, since the continuity of the running supremum implies from (1) that \(\int_0^t (\overline{X}_t - X_t)du(\overline{X}_t)\) is the null process. This results may be extended to locally bounded functions \(u\) via monotone class theorem. Alternatively, the equivalence can be argued using the general balayage formula, see Nikeghbali and Yor [18]. The case of locally integrable function \(u\) can be attained for continuous local martingale \(X\), as that has been shown by Oblój and Yor in [21].

The importance of the family of Azéma-Yor martingales is well exhibited by Oblój [20] who proves that in the case of continuous local martingale \((X_t)\) all local martingales which are functions of the couple \((X_t, \overline{X}_t)\), \(M_t = H(X_t, \overline{X}_t)\) can be represented by a \(M^U\) local martingale associated with a locally integrable function \(u\). We note that such processes are sometimes called max-martingales.

2.2 Monotonic transformation and Azéma-Yor process

We want to investigate further the structure of the set of \((U, X)\)-Azéma-Yor processes associated with max-continuous semimartingales. One of the most remarkable properties of these processes is that their running supremum can be easily computed, when the function \(U\) is non decreasing \((u \geq 0)\). We denote by \(\mathcal{U}_{an}\) the set of such functions, that is if \(U\) and \(F\) are
in $U_m$, and defined on appropriate intervals $U \circ F(x) = U(F(x))$ is in $U_m$. We let $G_m$ be the set of functions $U \in U_m$, with inverse function $V \in U_m$, or equivalently of functions $U$ such that $u > 0$ and both $u$ and $1/u$ are locally bounded.

In light of (2), then we have

**Proposition 2.** a) Let $U \in U_m$, $X$ be a max-continuous semimartingale and $(M^U(X))$ be the $(U,X)$-Azéma-Yor process. Then

\[ M_t^U(X) = U(X_t) = U(\overline{X}_t), \tag{4} \]

and $M^U(X)$ is a max-continuous semimartingale.

b) Let $F \in U_m$ defined on an appropriate interval so that $U \circ F$ is well defined. Then,

\[ M_t^U(M^F(X)) = M_t^{U \circ F}(X). \]

c) Moreover, the set of Azéma-Yor processes indexed by $U \in G_m$ defined on whole $\mathbb{R}$ with $U(\mathbb{R}) = \mathbb{R}$, is a group under the operation $\otimes$ defined by

\[ M^U \otimes M^F := M^{U \circ F}. \]

**Proof.** a) In light of (2), when $u$ is non negative, the Azéma-Yor process $M_t^U(X)$ is dominated by $U(X_t)$, with equality if $t$ is a point of increase of $\overline{X}_t$. So, $M_t^U(X) = U(\overline{X}_t)$ and (4) holds true since $U$ is non decreasing. Moreover, since $U(\overline{X})$ is a continuous process, $M^U$ is a max-continuous semimartingale and we may take an Azéma-Yor process of it.

b) Let $F$ be in $U_m$ such that $U \circ F$ is well defined, and $f = F' \geq 0$ its derivative. We have from (4)

\[ M_t^U(M^F(X)) = U(F(\overline{X}_t)) - u(F(\overline{X}_t))f(\overline{X}_t)(\overline{X}_t - X_t) \]

\[ = M_t^{U \circ F}(X), \tag{5} \]

where we used $(U(F(x)))' = u(F(x))f(x)$.

c) The set of $U \in G_m$ defined on $\mathbb{R}$ with $U(\mathbb{R}) = \mathbb{R}$ is itself a group for composition. Note that we can always chose to take $V'(y) = 1/U'(V(y))$.

The group structure of the set of Azéma-Yor processes follows from b) above, the neutral element is given by $X = M^{Id}(X)$ and the inverse of $M^U(X)$ is $M^{U^{-1}}(X)$, where $V = U^{-1}$.

The most important consequence for us is that we can take inverse in the set of Azéma-Yor processes, i.e. we can recover $(X_t)$ from $(M_t^U(X))$ when $U$ is in $G_m$. We phrase this as a separate corollary and specify to the setting of stopped processes which will be useful in the sequel.
Corollary 3. Let \( a < b \leq \infty \), \( U \in \mathcal{G}_m \) a primitive function of \( u : [a, b) \to (0, \infty) \) such that \( U(a) = a^* \). Let \( V : [a^*, U(b)) \to [a, b) \) be the inverse of \( U \) with locally bounded derivative \( v(y) = 1/u(V(y)) \).

Then for any max-continuous semimartingale \((X_t)\), \( X_0 = a \), stopped at the time \( T_b = T_b(X) = \inf\{t; X_t \geq b\} \) we have

\[
X_{t \wedge T_b} = M^V_{t \wedge T_b}(M^U(X)).
\]

(6)

From the differential point of view, on \([0, T_b)\),

\[
dM^U_t(X) := dY_t = u(X_t) dX_t, \quad dX_t = v(Y_t) dY_t.
\]

(7)

3 Bachelier equation and Drawdown constraints

In his paper "Théorie des probabilités continues", published in 1906, French mathematician Louis Bachelier [3] was the first to consider and study stochastic differential equations. In fact, obviously, he didn’t prove in his paper existence and uniqueness results but focused his attention on some particular types of SDE’s. In this way, he obtained the general structure of processes with independent increments and continuous paths, the definition of diffusions (in particular, he solved the Langevin equation), and generalized these concepts to higher dimensions.

3.1 The Bachelier equation

In particular, Bachelier considered and "solved", a particular type of SDE depending on the supremum of the solution, \( dY_t = \varphi(Y_t) dX_t \) which we call the Bachelier equation. Let \( U \in \mathcal{G}_m \) and \( V \in \mathcal{G}_m \) its inverse function with derivative \( v \). From (7), and the identity \( X_t = V(Y_t) \), we see that the \((U, X)\)-Azéma-Yor process \( Y \) verifies the Bachelier equation for \( \varphi(y) = 1/v(y) \). Now, we can solve the Bachelier equation as an inverse problem. We present a rigorous and explicit solution to this equation which proves to be surprisingly simple. We note that a similar approach is developed in Revuz and Yor [22, Ex.VI.4.21].

Theorem 4. Let \((X_t : t \geq 0)\), \( X_0 = a \), be a max-continuous semimartingale. Consider a positive Borel function \( \varphi : [a^*, \infty) \to (0, \infty) \) such that \( \varphi \) and \( 1/\varphi \) are locally bounded.

The function \( V \) defined by \( V(y) = a + \int_{a^*}^{y} \frac{dx}{\varphi(x)} \) belongs to \( \mathcal{G}_m \) and so does its inverse function \( U \) defined on \([a, V(\infty) = b]\).
a) The Bachelier equation

\[ dY_t = \varphi(Y_t) dX_t, \quad Y_0 = a^* \]  

(8)

has a strong, pathwise unique, max-continuous solution defined up to its explosion time \( \zeta_Y = T_b(X) \) given by \( Y_t = M_U(X), \quad t < T_b(X) \).

When \( X \) is a continuous local martingale it suffices to assume that \( 1/\varphi \) and \( \varphi \) are locally integrable functions.

b) Assume \( X \) to be positive, in particular \( V(a^*) = a > 0 \). Then, the Bachelier equation is equivalent to the Draw-Down equation (DD-equation)

\[ dY_t = (Y_t - w(Y_t)) \frac{dX_t}{X_t-}, \quad Y_0 = a^*, \]  

(9)

where \( (Y_t) \) is a max-continuous semimartingale satisfying \( Y_t > w(Y_t), \quad t < \zeta_Y \), and where

\[ w(y) = y - \frac{V(y)}{v(y)}, \quad \text{or equivalently} \quad V(y) = a \exp \left( \int_{a^*}^{y} \frac{1}{u - w(u)} du \right). \]  

(10)

Remarks. Under the stronger assumption that \( X \) has no positive jumps, any solution of the Bachelier equation has no positive jumps and hence is a max-continuous semi-martingale.

The name of DD-equation would be explained below, after having defined the notion of Draw Down constraint (DD).

Proof. a) – The assumptions on \( \varphi \) imply that \( V \) and therefore \( U \) are in \( \mathcal{G}_m \) with \( U(a) = a^* \). With the version of \( u \) we choose, Definition 1 gives that the \((U,X)\)-process \( M_U(X) \) (in short \( M^U \)) verifies

\[ dM^U_t := u(X_t) dX_t = \varphi(M^U_t) dX_t, \quad t < T_b(X). \]

Furthermore, \( M^U_{TV(n)}(X) = U(V(n)) = n \) and we see that if \( b = V(\infty) < \infty \) then \( T_b \) is the explosion time of \( M^U \). So, \( M^U \) is a solution of (8).

Now let \( Y \) be a max-continuous solution to the equation (8). Definition 1 and (8) imply that \( dM^Y_t(Y) = dX_t \) on \([0, \zeta_Y)\). It follows from Corollary 3 that \( Y_t = M^U_t(X) \) and \( T_b(X) \) is the explosion time \( \zeta_Y \) of \( Y \).

b) Assume \( X \) to be positive and let \( Y_t = M^U_t(X) \) be the solution of (8).

Recall that \( X_t = M^V_t(Y) \) and \( X_t = V(\bar{Y}_t) \). Direct computation yields

\[ Y_t - w(\bar{Y}_t) = Y_t - U(\bar{X}_t) + u(\bar{X}_t) \bar{X}_t = u(\bar{X}_t) X_{t-} = \varphi(\bar{Y}_t) X_{t-}. \]
Using the form of $w$ and since $u(X_t)X_t > 0$ we deduce instantly from (2) that $Y_{t^-} > w(Y_t)$ (see also (11) below) and it follows that $Y_t = M_t^U(X)$ solves (9). Expression for $V$ in terms of $w$ follows as $v(y) = \frac{V(y)}{y - w(y)}$.

Conversely, let $Y$ be a max-continuous solution of (9), $Y_{t^-} > w(Y_t)$ and $V$ a solution of $w(y) = y - \frac{V(y)}{v(y)}$. Then, using (2) and (3), we have

$$\frac{dY_t}{Y_{t^-} - w(Y_{t^-})} = \frac{v(Y_t)}{M_t^V(Y)} dY_t = \frac{dM_t^V(Y)}{M_t^V(Y)}.$$ 

On the other hand, since $Y$ is solution of (9)

$$\frac{dY_t}{Y_{t^-} - w(Y_{t^-})} = \frac{dX_t}{X_{t^-}}.$$ 

Therefore, $X$ and $M_t^V(Y)$ have the same relative stochastic differential, and the same initial condition. Then, there are undistinguishable processes and Corollary 3 yields $Y_t = M_t^U(X)$. \square

Remarks. The above extends naturally to the case when $a$ and $a^*$ are some $\mathcal{F}_0$-measurable random variables. It suffices to assume that $\varphi$ is well defined on $[l, \infty)$ where $-\infty \leq l$ is the lower bound of the support of $a^*$.

We could also consider $X$ which is only defined up to its explosion time $\zeta_X$ which would induce $\zeta_Y = \zeta_X \wedge T_b(X), b = V(\infty)$.

### 3.2 Drawdown constraint and DD-equation

In various applications, in particular in financial mathematics, one is interested in processes which remain above a (given) function $w$ of their running maximum. The purpose of this section is to show that Azéma-Yor processes provide a direct answer to this problem when the underlying process is positive. In particular we provide interpretation for the results in the part b) of Theorem 4.

Consider a function $w : \mathbb{R} \to \mathbb{R}$ with $w(y) < y$. We say that a process $M_t$ satisfies $w$-drawdown ($w$-DD) constraint if $M_t \geq w(M_t)$ for all $t \geq 0$ a.s. Azéma-Yor processes, $Y = M_t^U(X)$ defined from a positive max-continuous semimartingale $X$, and function $U \in \mathcal{G}_m$ are DD-constrained processes with constraint function $w$ defined in (10) by $w(y) = (U(x) - xu(x))_{x=V(y)} = y - V(y)/v(y)$, where $V = U^{-1}$, since thanks to the positivity of $X$ and $u$,

$$Y = U(X_t) - u(X_t)X_t + u(X_t)X_t \geq U(X_t) - u(X_t)X_t = w(Y_t).$$ (11)
Note that we can start with any such function $w$ with $y - w(y)$ locally bounded and locally bounded away from zero. Then $V$ in (10) is well defined and in $G_m$, its inverse $U \in G_m$ and their derivatives $v, u$ are locally bounded. In consequence we can define $Y_t = M_t^U(X)$ and it follows that $Y$ satisfies the $w$-DD constraint.

Moreover, $V$ is a convex function (and hence $U$ a concave function) if and only if $w$ is increasing. Note that, as a consequence of concavity of $U$, $Y$ stays above $U(X)$.

What is equally interesting is that we can revert the reasoning and solve for $X$ given $Y$ in (9).

**Theorem 5.** Given $w$ as above and $Y$ a max-continuous semimartingale, $Y_0 = a^*$, satisfying the $w$-drawdown constraint up to the time $\zeta^w(Y)$, there exists a unique max-continuous positive semimartingale $X, X_0 = a$, defined up to the time $\zeta^w(Y)$ by the DD-equation,

$$
\frac{dY_t}{Y_t - w(Y_t)} = \frac{dX_t}{X_t}, \quad t < \zeta^w.
$$

(12)

In fact, $X_t = M_t^V(Y)$, where $V(y) = a \exp(\int_0^y \frac{1}{w(u)} du)$ and $Y$ is the $(U, X)$-process, where $U$ is the inverse function of $V$.

When the function $w$ is increasing, $U$ is a concave function and $Y_t > U(X_t) = U(M_t^V(Y)), t < \zeta^w$.

**Proof.** The Theorem follows immediately from Theorem 4. Note that $V, U \in G_m$ and as $(Y_t)$ is a max-continuous semimartingale we can define a max-continuous semimartingale $X_t = M_t^V(Y)$ and it follows from the $w$-DD constraint on $Y$ that $X$ is positive for $t < \zeta^w$. Furthermore, $Y$ and $X$ solve (12) which is (9). The uniqueness also follows from Theorem 4 since for a given $X$ we have a unique solution $Y$ to (12) and $X_t = M_t^U(Y)$.

**Remarks.** The Draw-Down equation (12) was solved previously by Cvitanić and Karatzas [7] for $w(y) = ay, a \in (0, 1)$ and recently by Elie and Touzi [10]. The use of Azéma-Yor processes simplifies considerably the proof and allows for a general $w$ and $X$ since we have shown that this equation is equivalent to the Bachelier equation and so has a unique strong solution.

Note that we assumed $X$ is positive. The quantity $dX_t/X_{t-}$ has a natural interpretation as the differential of the stochastic logarithm of $X$. In various applications, such as financial mathematics, this logarithm process is often given directly since $X$ is defined as a stochastic exponential in the first place.
An Illustrative Example. Let $X$ be a positive max-continuous semi-martingale such that $X_0 = 1$. Let $U$ be the power utility function defined on $\mathbb{R}^+$ by $U(x) = \frac{1}{1-\gamma} x^{1-\gamma}$ with $\gamma < 1$ and $u(x) = x^{-\gamma}$ its derivative. Put $\beta = \frac{1}{1-\gamma}$, and $C_{\beta} = (1-\gamma)\beta = \beta^{-\beta}$. Then the inverse function $V$ of $U$ is $V(y) = C_{\beta} y^{\beta}$ and its derivative is $v(y) = (C_{\beta} y^{\beta})^{-\gamma}$.

Then the (power) Azéma-Yor process is

$$M_t^U(X) = Y_t = \frac{1}{1-\gamma} (X_t)^{1-\gamma} \left( \gamma + (1-\gamma) \frac{X_t}{X_t} \right) = Y_t \left( \gamma + (1-\gamma) \frac{X_t}{X_t} \right)$$

Since $X$ is positive, $Y_t > w(Y_t) = \gamma Y_t$. The process $(Y_t)$ is a semimartingale (local martingale if $X$ is a local martingale) starting from $Y_0 = 1$, and staying in the interval $[\gamma Y_t, Y_t]$.

The Bachelier-drowdown equation (8)-(9) becomes

$$dY_t = X_t^{-\gamma} dX_t = (1 - \gamma)^{-\gamma} (Y_t)^{-\gamma} dX_t = (Y_t - \gamma Y_t) \frac{dX_t}{X_t}.$$ 

As noted above, this equation, for a class of processes $X$, was studied in Cvitanic and Karatzas [7]. Furthermore, in [7] authors in fact introduced processes $M_t^U(X)$ and used them to solve the portfolio optimisation problem with drawdown constraint of Grossman and Zhou [13]. Using our methods we can simplify and generalise their results and show that the portfolio optimisation problem with drawdown constraint, for a general utility function and a general drawdown function, is equivalent to an unconstrained portfolio optimisation problem with a modified utility function. We develop these ideas in a separate paper.

4 Maximum distribution and Skorohod embedding problem

We study now in more details the Azéma-Yor processes associated with non-negative local martingales, which appear to be more convenient than the classical Brownian motion. In particular, as a direct consequence, we’ll obtain in the next section the Azéma-Yor [2] solution of the Skorokhod embedding problem.

4.1 Distribution of the Maximum and last passage times

At first we recall some well known results on the distribution of the maximum of such processes (see Exercice III.3.12 in Revuz-Yor [22]).
Proposition 6. Let \( (N_t), N_0 = \kappa > 0 \) be a non-negative max-continuous local martingale with \( N_t \xrightarrow{t \to \infty} 0 \) a.s.

a) The random variable \( \kappa / N_\infty \) is uniformly distributed on \([0, 1]\).
b) Let \( \zeta = \inf\{t : N_t \in \{0, b\}\}, b > \kappa \). Then \( N_{t \wedge \zeta} \) is a bounded martingale and \( N_\zeta \in \{0, b\} \). Given the event \( \{N_\zeta < b\} = \{N_\zeta = 0\} \), \( \kappa / N_\zeta \) is uniformly distributed on \([\kappa / b, 1]\). Moreover, the probability of the event \( \{N_\zeta = b\} \) is \( \kappa / b \).

Proof. a) Let us consider the Azéma-Yor martingale associated with \( (N_t) \) and the function \( U(x) = (K - x)^+ \), where \( K \) is a fixed real \( \geq 1 \). Thanks to the positivity of \( (N_t) \), this martingale is bounded by \( K \).

\[
0 \leq M_t^U(N) = (K - N_t)^+ + 1_{\{K > N_t\}}(N_t - N_t) = 1_{\{K > N_t\}}(K - N_t) \leq K.
\]

So \( M_t^U(N) \) is a u.i. martingale, and \( \mathbb{E} M_\infty^U(N) = M_0^U(N) \).

In other terms, for \( K > \kappa \), \( K \mathbb{P}(K > N_\infty) = K - \kappa \), or equivalently \( \mathbb{P}\left(\frac{\kappa}{N_\infty} \leq \frac{\kappa}{K}\right) = \frac{\kappa}{K} \), for any \( K \geq \kappa \). That is exactly the desired result.

b) Assume now that \( N \) is stopped at the stopping time \( \zeta \), with \( N_\zeta = b \), or \( 0 \). By the same argument, for any \( \kappa \leq K < b \), \( \mathbb{P}\left(\frac{\kappa}{N_\zeta} \geq \frac{\kappa}{K}\right) = 1 - \frac{\kappa}{K} \), and \( \mathbb{P}\left(\frac{\kappa}{N_\zeta} > \frac{\kappa}{b}\right) = 1 - \frac{\kappa}{b} \). \( N_\zeta \) has a Dirac distribution at \( b \) with the probability \( \kappa / b \).

Recently, Madan, Roynette and Yor ([17]) have proposed an alternative expression of the Black-Scholes formula in terms of last passage times (see also Benata and Yor [4]). Let \( X \) be a max-continuous positive local martingale with \( X_0 = 1 \), going to 0 at \( \infty \). The last passage time of \( X \) over the level \( K \) is defined by

\[
g_K(X) = \sup\{t \geq 0 : X_t = K\} = \sup\{t \geq 0 : X_{t, \infty} = K\}, \tag{13}
\]

where \( \sup\{0\} = 0 \), so that \( 1_{\{g_K(X) < t\}} = \inf_{u \geq t} 1_{\{X_u < K\}} = 1_{\{X_{t, \infty} < K\}} \).

Let \( U_K(x) = (K - x)^+ \) be the Put function with strike \( K > 0 \), and \( M_t^K \) the associated Azéma-Yor process. Then \( M_t^K \) is a bounded martingale given by

\[
M_t^K = 1_{\{X_t < K\}}(K - X_t) = 1_{\{T_K(X) > t\}}(K - X_t) = K - X_{t \wedge T_K(X)}
\]

In terms of last passage times, we have

\[
M_t^K = \mathbb{E}[M_\infty^K | \mathcal{F}_t] = K \mathbb{P}(X_\infty < K | \mathcal{F}_t) = K \mathbb{P}(g_K(X) = 0 | \mathcal{F}_t).
\]

In particular, \( \mathbb{P}(g_K(X) > 0 | \mathcal{F}_t) = X_{t \wedge T_K(X)}/K \).
We start now at time $t$ from $X_t$ and define the translated $\mathcal{F}_{t+h}$-local martingale $X_t^h = X_{t+h}$. Let $M_{h,t}^{K}$ the Azéma-Yor process associated with the Put function $U_K(x) = (K - x)^+$. Then $M_{h,t}^{K}$ is an $\mathcal{F}_{t+h}$ bounded martingale given by

$$M_{h,t}^{K} = 1_{\{X_{t+h} < K\}}(K - X_{t+h}) = K P(\overline{X}_{t,\infty} < K | \mathcal{F}_{t+h})$$

In particular, $(1 - X_t/K)^+ = \mathbb{P}(g_K(X) < t | \mathcal{F}_t)$. The supermartingale $\mathbb{P}(g_K(X) > t | \mathcal{F}_t) = X_t/K \wedge 1$ is the conditional running supremum of $1_{\{X_u > K\}}$

$$\mathbb{P}(g_K(X) > t | \mathcal{F}_t) = X_t/K \wedge 1 = E(\sup_{u \geq t} 1_{\{X_u > K\}} | \mathcal{F}_t).$$

The distribution function of $g_K(X)$ is given by the normalized Put price,

$$\mathbb{P}(g_K(X) < t) = \text{Put}(1, K, t)/K = E((1 - X_t/K)^+).$$

When $X$ is a geometrical Brownian motion with volatility 1, $E((1 - X_t/K)^+)$ is given by the Black-Scholes formula for the Put option and this result is exactly the Th 1.1 of [4].

$$\mathbb{P}(g_K(X) < t) = N(\log K/\sqrt{t} + \sqrt{t}/2) - 1/K N(\log K/\sqrt{t} - \sqrt{t}/2)$$

where $N$ is the Gaussian distribution function $N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du$. In particular, $\mathbb{P}(g_1 < t) = N(\sqrt{t}/2) - N(-\sqrt{t}/2) = \mathbb{P}(4B_1^2 \leq t)$, where $B_1$ is a standard Gaussian random variable.

### 4.2 Azéma-Yor martingales with prescribed terminal laws

Using Proposition 6 jointly with an analytic lemma below, we describe all martingales whose terminal values are Borel functions of the maximum of some non-negative local martingale. We then construct Azéma-Yor martingales with given terminal laws.

**Lemma 7.** Let $h$ be a locally bounded Borel function such that $h(x)/x^2$ is integrable away from 0, $U_\infty$ be the solution of the ordinary differential equation

$$U(x) - x U'(x) = h(x), \quad \lim_{x \to \infty} U(x)/x = 0. \quad (14)$$

a) The solution $U_\infty$ is given by

$$U_\infty(x) = x \int_{x}^{\infty} \frac{h(s)}{s^2} ds = \int_{0}^{1} h(\frac{x}{s}) ds \quad (15)$$
Moreover, if \( h_m \) is the function \( h(\cdot \lor m) \) constant on \((0, m)\), then \( U_\infty(m, x) \), the associated solution of (14), is affine on \((0, m)\) and

\[
U_\infty(m, x) = U_\infty(m) - U'_\infty(m)(m - x), \quad \text{if} \ x < m,
\]

\[
U_\infty(m, x) = U_\infty(x), \quad \text{if} \ x \geq m.
\]

(16)

b) Assume \( h \) to be locally bounded on \((0, b]\) and denote by \( U_b \) the solution of (14) for \( x \leq b \) with boundary condition \( U_b(b) = h(b) \). Then,

\[
U_b(x) = x \int_x^b \frac{h(s)}{s^2} ds + \frac{h(b)}{b} x = \int_0^1 h\left(\frac{x}{s} \land b\right) ds, \quad x \leq b.
\]

(17)

For \( m < b, h_m(x, b) = h(m \lor (x \land b)) \), the associated solution \( U_b(m, x) \) is affine on \([0, m]\), \( U_b(m, x) = U_b(m) - U'_b(m)(m - x) \).

If \( h(x)/x^2 \) is integrable away from zero then \( U_b(x) = U_\infty(x) - x U'_\infty(b) \), for \( x \leq b \).

**Remark 8.** In the sequel it is sometimes convenient to consider \( U_b(x) \) for \( x > b \) and then we take \( U_b(x) = U_b(b) = h(b) \) for \( x > b \).

Suppose the function \( h \) to be non decreasing. The solutions of the ODE (14) are concave functions on the appropriate intervals. Moreover, since \( m \geq x \), we also have \( U_b(m, x) \geq U_b(x) \). Finally, if additionally \( h > 0 \), then \( U_\infty(x) \) is non-decreasing and in fact it is strictly increasing if \( h(\infty) > h(x), \ x < \infty \).

**Proof.** a) (15) and (17) are easy to obtain using the transformation \( (U(x)/x)' = -h(x)/x^2 \). Assume \( h = h_m \). When \( x \geq m \) we clearly have \( U_\infty(m, x) = U_\infty(x) \). For \( x < m \), \( (U_\infty(m, x)/x)' = -h(m)/x^2 \). Using \( h(m) = U_\infty(m) - m U'_\infty(m) \) and \( U_\infty(m, m) = U_\infty(m) \) we deduce (16). The results for \( U_b \) and \( U_b(m, x) \) are analogous. \( \Box \)

This analytical lemma allow us to characterize Azéma-Yor martingales from their terminal values. This extends in more details the ideas presented in El Karoui and Meziou [9, Propositon 5.8].

**Proposition 9.** Let \( h \) be a Borel function such that \( h(x)/x^2 \) is integrable away from 0, and \( U_\infty \) the solution of the ODE (14).

Let \( N \) be a max-continuous non negative local martingale going to 0 at \( \infty \) with an integrable initial value \( N_0 > \epsilon > 0 \).

a) Then, \( h(\overline{N}_\infty) \) is an integrable random variable and the closed martingale \( H_t = \mathbb{E}(h(\overline{N}_\infty)|\mathcal{F}_t) \) is the Azema-Yor martingale \( M^{U_\infty}(N) \).

b) For a function \( U \) with locally bounded derivative \( u \) and with \( U(x)/x \to 0 \)
as \( x \to \infty \), the Azéma-Yor local martingale \( M^U(N) \) is a uniformly integrable martingale if and only if \( h(x)/x^2 \) is integrable away from zero, where \( h \) is now defined via (14).

\( \text{c) Assume } h \text{ to be locally integrable on } (0,b], N_0 < b \text{ a.s., and let } \zeta_b = \inf\{t \geq 0 : N_t \in \{0,b\}\}. \text{ Then, } h_b(N_{\zeta_b}) \text{ is an integrable random variable and the closed martingale } H_t = \mathbb{E}\left(h_b(N_{\zeta_b})|\mathcal{F}_{t \wedge \zeta_b}\right) \text{ is the Azema-Yor martingale } M^U_{t \wedge \zeta_b}(N).

\begin{proof}
We start with the proof of a) and assume that \( h(x)/x^2 \) is integrable away from 0. It is easy to calculate the expectation of \( h(N_{\infty}) \) since

\[
\mathbb{E}\left(|h(N_{\infty})|\right) = \mathbb{E}\int_0^1 |h(N_0/s)|ds \leq \mathbb{E}N_0 \int_s^\infty |h(s)|/s^2 ds < \infty.
\]

To study the martingale \( H_t = \mathbb{E}\left(h(N_{\infty})|\mathcal{F}_t\right) \), we use that \( N_{\zeta_b} \) is the maximum process of the translated martingale \( N_{t+s} \) issued from \( N_t \) at time \( s = 0 \), with respect to the filtration \( \mathcal{F}_{t+s} \). Applying Proposition 6 conditionally on \( \mathcal{F}_t \), with \( \kappa = N_t \), we see that \( N_{t,\infty} \) has the same \( \mathcal{F}_t \)-conditional distribution as \( N_t/U \) where \( U \) is an independent uniform r.v. on \([0,1] \).

The martingale \( H_t \) is given by the following closed formula \( H_t = \mathbb{E}\left(h(N_{t}\lor (N_t/U)|\mathcal{F}_t\right) \) that is

\[
H_t = \int_0^1 h(N_t\lor (N_t/u))du = U_{\infty}(N_t, N_t) = U_{\infty}(N_t) - U'_{\infty}(N_t)(N_t - N_t)
\]

In the last equality, we have used Lemma 7.

To prove part b) it suffices to observe that \( M^U_t(X) \rightarrow h(N_{\infty}) \) a.s. and hence integrability of \( h(N_{\infty}) \), i.e. integrability of \( h(x)/x^2 \) away from zero, is necessary for uniform integrability of \( M^U(N) \). That it is sufficient we proved in part a).

Part c) is analogous to part a).
\end{proof}

**Remark 10.** We stress that the boundary condition \( U(x)/x \to 0 \) as \( x \to \infty \) in (14) is essential for part a). Indeed, consider \( N_t = 1/Z_t \) the inverse of a three dimensional Bessel process. Note that \( N_t \) satisfies our hypothesis and it is well known that \( N_t \) is strict local martingale (cf. Exercise V.2.13 in Revuz and Yor [22]). Then for \( U(x) = x \) we have \( M^U_t(N) = N_t \) is also a strict local martingale but obviously we have \( U(x) - U'(x)x = 0 \).

We note that similar consideration as in a) above were independently made in Nikeghbali and Yor [18].
The first consequence of this calculation is the construction of an Azéma-Yor martingale with given terminal distribution, which is an important step towards the embedding theorem. We start by studying the properties of the solution of (14) when \( h(x) = \overline{q}(1/x) \) where \( \overline{q} \) is the quantile function of some centered probability measure \( \mu \).

**Lemma 11.** Let \( \mu \) be a centered probability measure on \( \mathbb{R} \). Define the right continuous tail distribution function \( \underline{\mu}(x) = \mu([x, \infty)) \). Let \( \overline{q} : [0, 1] \to \mathbb{R} \) the tail quantile function defined as the left-continuous inverse of \( \underline{\mu} \), that is \( \underline{\mu}(x) < y \) iff \( \overline{q}(y) < x \).

a) Let \( r = \overline{q}(0^+) \leq \infty \) be the supremum of the support of \( \mu \) and denote \( U_{\mu} \) the solution \( U_1/\underline{\mu}(r) \) of (14), with \( h(x) = \overline{q}(1/x) \), given by (15) or (17) when \( \underline{\mu}(r) = 0 \) or \( \underline{\mu}(r) > 0 \) respectively. Then \( U_{\mu} \) is strictly increasing on \( [1, 1/\underline{\mu}(r)] \) and \( U_{\mu}(1/\lambda) \) is the average value at risk (AVaR) of \( \mu \) at level \( \lambda \) given by

\[
U_{\mu}(1/\lambda) = \int_0^1 \overline{q}(\lambda s) ds = \frac{1}{\lambda} \int_0^\lambda \overline{q}(s) ds := \text{AVaR}(\lambda), \quad \lambda \in (0, 1),
\]

where, if needed, we extend \( U_{\mu} \) via \( U_{\mu}(x) = U_{\mu}(1/\underline{\mu}(r)) \) for \( x \in (1/\underline{\mu}(r), \infty) \).

b) For the barycentre function \( \Psi_{\mu}(\cdot) \) defined as

\[
\Psi_{\mu}(x) = \frac{1}{\overline{\mu}(x)} \int_{[x, \infty)} s \mu(ds),
\]

we have \( \text{AVaR}(\overline{\mu}(x)) = \psi_{\mu}(x), \ x < r, \) and in consequence \( \text{AVaR}(\lambda) = \psi_{\mu}(\overline{q}(\lambda)) \overline{q}(\lambda) -\text{a.e.} \)

b) Let \( w_{\mu} \) be the draw-down function associated with \( \mu \) by (10) with \( V \) the inverse of \( U_{\mu} \). Then \( w_{\mu}(U_{\mu}(x)) = \overline{q}(1/x), \ x \leq r, \) is non-decreasing, right-continuous and \( w_{\mu}(\text{AVaR}(\lambda)) = \overline{q}(\lambda), \ \lambda \in (0, 1). \) Furthermore, \( w_{\mu} \) is the right-continuous inverse of the barycentre function \( \psi_{\mu} \).

**Remark 12.** The average value at risk (AVaR) has been intensively studied by many authors, from the famous paper of Hardy and Littlewood [14]; recently Foellmer and Schied (2004) studied their properties as coherent risk measures [11](Appendix A.3 p406), and (pp179-182). It is also called expected shortfall, or Conditional Value at Risk. We note however that the characterization of the Average Value at risk given in a) above is not so classical. Finally note that here \( \mu \) is the law of losses (i.e. negative of gains) and some authors refer to \( \text{AVaR}_{\mu}(\lambda) \) as \( \text{AVaR}_{\mu}(1-\lambda). \)
Proof. a) The properties of the function $U_\mu$ result immediately from its definition and Lemma 7. In the case $\overline{\mu}(r) = 0$ the formula (18) is just (15).

In the case $\overline{\mu}(r) > 0$ we write (17), with $b = 1/\overline{\mu}(r)$,

$$\int_0^1 h\left(\frac{x}{s} \wedge b\right) ds = \int_0^1 \overline{\mu}(s) \wedge \overline{\mu}(r) x + \int_0^1 \overline{\mu}(s) ds = x \int_0^1 \overline{\mu}(s) ds,$$

which is again (18) and where we used that $\overline{\mu}(s) \equiv r$ for $s \in [0, \overline{\mu}(r)]$.

b) This follows by a change of variables and is rather classical.

c) We have $w_\mu(y) = y - V(y)/v(y)$ and hence $w_\mu(U_\mu(x)) = U_\mu(x) - xu_\mu(x) = q(1/x)$ by (14). Right-continuity of $w_\mu$ follows then from the left-continuity of $q$. More precisely, from (14), $u_\mu = U_\mu'$ is right continuous and hence also $V'(y) = 1/y + \mu(V(y))$ is right-continuous. Note that $\psi_\mu$ has a jump at $x_0$ if and only if $\mu\{x_0\} > 0$. Then put $y = 1/\overline{\mu}(x_0) < z = 1/\overline{\mu}(x_0+)$ and observe from point b) above that $U_\mu(r)$ for $r \in [y, z]$ continuously interpolates $\psi_\mu(x_0)$ and $\psi_\mu(x_0+)$. Hence $w_\mu(y)$ is constant for $y \in [\psi_\mu(x_0), \psi_\mu(x_0+))$ and the levels of constancy of $w_\mu$ correspond precisely to the jumps in $\psi_\mu$. Further, we have $w_\mu(\psi_\mu(x)) = w_\mu(U_\mu(1/\overline{\mu}(x))) = q(\overline{\mu}(x)), x \leq r$. It follows that the jumps of $w_\mu$ correspond to the levels of constancy of $\psi_\mu$ and $w_\mu$ is the right-continuous inverse of $\psi_\mu$.

Now we can easily select an Azéma-Yor process having a terminal value with given distribution.

**Proposition 13.** Let $\mu$ be a centered probability measure on $\mathbb{R}$ and $\overline{q}, \overline{r}, U_\mu$ as defined in Lemma 11. Consider a non-negative max-continuous local martingale $(N_t)$ with $N_0 = 1$ which converges to 0.a.s. when $t \to \infty$ and let $\zeta = \inf\{t \geq 0 : N_t \in (0, b]\}$, $b = 1/\overline{\mu}(r)$.

a) Let $Y_t^\mu = M_{t,\zeta}^{U_\mu}(N)$ be the Azéma-Yor martingale given via (2). Then $Y_\zeta^\mu = Y_\zeta^\mu = \overline{q}(1/\overline{N}_\zeta)$ is distributed according to $\mu$.

b) We have $\zeta = \inf\{t \geq 0 : Y_t \leq w_\mu(\overline{Y}_t)\} = \inf\{t \geq 0 : \psi_\mu(Y_t) \leq \overline{Y}_t\}$ which is the Azéma-Yor stopping time $[2]$.

d) $\overline{Y}_\zeta = U_\mu(\overline{N}_\zeta) = A\text{VaR}_\mu(1/\overline{N}_\zeta)$ is the Hardy and Littlewood maximal r.v. associated with $\mu$ (cf. Gilat and Meilijson [12]), that is a r.v. $X^* = A\text{VaR}_\mu(\xi)$ where $\xi$ is uniformly distributed on $[0, 1]$.

**Proof.** a) We have

$$Y_\zeta = \begin{cases} U_\mu(\overline{N}_\zeta) - u_\mu(\overline{N}_\zeta)\overline{N}_\zeta, & \text{on } \overline{N}_\zeta \leq b, \\ U_\mu(b), & \text{on } \overline{N}_\zeta = b, \end{cases} = \overline{q}(1/\overline{N}_\zeta),$$
using the properties of $U_\mu$ from Lemmas 7 and 11. Note that if $b = \infty$ then we always have $N_\zeta = 0$. The result follows as $1/N_\zeta$ has uniform distribution on $[0,1]$, see Proposition 6.

b) From Lemma 11 we have that $w_\mu$ is the right-continuous inverse of $\psi_\mu$, which is itself left-continuous. It follows that $\{x : \psi_\mu(x) \leq z\} = (-\infty, w_\mu(z)]$ for any $z \leq r$, where $r = \overline{q}(0+)$. Using the fact that $N_{t \wedge \zeta} = M^V_t(Y)$ where $V$ is the inverse of $U_\mu$, we have

$$
\zeta = \inf\{t : N_t \leq 0\} \wedge T_b(N) = \inf\{t : V(Y_t) - v(Y_t)(Y_t - Y_{t^-}) \leq 0\} \wedge T_r(Y)
= \inf\{t : Y_t \leq w_\mu(Y_t)\} \wedge T_r(Y) = \inf\{t : Y_t \leq w_\mu(Y_t)\},
$$
since $w_\mu(r) = r$.

c) It suffices to note that even when $b < \infty$ i.e. when $\mu(\{r\}) > 0$ we still have $AVaR(\xi) \sim AVaR(1/N_\zeta)$. \hfill \Box

4.3 The Skorohod embedding problem revisited

The Skorokhod embedding problem can be phrased as follows: given a probability measure $\mu$ on $\mathbb{R}$ find a stopping time $T$ such that $X_T$ has the law $\mu$, $X_T \sim \mu$. One further requires $T$ to be small in some sense, typically saying that $T$ is minimal. We refer the reader to Oblój [19] for further details and the history of the problem.

In [2] Azéma and Yor introduced the family of martingales described in Definition 1 and used them to give an elegant solution to the Skorokhod embedding problem for $X$ a continuous local martingale (and $\mu$ centered). Namely, they proved that

$$
T_\mu = \inf\{t \geq 0 : \psi_\mu(X_t) \leq X_t\}, \quad (20)
$$

where $\psi_\mu$ in the barycentre function (19), solves the embedding problem. We propose to rediscover their solution using Proposition 13. The key observation is the identification of the Azéma-Yor stopping time there in $b$ as the first time a certain local martingale hits zero or as the first time a certain local martingale violates the DD-constraint.

**Theorem 14** (Azéma and Yor). Let $(X_t)$ be a continuous local martingale, $X_0 = 0$, $\langle X \rangle_\infty = \infty$ a.s. and $\mu$ a centered probability measure on $\mathbb{R}$: $\int |x| \mu(dx) < \infty$, $\int x \mu(dx) = 0$. Then $(X_t \wedge T_\psi)$ is a UI martingale and $X_{T_\psi} \sim \mu$, where $T_\psi$ is defined via (20).

Let $U_\mu$, $b$, $w_\mu$ be as in Proposition 13, $V_\mu$ the inverse function of $U_\mu$ and define $N_t = M^V_{t \wedge T_\psi}(X)$. We have

$$
T_\psi = \inf\{t \geq 0 : X_t \leq w_\mu(X_t)\} = \inf\{t \geq 0 : N_t \leq 0\} \wedge T_b(N), \quad (21)
$$
and $X_{t \wedge T_\psi} = M_{t \wedge T_\psi}^U(N)$.

Proof. Let $\zeta = \inf\{t \geq 0 : N_t \in \{0, b\}\}$ and note that $N_t \wedge \zeta$ is a non-negative continuous local martingale with $N_0 = 1$. It remains to show that $N_\zeta \in \{0, b\}$ a.s. since then the identification in (21) as well as the embedding property $X_{T_\psi} \sim \mu$ follow immediately from Proposition 13. If $r < \infty$ then $\zeta \leq \inf\{t : N_t = b\} = T_r(X) < \infty$ a.s. regardless of whether $b < \infty$ or $b = \infty$. If $r = \infty$ then $b = \infty$ and hence $\overline{N}_\infty = V_\mu(\overline{X}_\infty) = \infty$ a.s. which readily implies (cf. Proposition V.1.8 in Revuz and Yor [22]) that $P(N_\infty = \infty \text{ or } \langle N \rangle_\infty = \infty) = 1$. Since $T_0(N) < \infty$ on $\{\langle N \rangle_\infty = \infty\}$ we conclude that $N_\zeta \in \{0, b\}$.

We do not prove here that $(X_{t \wedge T_\psi})$ is a UI martingale as we have no new method for doing this. We could only repeat the proof of Azéma and Yor [2] or a potential theoretic proof as in Chacon and Walsh [6] (cf. Obłój [19]).

Remark 15. Note that the continuity of $(X_t)$ is important here and max-continuity would not be enough. More specifically, we need the process $N_t$ to cross zero continuously so that $\zeta = \inf\{t : N_t \leq 0\} \wedge T_b(N)$.

5 On some optimal properties relative to the concave order

The optimality of the Azéma-Yor stopping time among the stopping times solving the Skorohod embedding problem has been studied by several authors ([1], [12], Kertz and Rössler [16] and Hobson [15]). Based on a Blackwell and Dubins result [5], and on Proposition 13, we observe that the $\overline{X}_{T_\psi}$ is a Hardy-Littlewood maximum r.v. associated with $\mu$, and so $\overline{X}_{T_\psi}$ stochastically dominates the maximum of every martingale with terminal distribution $\mu$:

$$P(\overline{X}_{T_\psi} \geq x) \geq P(\overline{N}_\infty \geq x),$$

where $N$ is a uniformly integrable martingale such that $N_\infty$ is distributed according to $\mu$. We give a simple derivation of this property, based on Azéma-Yor martingales, following Oblój and Yor [21]. Suppose for simplicity that $\mu$ has a positive density, which is equivalent to $\Psi_\mu$ being continuous and strictly increasing. Azéma-Yor process $M_U(N)$, for $U(x) = (x - \lambda)^+$, is a UI martingale and hence

$$\lambda P(\overline{N}_\infty \geq \lambda) = \mathbb{E}\left[N_\infty 1_{\overline{N}_\infty \geq \lambda}\right],$$

(22)
which is Doob’s maximal equality for continuous-time martingales. Let \( p := P(N_\infty \geq \lambda) \). As \( N_\infty \sim \mu \), then the RHS is smaller than \( \mathbb{E}[N_\infty \mathbf{1}_{(N_\infty \geq \mu^{-1}(p))}] \) which, by definition in (19), is equal to \( p\Psi(\mu^{-1}(p)) \). We obtain therefore:

\[
\lambda P(N_\infty \geq \lambda) = \lambda p \leq \mathbb{E}[N_\infty \mathbf{1}_{(N_\infty \geq \mu^{-1}(p))}] = p\Psi(\mu^{-1}(p)),
\]

thus \( p \leq \Psi^{-1}(\lambda) \) since \( \Psi \) is decreasing. (23)

To end the proof is suffice to note that \( P(X_{T_\mu} \geq \lambda) = \mu(\Psi^{-1}(\lambda)) \), which is obvious from the definition of \( T_\mu \).

5.1 Floor Constraint and concave order

Consider \( g \) an increasing function on \( \mathbb{R}^+ \) whose increasing concave envelope \( U \) is finite and such that \( \lim_{x \to \infty} U(x)/x = 0 \). We close this paper with some more properties of the martingales \( M^U \). Our insight comes from constrained portfolio optimization problems arising in mathematical finance discussed by El Karoui and Meziou [8]. Let \( N_t \) be a non-negative continuous local martingale converging to zero \( N_t \xrightarrow{t \to \infty} 0 \) a.s. with \( \mathbb{E} N_0 < \infty \). In the financial context, the underlying process is modeled by \( Y_t = g(N_t) \). Recall the future supremum process \( \overline{N}_{t,s} = \sup_{t \leq u \leq s} N_u \) and define \( h \) via (14) i.e. \( h(x) = U(x) - xu(x), u = U' \), which is non-decreasing on \( \mathbb{R}^+ \).

**Lemma 16.** We have

\[
U(N_t) = \mathbb{E}
\left[
\left.
h(N_{t,\infty})\right| \mathcal{F}_t
\right],
\]

and the Azéma Yor process \( M^U(N) \) is a UI martingale with \( M^U_t(N) = \mathbb{E}
\left[
\left.
h(N_{0,\infty})\right| \mathcal{F}_t
\right] \geq U(N_t) \).

**Proof.** The Lemma follows instantly from Proposition 9 part a) applied to local martingales \( (N_{t+u} : u \geq 0) \) and \( (N_t : t \geq 0) \). The last statement follows since \( h \) is increasing and \( \overline{N}_{\infty} \geq \overline{N}_{t,\infty} \). Alternatively, it is a simple consequence of concavity of \( U \).

The process \( U(N_t) \) of conditional expectations of future maxima serves as the floor process in optimization problems. We are naturally led to investigate martingales which dominate it. As observed above \( M^U(N) \) is such martingale and it turns out that the martingale \( M^U(N)_t \) is optimal in the concave order in the terminal value.
Theorem 17. Let $P_t$ be a UI martingale with $P_t \geq Z_t = U(N_t)$, $P_0 = Z_0$, and write $M_t = M^U_t(N)$. Then $P_t \geq M_t = Z_t$ and for any increasing concave function $G$, $E G(M_\infty) \geq E G(P_\infty)$.

Proof. From Lemma 16 we know that $M_t = E[h(N_\infty)|\mathcal{F}_t]$ is a UI martingale and we also have $M_t = U(N_t) = Z_t$ (cf. Proposition 2). We assume $G$ is twice continuously differentiable, the general case following via a limiting argument. Since $h$ is concave, $G(y) - G(x) \leq G'(x)(y - x)$ for all $x, y \geq 0$. In consequence

$$E \left[ G(P_\infty) - G(M_\infty) \right] \leq E \left[ G'(M_\infty)(P_\infty - M_\infty) \right] = E \left[ G'(h(N_\infty))(P_\infty - M_\infty) \right]$$

$$\leq E \int_0^\infty G'(h(N_t))d(P_t - M_t) + E \int_0^\infty (P_t - M_t)G''(h(N_t))d(h(N_t)).$$

The first integral is a difference of two UI martingales (note that $N_0 > 0$) and its expectation is zero. For the second integral, recall that $h$ is increasing and the support of $d(h(N_t))$ is contained in the support of $dN_t$ on which $M_t = M_t = Z_t \leq P_t$. As $G$ is concave we see that the integral is a.s. negative which yields the desired inequality. $\square$

References


