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PERFORMANCE OF ROBUST HEDGES FOR DIGITAL DOUBLE BARRIER OPTIONS

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We analyse the performance of robust hedging strategies of digital double barrier options of Cox and Obłój against that of traditional hedging methods such as delta and delta/vega hedging. Digital double barrier options are financial derivative contracts which pay out a fixed amount on the condition that the underlying asset remains within or breaks into a range defined by two distinct barrier levels. We perform the analysis in hypothetical forward markets driven by models with stochastic volatility and jumps, calibrated to the AUD/USD foreign exchange rate market. Our findings are strikingly unanimous and suggest that, in the presence of model uncertainty and/or transaction costs, robust hedging strategies typically outperform in a substantial way model-specific hedging methods.

Keywords: Robust hedging; Evaluation of hedging strategies; Skorokhod Embedding; Digital Double Barrier option

1. Introduction

One way to see the fundamental idea of Black and Scholes [4], which gave momentum to the whole financial industry, is that pricing is done through hedging. The unique fair price for a payoff at some future date is equal to the capital needed to replicate, through trading, its cashflow. In this idea is embedded an assumption of a fixed model for the dynamics of the price process. The price and, more importantly, the hedging strategy depend crucially on this model. Naturally, they also depend on a number of simplifying assumptions about transaction costs, dividends, interest rates, etc. However, if the trader can have a feel for the impact of the latter, it is very hard to estimate the impact of wrong modelling assumptions. This issue was accentuated through the crisis in 2008 and resulted in a growing interest in robust hedging strategies as alternatives to model-specific hedging strategies.

The generic idea behind robust hedging is as follows. One starts with the market input, namely the prices of liquidly traded options. Using these options and trading strategies in the underlying S, one looks for the cheapest superhedge (and/or most expensive subhedge) for an exotic option which works under any dynamics of S, possibly under some generic assumptions (e.g. continuity of paths of S). The bounds are tight if one can further construct a model which matches the market input and in which the superhedge (resp. subhedge) is a perfect hedge.

One of the methods to obtain such robust hedging strategies is based on the Skorokhod embeddings and pathwise inequalities (see Hobson [20] and Obłój [31, 30]). The underlying ideas go back to Hobson [21] and Brown, Hobson and Rogers [5] and they were more recently explored in depth by Cox and Obłój [11,10]. In there, the authors developed robust pricing and hedging for digital double touch and double no-touch barrier options. Digital double barrier options are financial derivative contracts, most commonly traded in the foreign exchange markets, which pay out a fixed amount on the condition that the underlying asset remains within or breaks into a range defined by two distinct barrier levels.

The aim of this paper is to compare the performances of the robust hedging methods of Cox and Obłój [10,11] against that of some model-specific hedging strategies. A preliminary comparison was carried out in [10] where the authors concluded that for Heston dynamics and in the presence of transaction costs, robust hedging may well outperform delta/vega hedging. At least three interesting avenues for further research were also suggested. First, the methods in [10,11] were developed under the assumption of continuous paths and one should explore what happens when the true model has jumps. Second, Cox and Obłój only considered deta/vega hedging with fixed rebalancing frequency and a better benchmark would be set by a strategy with random, delta-dependent, rebalancing times. Finally, methods of [10,11] are tailored to forward markets and extending them to spot markets is a challenging problem.

In this paper we deal with the first two of these three questions. We present a comprehensive analysis of the performance of robust versus classical hedging strategies of digital double touch and double no-touch barrier options under a number of market scenarios, which include stochastic volatility and jumps. We analyse risk-adjusted performance using standard deviation, VaR, CVaR and maximum loss as risk measures as well as expected utility when needed. Market assumptions and parameters are calibrated to AUD/USD foreign exchange rate market on 14 January 2010. Our findings are strikingly homogeneous: robust hedging strategies tend to outperform classical delta or delta/vega hedging strategies under nearly all market scenarios.

The remainder of the paper is structured as follows. In the following section we describe our setup: Section 2.1 discusses market scenarios which drive the dynamics of the underlying, Sections 2.2 and 2.3 present robust hedging strategies and classical hedging strategies respectively, and finally Section 2.4 introduces risk measures used to evaluate the performance of different hedging strategies. Then in Section 3 we describe our findings and Section 4 concludes. Appendix A contains tables and figures which summarise our results and Appendix B has a short discussion of the numerical procedures employed in our simulations.

2. Market models and hedging strategies

The aim of this paper is to compare the performances of the robust hedging methods of Cox and Obłój [10,11] when applied to digital double touch options and digital double no-touch options against that of some model-specific hedging methods. We want to consider a situation when the trader is uncertain about the real dynamics and will either decide to use a robust hedge or some Black-Scholes-based delta or delta/vega hedge. We will simulate their hedging performance in a number of market scenarios and examine the distributions of hedging errors of the considered hedging strategies. This methodology and setup have several important ingredients, which are now described in detail: market scenarios, robust hedging, classical hedging strategies and risk-adjusted performance measures of hedging errors.

2.1. *Market scenarios*

We assume a trader sells or buys for a premium p a one-year digital double barrier option with continuously monitored barriers $\underline{b}$ and $\overline{b}$ written on an underlying process $S := \{S_t : t \geq 0\}$ which has zero cost of carry. We consider both digital double touch option, which pays one on the event $\{\min_{t\leq 1} S_t \leq \underline{b} \;\&\; \max_{t\leq 1} S_t \geq \overline{b}\}$ and zero otherwise, and digital double no-touch option, which pays one on the event $\{\min_{t\leq 1} S_t > \underline{b} \;\&\; \max_{t\leq 1} S_t < \overline{b}\}$ and zero otherwise. The premium p is set to the Black-Scholes price corresponding to the implied volatility of the at-the-money call option at time $t = 0$. We think of this as the price dictated by the market. The values of the premium are reported in Tables 1–5. We stress that changing p is equivalent to a constant additive shift of hedging errors (of all strategies) and does not affect our relative performance comparisons.

We assume that transactions in the asset S carry a 4bps cost and that buying or selling call or put options carries a 100bps cost. Both rates are applied to the volume traded (i.e. the number of units traded multiplied by the unit price) and approximate the proportional transaction costs observed in the real-life market data set used below. We further assume that there are 250 trading days in one year.

We consider three base scenarios for the dynamics of S: a diffusion process driven by the Black-Scholes model ([4]), a diffusion process with stochastic volatility and finite activity jumps driven by the Bates model ([3]) and a pure-jump Lévy process with stochastic volatility and infinite activity jumps driven by the Variance Gamma with CIR stochastic clock model ([7]). All modelling is done under the risk-neutral measure so that S is a martingale (since it has zero cost of carry). We summarise these models hereafter.

Black-Scholes (BS) model [4] assumes that S is the solution to the following stochastic differential equation:

$$\frac{\mathrm{d}S_t}{S_t} = \sigma \mathrm{d}W_t, \tag{2.1}$$

where $\sigma > 0$ is a constant referred to as the volatility of S and (W_t) is a Brownian motion.

Bates model [3] describes the dynamics of S via the following stochastic differential equations:

$$\begin{cases} \dfrac{\mathrm{d}S_t}{S_t} = -\lambda\mu_j \mathrm{d}t + \sqrt{v_t}\mathrm{d}W_t^{(1)} + J_t \mathrm{d}L_t, \\ \mathrm{d}v_t = \kappa(\theta - v_t)\mathrm{d}t + \zeta\sqrt{v_t}\mathrm{d}W_t^{(2)}, \end{cases} \tag{2.2}$$

with

$$\begin{cases} \mathrm{d}\langle W^{(1)}, W^{(2)}\rangle_t = \rho \mathrm{d}t, \ \mathbb{E}(\mathrm{d}L_t) = \lambda \mathrm{d}t, \\ \log(1+J_t) \sim \mathcal{N}(\log(1+\mu_j) - \frac{1}{2}\sigma_J^2, \sigma_J^2), \end{cases} \tag{2.3}$$

where $(W_t^{(1)})$ and $(W_t^{(2)})$ are correlated Brownian motions with correlation coefficient ρ, (L_t) is an independent Poisson process with intensity $\lambda > 0$ and J_t are lognormally, identically and independently distributed over time with mean μ_j.

Variance Gamma with CIR stochastic clock (VGSV) model [7] considers S given by:

$$S_t = S_0 \exp\Big(X\big(Y(t)\big) - Y(t)\psi_X(-i)\Big), \tag{2.4}$$

where $X\big(Y(t)\big)$ is a Variance Gamma process $X(t)$ with parameters C, G and M (cf. Madan and Seneta [26] and Carr, Geman, Madan and Yor [6]) time-changed with an integral of a CIR process (cf. [12]): $Y(t) = \int_0^t y(u)\mathrm{d}u$,

$$\mathrm{d}y(u) = \kappa(\eta - y(u))\mathrm{d}u + \lambda\sqrt{y(u)}\mathrm{d}W(u), \tag{2.5}$$

where (W_u) is a standard Brownian motion independent of all other processes used in the model. $\psi_X(u)$ in (2.4) is the characteristic exponent of X, which is the logarithm of the characteristic function of X_1 (see Carr, Geman, Madan and Yor [6], [7]) and is given by

$$\psi_X(u) = C\log\left(\frac{GM}{GM + (M-G)iu + u^2}\right). \tag{2.6}$$

The market and model parameters we used are based on real-life global spot foreign exchange interbank market data observed on 14 January 2010 for the AUD/USD foreign exchange spot rate. We calibrated the models to bid and ask implied volatility quotes on standard European options on AUD/USD for different maturities and spot delta points[a], without premium adjustment. This way, we obtain realistic parameters for our models and we also have different models which all match market observed prices so each of them could potentially drive the true dynamics and the trader should consider all these models.

As a result of the calibration, we have $S_0 = 0.9308$ for all models whilst the other model parameters are summarised below:

$$\begin{aligned}
&\text{BS: } \sigma = 0.1506.\\
&\text{Bates: } \begin{cases} v_0 = 0.0107,\ \kappa = 0.9952,\ \theta = 0.0451, \zeta = 0.4074,\\ \rho = -0.4031,\ \lambda = 1.2563,\ \mu_j = -0.0112,\ \sigma_j = 0.0695. \end{cases}\\
&\text{VGSV: } \begin{cases} C = 17.2562,\ G = 41.9420,\ M = 50.5294,\\ \kappa = 4.7198,\ \lambda = 4.7295,\ \eta = 1.6226. \end{cases}
\end{aligned} \tag{2.7}$$

The calibrated models are then used to drive our hypothetical market scenarios and generate option prices available to the trader. At any time $t \geq 0$, the trader is given prices of call and put options co-maturing with the barrier option and at strike prices evenly spaced from 0.641 to 1.153 with an increment of 0.0005, thereby obtaining 1,024 call option prices from which we derive a volatility smile[b]. These prices are computed using the same model which drives the evolution of S. As a result of our calibration, under Bates and VGSV models, the initial prices are close to our market data (naturally BS model only produces a flat volatility surface).

In addition to the three models above, we consider two rather unrealistic models which are designed to examine the performance of various hedging strategies when the volatility of the underlying dramatically and suddenly changes, e.g. due to the arrival of some unforeseen news. The new values of volatility are chosen here in an arbitrary manner to obtain a significant change whilst remaining within the interval commonly observed in practise for the volatility of the underlying.

BS HV model is the BS model described above, only σ at time $t > 0$ is set to 0.4518 (which is three times its value at time $t = 0$).

BS LV model is the BS model described above, only σ at time $t > 0$ is set to a value of 0.1.

In both models, the initial European option prices in the market are computed using the BS model with a constant $\sigma = 0.1506$ and are thus mispriced, which reflects the unforeseen nature of the news. Prices at $t > 0$ are computed using the appropriate value of σ.

[a] A total of 176 data points: 11 options (strikes ranging between 5Δ put to 35Δ put strike, ATM and from 35Δ call strike to 5Δ call strike) for each of 16 maturities ranging from 1 week to 1 year.

[b] The lower and upper bounds of the range of strike prices approximate the strike prices of options from our real-life market data set at delta strikes 5Δ put and 5Δ call, respectively.

2.2. *Robust hedging strategies*

Robust hedges of digital double barrier options described in [10,11] are quasi-static. They involve a carefully chosen initial position in co-maturing European vanilla options (at most 7 among the 1024 options available to the trader), cash and forward, and at most two dynamic forward transactions when barriers are breached. The nature of the hedge depends on the position of the barriers relative to the spot and market prices of European options. For example, if $\overline{b}$ is very far from the spot compared with $\underline{b}$ then it is shown that the optimal robust hedge for the digital double touch options is in fact the one developed in [5] for a one-touch option struck at $\overline{b}$. Although this is a possible case, we are interested here in examining the strategies which are genuinely new in [10,11] and tailored to double barrier options. We consider four possible positions: long or short in digital double touch or double no-touch option. For each of these, we set barrier levels so that we are in the *typical* case when the genuine double barrier hedging strategy should be used[c]. More precisely, we have the following four scenarios:

(i) *The trader sells a digital double touch option with barriers* $\underline{b} = 0.85$ *and* $\overline{b} = 1.01$. The optimal robust super-hedge is type III in [10]. The initial portfolio is composed of $\alpha_i > 0$ calls with strike K_i, $i = 1, 2$ and $\alpha_j > 0$ puts with strike K_j, $j = 3, 4$ where $0 < K_4 < \underline{b} < K_3 < K_2 < \overline{b} < K_1$. The dynamic part involves at most two forward transactions. If $\overline{b}$ is reached first the trader sells β_1 forwards and if then the price reaches $\underline{b}$ they buy β_3 forwards. Alternatively, if $\underline{b}$ is reached first the trader buys β_2 forwards and then if the price reaches $\overline{b}$ they sell β_4 forwards. The choice of α_i, β_i ensures superreplication. We refer to [10] for details and explicit formulae. The optimal hedge $\overline{H}^{III}_{DT}$ is found by minimising the price over all admissible strikes K_1, K_2, K_3, K_4 and the optimal values are reported in Table 1.

(ii) *The trader sells a digital double no-touch option with barriers* $\underline{b} = 0.85$ *and* $\overline{b} = 1.01$. The optimal robust super-hedge is type I in [11]. It is simply given by $\overline{H}^{I}_{DNT} = \mathbf{1}_{S_T \in (\underline{b}, \overline{b})}$, i.e. the superhedge is static and consists simply of buying digital options paying 1 when $\underline{b} < S_T < \overline{b}$.

(iii) *The trader buys a digital double touch option with barriers* $\underline{b} = 0.875$ *and* $\overline{b} = 0.985$. The optimal robust sub-hedge is type I, described explicitly in [10] to which we refer for the exact formulae. The initial portfolio has judiciously chosen amounts of: cash, forward, long positions in calls with strikes $K_2 < \underline{b}$, $K_3 \in (\overline{b}, \underline{b})$ and $K_1 > \overline{b}$ and in a digital call struck at $\underline{b}$, and short positions in calls with strikes $\underline{b}$, $\overline{b}$ and in a digital call struck at $\overline{b}$. The dynamic part is analogous to the one in $\overline{H}^{III}_{DT}$ described above, i.e. it involves forward transactions when the barriers are first breached. The value of K_3 is given as a function of K_1 and K_2, and the optimal hedge $\underline{H}^{I}_{DT}$ is hence found by maximising

[c]The particular choice of barrier levels, as long as it guarantees we are in the interesting case, does not seem to affect our results qualitatively, cf. [35].

the price over admissible values of K_1, K_2. The optimal values are reported in Table 3.

(iv) *The trader buys a digital double no-touch option with barriers* $\underline{b} = 0.71$ *and* $\overline{b} = 1.1$. The optimal robust sub-hedge is type II in [11]. The initial portfolio consists of -1 (cash) and is long $1/(\overline{b} - K_2)$ calls with strike K_2 and $1/(K_1 - \underline{b})$ puts with strike K_1, for arbitrary $\underline{b} < K_1 < K_2 < \overline{b}$. The dynamic component involves at most one transaction: the trader sells $1/(\overline{b} - K_2)$ forwards when $\overline{b}$ is breached first or buys $1/(K_1 - \underline{b})$ forwards if $\underline{b}$ is breached first. The optimal hedge $\underline{H}^{II}_{DNT}$ is found by maximising the price over admissible values of K_1, K_2 and the optimal values are reported in Table 4.

Three out of four strategies described above require the trader to monitor barrier crossings continuously. In practice this may be hard or costly. We consider thus the following strategies:

(i) ***Robust hedge exact***: the trader implements the optimal robust hedge, as described above. They monitor continuously the breaching of the barriers[d].

(ii) ***Robust hedge*** N: the trader implements the optimal robust hedge, as described above, but only monitors the breaching of the barriers N times, at even intervals, during the life of the option. This means that they may carry out the relevant forward transactions with some delay.

We consider the cases where $N = 250$ and $N = 1000$, corresponding to monitoring daily and every six hours, respectively, during the life of the option.

Let us examine the distribution of hedging errors resulting from robust hedging under continuous monitoring of barrier crossing. Denote $O(S)_T$ the payoff of the digital double barrier option at maturity. The trader sells or buys the option for p, as discussed in Section 2.1, whereas the (frictionless) correct price[e] at time zero is $p^* = \mathbb{E}[O(S)_T]$. The trader then sets up the initial portfolio $F(S_T)$ prescribed by their super- or sub- hedge for an initial premium $p^{RH} = \mathbb{E}[F(S_T)]$ and pays transaction costs ϵ_0. Typically, $p^{RH} > p^*$ if the trader sold the digital double barrier option and $p^{RH} < p^*$ if they bought it. When barriers are breached the trader enters into forward transactions. At maturity T, the trader thus holds

$$\Pi = \gamma(-O(S)_T + F(S)_T + V_T + p - p^{RH}) - \epsilon_0 - \epsilon_1, \tag{2.8}$$

where $\gamma = 1$ or -1 if the trader respectively sold or bought the digital double barrier option and V_T is the payoff from the self-financing dynamic trading (i.e. forward

[d]The continuous monitoring of the breaching of the barriers is approximated by monitoring the breaching of the barriers at each one of the 200,000 time steps used for the simulation of the underlying price.

[e]In any given scenario the underlying and option prices are generated by an arbitrage-free model which is complete if European options and the underlying can be traded with no transaction costs. Frictionless prices are then uniquely specified as the risk-neutral expectation of the payoffs.

transactions) which carried (aggregated) transaction costs ϵ_1. We have

$$\mathbb{E}[\Pi] = \gamma(p - p^*) - \epsilon_0 - \epsilon_1, \quad \text{and} \quad \Pi \geq \gamma(p - p^{RH}) - \epsilon_0 - \epsilon_1.$$

The mean of Π is thus the same as that of an unhedged position less the transaction costs. However, these transaction costs are small, as we will see, since they come from single transactions as opposed to continuous rebalancing. On the other hand, Π is bounded below, which may be a greatly appealing feature and will be rewarded by risk measures.

If barrier crossing is monitored discretely (*Robust hedge N*) then Π in (2.8) has an additional error term which has zero mean but is not theoretically bounded. In practice this term is very small, as one would expect, but it does affect the risk profile of hedging errors.

2.3. *Benchmark (classical) hedging strategies*

We want to benchmark the robust hedging strategies described above against classical delta or delta/vega hedging. The trader needs to decide on a model-specific hedge facing uncertainty about the market conditions and hence the model. As in [10], we assume they settle for the simplest (and possibly most universal, see e.g. El Karoui, Jeanblanc-Picqué and Shreve [15]) solution, i.e. they construct the delta and delta/vega hedges within the Black-Scholes framework using the same Black-Scholes volatility input which is used to determine the option price p at time $t = 0$. We consider the following strategies:

(i) ***No hedging***: the trader does not set up any replicating portfolio after the sale of the digital double barrier option and the position is left unhedged until expiry.
(ii) ***Delta*** N (Δ N): the trader delta hedges the digital double barrier option by rebalancing the replicating portfolio N times, at even intervals, during the life of the option.
(iii) ***Delta/vega*** N ($\Delta/\mathcal{V}$ N): the trader delta/vega hedges the digital double barrier option and rebalances the position N times, at even intervals, during the life of the option. When rebalancing the trader aims to keep his portfolio consisting of the digital double barrier option and the hedging portfolio delta– and vega– neutral by trading ATM options, the asset and the risk-free bond.

We consider the cases where $N = 250$ and $N = 1000$, corresponding to rebalancing daily and every six hours, respectively, during the life of the option.

We also examine modifications of the *delta N* and *delta/vega N* hedging strategies intended to improve their performance in the presence of transaction costs. One of the most popular methods to improve delta or delta/vega hedging in markets with transaction costs is the utility-based stochastic optimisation approach first proposed by Hodges and Neuberger [22]. They define the price of an option as that which results in the investor achieving the same terminal utility as when not trading the option. Mohamed [28], Clewlow and Hodges [9], and Martellini and Priaulet [27] studied the performance of the utility-based approach and showed that it achieves very good empirical performance. In this project, however, we do not implement this approach because it would have been too computationally intensive. Instead, we looked for methods that provide a closed-form formulae and are, in spirit and in results, reasonably good approximations to the utility-based stochastic optimisation approach.

Whalley and Wilmott [36] performed an asymptotic analysis of the stochastic optimisation problem under small transaction costs and proposed a closed-form formula that can be used to arrive at an approximately optimal trading rule for delta hedging. More precisely, assuming the trader has an exponential utility function with absolute risk aversion γ, [36] showed that it is optimal for the hedger to not rebalance their delta hedge when the current delta of their replicating portfolio is within a given bandwidth centred about the target delta Δ_t^{BS} derived in the Black-Scholes framework. In the case of zero interest rates the formula simplifies considerably and the bandwidth is given by

$$\Delta_t = \Delta_t^{BS} \pm \left(\frac{3}{2\gamma}\right)^{\frac{1}{3}}. \tag{2.9}$$

Zakamouline [37] compared the risk-adjusted performances of different optimised hedging strategies and found that the Whalley and Wilmott method gives very good results considering its ease of use. We obtained similar results as a by-product of our simulations.

We were not able to find any analogous asymptotic analysis of the stochastic optimisation problem under transaction costs applicable to vega hedging. We chose to implement a rule similar to (2.9), which is rather widespread in practice: a fixed vega band rule. The trader does not rebalance their vega hedge as long as

$$\mathcal{V}_t = \mathcal{V}_t^{BS} \pm H, \tag{2.10}$$

where $\mathcal{V}_t^{BS}$ is the Black-Scholes vega at time t and H is a constant. The more risk-averse the hedger is, the lower H should be. The higher transaction costs are, the higher H should be. Apart from those two intuitions, the optimal H must be found either by running the simulation with many different values of H and selecting the value achieving the highest risk-adjusted performance or by applying a rule of thumb. We believe the former would give an unfair advantage to vega hedging and would not be representative of a practical trading rule, as we would be setting in advance the parameters of the true distribution of the underlying and selecting the best value of H for that specific distribution. Instead, we decided to use a rule of thumb, which is closer to the way the trading rule would be applied in practice. The idea is to set the maximum loss due to volatility movements the trader is willing to incur between rebalancing and infer the corresponding value of H. Let us assume that the trader sells a digital double barrier option for a premium p and is not willing to incur a loss exceeding $x\%$ of this premium between any rebalancing. Let us further assume that the trader believes that the volatility of the asset can move by up to $\Delta\sigma$ in one day during the life of the option. We may then compute H as follows: $H = xp/\Delta\sigma$. In our implementation, we chose $x = 50\%$ and $\Delta\sigma = 0.1$ after having experimented with different values and found these chosen values to provide significant improvements in performance. Summarising, in addition to the *delta N* and *delta/vega N* hedging strategies previously described, we have two additional strategies:

(iv) ***Optimised delta*** N ($\Delta+$ N): identical to *delta* N, only the trader implements the aforementioned delta trading rule (2.9). The trader's absolute risk aversion (ARA) level is assumed equal to 1.

(v) ***Optimised delta/vega*** N ($\Delta/\mathcal{V}+$ N): identical to *delta/vega* N, only the trader uses the delta and vega trading rules (2.9) and (2.10) described above.

2.4. *Performance measures*

We run Monte Carlo simulations, each with 200,000 simulated paths and 200,000 time steps and carry out in parallel the twelve hedging strategies described above. In order to compare the performances of the simulated hedging strategies, we consider their expected profit and their risk. We record the distribution of hedging errors for each of the hedging strategies and mean-correct them in the spirit of the control variate approach suggested in Hull and White [24] (see also Davis, Schachermayer and Tompkins [14] or [10]). The procedure is detailed in Appendix B.3 and the adjustments (Adj[f]) are reported. We call these the *realised hedging errors* and compute their sample mean (Mean), standard deviation (SD), skewness (Skew), kurtosis (Kurt), minimum (Min) and maximum (Max).

We chose the sample mean of the hedging errors as an obvious measure of expected profit. Note that the sample mean for *no hedging* is simply the difference between the sell/buy price[g] and the true price. The means for other strategies additionally capture the average level of transaction costs paid by each strategy as well as its exposure to mispriced European options (the latter only happens in the BS HV and BS LV models).

With respect to measuring the risk, there is no obvious unique choice. The distributions of the hedging errors for the considered hedging strategies substantially deviate from the normal distribution and, as a result, it would be misleading to only use the sample standard deviation as a measure of risk. Instead, we chose as risk measures the maximum observed loss, the Value-at-Risk (VaR) and the Conditional Value-at-Risk (CVaR). Both VaR and CVaR measures were computed at the 99% confidence level and with a time horizon equal to the life of the digital double barrier option. These measures are typically used in practice to assess the economic capital

[f] Abbreviations given in parenthesis are used in Tables 1–5.
[g] Recall that it was set to the BS price with the at-the-money call option implied volatility.

of unfunded transactions such as the sale of an exotic option. We examine those risk measures in combination rather than in isolation when subjectively assessing the risk of a particular hedging strategy. To illustrate, whenever a hedging strategy H1 achieves lower VaR, CoVaR and maximum loss measures than another strategy H2, we conclude that H1 is less risky than H2. Furthermore, if that same H1 achieves an expected return equal to or higher than H2, we conclude that H1 achieves a higher risk-adjusted performance than H2.

We recall that VaR(α) of a portfolio at a confidence level α is given by the smallest number l such that the probability that the portfolio loss exceeds l is not greater than $(1-\alpha)$. The Conditional Value-at-Risk (CVaR), also called expected shortfall, is given by

$$\mathrm{CVaR}(\alpha) = \frac{1}{1-\alpha}\int_{\alpha}^{1} \mathrm{VaR}(u)\mathrm{d}u \tag{2.11}$$

and has the advantage of being a convex risk measure, see Artzner, Delbaen, Eber and Heath [2] or Föllmer and Schied [17].

Finally, to complement the above, we compute the trader's expected utility assuming a negative exponential utility function: $U(X) = 1 - e^{-\gamma X}$ for two values of absolute risk aversion $\gamma = 1$ and $\gamma = 2$. These values are reported as EUM and EUH respectively.

Observe that all the statistics and risk measures considered are either invariant or monotone under a constant shift of the distribution and hence the choice of the initial premium p does not affect the relative performance of hedging strategies.

All the computations are done under the risk-neutral measure (RNM), which is uniquely specified through the calibration procedure to the market data. It may be argued that for risk assessment one should use the real-world (physical) measure (RWM) and not the pricing measure. However, we are here concerned with the relative evaluation of different hedging methods and there are good reasons to do it under the RNM. Firstly, a canonical example for an underlying here would be the exchange rate for a pair of stable currencies with very similar domestic interest rates. In such a situation one might expect a drift close to zero, even under the RWM, making the RNM a plausible guess for the RWM. Secondly, to estimate the dynamics under the RWM one would have to turn to time series of historical prices. Such model estimation may be inaccurate, drift estimation poses a significant difficulty (see e.g. Monoyios [29]) and one would have to assess how sensitive the results are with respect to e.g. drift misspecification. Further, we think here of a situation of model uncertainty when the trader has doubts about future dynamics. In such a case, using historical prices to forecast future dynamics does not seem appropriate. The RNM in contrast has the benefit of being well specified and calibrated to prices of future payoffs. Naturally, these are interesting problems that merit independent research and we leave it for future investigations.

3. Results

We discuss now the risk-adjusted performance of hedging strategies under different market scenarios applying the methodology described above. We refer to Tables 1–5 for the detailed results of our simulations.

We begin by looking at how the robust hedging strategies performed relative to traditional hedging methods in the ideal case for the latter, namely when the underlying is driven by the Black-Scholes model. In this case there is no model misspecification and the only error in delta-hedging is due to discrete rebalancing and transaction costs. One might expect the robust hedges not to be competitive in such a scenario. Indeed, for a long position in a digital double touch (DT) option and a short position in a digital double no-touch (DNT) option, classical strategies, especially *delta N* or *optimised delta N* hedges, tend to outperform robust hedges with the exception that robust strategies achieve lower maximum observed losses. In contrast, for a short position in a DT option and a long position in a DNT option, the situation is reversed and the robust strategies achieved a higher risk-adjusted performance than all considered traditional hedging methods. We note, however, that the standard deviation of the hedging errors for robust strategies is between 2 and 3 times greater than that of a typical classical hedging strategy.

If we now look at the results for Bates and VGSV models, we find that the robust hedging strategies lead to an often dramatic reduction in risk relative to traditional hedging methods, whilst achieving similar or higher expected profits[h]. Robust hedges for a short position in a DT option and short or long position for a DNT option outperformed all other considered hedging strategies on a risk-adjusted basis. The results are less conclusive for a long position in a DT option. Here, the $\Delta+1000$ strategy under the Bates model and the $\Delta/\mathcal{V}+1000$ strategy under the VGSV model achieved significantly higher means than robust hedges (the differences are small and equal 3.7% and 0.5% of option's premium respectively). They performed worse on risk measures and achieved similar values for the expected utility.
The standard deviation of hedging errors for robust strategies was either comparable to that of classical hedges (short or long DT option) or higher by around 30% (long DNT option) up to a factor of 2 (short DNT option). Finally, in terms of expected utility, for higher risk aversion (EUH) robust strategies typically decisively outperformed classical strategies, while achieving higher or similar expected utility for lower risk aversion (EUM). We emphasise that both Bates and VGSV models have jumps and these results show that the robust strategies, originally designed for markets with continuous paths, perform remarkably well also in the presence of jumps.

We now turn to market scenarios with dramatic changes of volatility: BS HV and BS LV. Robust hedges use market traded European options to construct the

[h] To compare the means of hedging errors we run paired t-test at a 99% confidence level. We say that a strategy has a significantly higher mean if this is true for both raw and mean-adjusted hedging errors (cf. Appendix B.3).

static parts of the hedge. At time $t = 0$, these options are underpriced in BS HV and overpriced in BS LV. We would thus expect robust hedges to perform well or poorly under BS HV/LV depending on whether they go long or short European options. Naturally, a similar reasoning applies to delta/vega hedges. Our results confirm this intuition. Under BS HV for a short position in a DT option and a long position in a DNT option and under BS LV for a short position in a DNT option, robust hedges decisively outperformed all other hedging strategies. For a long position in a DT option under BS LV, robust hedges outperformed delta hedges but comparison with delta/vega strategies is inconclusive: robust hedges came first under our three risk measures but delta/vega hedges achieved higher means and expected utility. We note finally that in all four situations the hedging errors of robust strategies had a considerably larger standard deviation than other strategies (about 2 to 3 times larger up to 5 times larger for the long position in DNT option under BS HV). This is mainly due to comparatively large maximal hedging errors (gains) attained by robust strategies.

Table 5 contains the simulation results for the remaining four cases: long DT or short DNT under BS HV and short DT or long DNT under BS LV. In these cases, both the robust hedges and the delta/vega hedges had a worse risk-adjusted performance than delta hedges or *no hedging*. This was to be expected: frictionless delta hedging would be guaranteed to super/sub-replicate (cf. El Karoui, Jeanblanc-Picqué and Shreve [15]) while robust or delta/vega hedging methods have to lead to an expected loss. Remarkably, in all four cases, we find that robust hedges achieved a better risk-adjusted performance than delta/vega hedging. As previously, the hedging errors of robust strategies had higher standard deviations than other strategies but this was less pronounced: of the order of 1.5 to 2 times higher, except for the short position in DNT under BS HV where other strategies had almost zero variance.

In all market scenarios and for all positions, the distribution of hedging errors resulting from the robust methods is significantly positively skewed. In other words, robust hedging tends to lead to poor returns more frequently than good returns, but the poor returns are small whilst the good returns are large. In contrast, the distribution of hedging errors for traditional hedging methods is typically negatively skewed, leading to the opposite conclusion regarding the frequency and size of poor and good returns.

Figures 1–4 show the cumulative distribution functions of the hedging errors achieved by some of the simulated hedging strategies. Those figures highlight how the hedging errors are practically bounded on the loss side for the robust hedges whilst all the traditional hedging methods have important left tails. The figures also highlight the extent to which the probability of large returns is much higher for the robust hedges than for the traditional ones: the right tails of the cumulative distribution functions for the traditional methods converge to 1 on the right-hand side of the figures much faster than those of the robust methods.

Finally, we examine in more detail the effect of jumps and discrete monitoring of barrier crossing, both of which violate assumptions in [10,11], on the performance

of robust hedges. We are interested in how well the theoretical property of super- or sub-hedging is preserved and hence we need to examine the left tail of the distribution of hedging errors. For the short position in a DNT option the robust hedge is fully static and hence invariant to these factors. The hedging error is bounded below, as in (2.8), which is clearly visible in Figure 2. Jumps in the underlying alone do not affect the performance of the robust hedge of a short position in a DT option or a long position in a DNT option, see [10,11]. However, the hedging error is no longer bounded below because the discrete monitoring of barrier crossing introduces an additional error term to (2.8). We see this in Figures 1 and 4 – robust hedges exhibit a small left tail. It is, however, roughly similar in all four figures, which confirms that jumps in the underlying do not introduce an additional error. We see also that the tail grows with the decrease of monitoring frequency, which is summarised by worsening risk measures, cf. Tables 1 and 4. Lastly, in Figure 3 we see the cumulative distribution function of hedging errors from a long position in a digital double touch option. In this case the theory indicates that the robust hedging strategy is sensitive to the assumption of continuity of paths. This is confirmed by our simulations. Under BS and BS LV robust hedging errors exhibit a small left tail which grows with decreasing monitoring frequency. Under Bates and VGSV models the left tail becomes more pronounced and the changing frequency of monitoring has little effect, indicating that the presence of jumps is a more important factor. This is confirmed by the risk measures computed in Table 3.
We conclude that the introduction of jumps in the underlying and the discrete monitoring of barrier crossing may affect the performance of robust hedges, along the lines predicted by the theory. We stress, however, that in our simulations this effect is of second order compared with the differences between robust and classical hedges and all our main findings above apply equally well to all considered versions of the robust hedges.

4. Conclusions and further research

The results of our simulations, as described above, are strikingly unanimous. We compared the hedging errors resulting from robust hedging strategies of Cox and Obłój [10,11] and from Black-Scholes delta or delta/vega hedges with fixed or optimally adjusted rebalancing times. In the vast majority of considered market scenarios and trader's positions, robust hedges achieved an expected return similar to or higher than the best classical hedge whilst delivering an often dramatic reduction in risk measured with VaR, CVar and Maximum Loss. The hedging errors of robust strategies typically have higher variance than classical model-specific hedges but this is mainly due to infrequent large gains.

Theoretical properties of robust hedging strategies of digital double barrier options were developed in [10,11] assuming continuity of paths of the underlying and continuous monitoring of barrier crossings. Our market scenarios included models with finite and infinite activity jumps component and we implemented robust strategies with discrete monitoring of barrier crossings. Results of our simulations indicate the robust hedges are truly robust, even with respect to these two simplifying assumptions in [10,11].

We believe that our results present convincing evidence of the benefits of robust hedging methods. Nevertheless, the setup of our paper was restrictive in two ways. Firstly, we only considered an underlying with zero cost of carry. This may be appropriate for futures markets or currency pairs with the same domestic and foreign interest rates but not in all generality. Implementation of robust hedging methods of [10,11] for generic spot markets and a numerical analysis of their performance remains, in our view, an important research avenue. Secondly, all our analysis apply to a single position in a digital double barrier option. It remains an open problem whether, and how, this discussion may be extended to a portfolio of options.

Appendix A. Tables and Figures

Table 1: Simulation results (cf. Section 2.4) for the short position in the considered digital double touch barrier option.

Strategy	Mean	SD	Skew	Kurt	Min	Max	VaR	CVaR	EUM	EUH	Adj
BS model; $p = 0.1644$; Robust Strategy: $\overline{H}^{III}_{DT}$, $K_1 = 1.0425$, $K_2 = 0.9805$, $K_3 = 0.8795$, $K_4 = 0.819$											
No Hedging	0.0000	0.37	-1.8	4.3	-0.84	0.16	0.84	0.84	-0.09	-0.47	-0.0010
Δ 250	-0.0209	0.07	-1.8	39.1	-1.80	1.25	0.24	0.38	-0.02	-0.06	-0.0011
Δ 1000	-0.0397	0.05	-4.6	99.1	-1.95	0.94	0.19	0.30	-0.04	-0.09	-0.0011
$\Delta/\mathcal{V}$ 250	-0.0549	0.07	-2.0	35.6	-1.83	1.21	0.28	0.43	-0.06	-0.13	-0.0011
$\Delta/\mathcal{V}$ 1000	-0.0999	0.07	-2.0	23.1	-2.04	0.78	0.30	0.41	-0.11	-0.24	-0.0011
$\Delta+$ 250	-0.0092	0.10	-1.2	19.0	-1.93	1.14	0.30	0.44	-0.01	-0.04	-0.0011
$\Delta+$ 1000	-0.0101	0.08	-1.3	23.5	-2.10	1.29	0.24	0.35	-0.01	-0.03	-0.0011
$\Delta/\mathcal{V}+$ 250	-0.0248	0.10	-0.9	15.0	-1.85	1.14	0.31	0.46	-0.03	-0.08	-0.0023
$\Delta/\mathcal{V}+$ 1000	-0.0266	0.09	-0.9	15.1	-2.25	1.27	0.26	0.37	-0.03	-0.07	-0.0026
Robust 250	-0.0081	0.20	1.7	6.2	-0.45	2.47	0.23	0.26	-0.03	-0.08	-0.0006
Robust 1000	-0.0081	0.20	1.8	6.5	-0.29	2.45	0.18	0.20	-0.03	-0.08	-0.0006
Robust Exact	-0.0081	0.20	1.8	6.5	-0.15	2.45	0.14	0.14	-0.03	-0.08	-0.0006
BS HV model; $p = 0.1644$; Robust Strategy: $\overline{H}^{III}_{DT}$, $K_1 = 1.0425$, $K_2 = 0.9805$, $K_3 = 0.8795$, $K_4 = 0.819$											
No Hedging	-0.5316	0.46	0.8	1.7	-0.84	0.16	0.84	0.84	-0.86	-2.93	-0.0027
Δ 250	-0.5482	0.45	-1.8	8.7	-9.68	2.25	2.15	2.59	-1.03	-520.56	-0.0026
Δ 1000	-0.5621	0.45	-1.9	8.8	-7.05	0.77	2.19	2.64	-1.04	-10.92	-0.0028
$\Delta/\mathcal{V}$ 250	0.4405	0.51	-1.5	7.1	-8.64	2.94	1.29	1.73	0.22	-65.23	-0.0026
$\Delta/\mathcal{V}$ 1000	0.3355	0.59	-1.6	6.6	-6.40	1.51	1.65	2.13	0.08	-2.52	-0.0027
$\Delta+$ 250	-0.5439	0.46	-1.9	9.0	-9.58	2.21	2.16	2.61	-1.03	-430.30	-0.0025
$\Delta+$ 1000	-0.5468	0.45	-2.0	9.2	-6.96	0.78	2.16	2.62	-1.00	-10.28	-0.0027
$\Delta/\mathcal{V}+$ 250	0.4925	0.46	-1.6	7.8	-8.45	2.99	1.07	1.51	0.29	-43.63	-0.0026
$\Delta/\mathcal{V}+$ 1000	0.4804	0.46	-1.7	8.0	-6.04	1.90	1.11	1.57	0.28	-0.48	-0.0027
Robust 250	0.6414	1.34	3.8	29.4	-1.13	31.90	0.42	0.50	0.23	0.23	-0.0006
Robust 1000	0.6414	1.34	3.9	29.8	-0.56	31.97	0.27	0.31	0.23	0.26	-0.0007
Robust Exact	0.6414	1.34	3.9	30.0	-0.15	31.92	0.14	0.14	0.24	0.26	-0.0006
Bates model; $p = 0.1831$; Robust Strategy: $\overline{H}^{III}_{DT}$, $K_1 = 1.046$, $K_2 = 0.978$, $K_3 = 0.8845$, $K_4 = 0.8105$											
No Hedging	0.0179	0.37	-1.8	4.3	-0.82	0.18	0.82	0.82	-0.07	-0.42	0.0000
Δ 250	0.0029	0.25	-1.9	15.7	-4.90	3.58	0.90	1.23	-0.04	-0.37	0.0005
Δ 1000	-0.0100	0.25	-2.0	15.9	-4.48	3.55	0.90	1.24	-0.05	-0.40	0.0006
$\Delta/\mathcal{V}$ 250	-0.0240	0.23	-1.8	18.4	-4.99	3.52	0.84	1.15	-0.06	-0.38	0.0728
$\Delta/\mathcal{V}$ 1000	-0.0532	0.23	-1.9	17.9	-4.51	3.55	0.89	1.20	-0.09	-0.45	0.0728
$\Delta+$ 250	0.0107	0.25	-2.0	15.2	-4.63	3.47	0.91	1.25	-0.03	-0.32	0.0006
$\Delta+$ 1000	0.0098	0.25	-2.0	16.3	-4.58	3.07	0.89	1.22	-0.03	-0.37	0.0004
$\Delta/\mathcal{V}+$ 250	-0.0056	0.23	-1.7	17.1	-4.97	3.66	0.82	1.13	-0.04	-0.32	0.0763
$\Delta/\mathcal{V}+$ 1000	-0.0071	0.22	-1.7	18.2	-4.47	3.27	0.79	1.10	-0.04	-0.29	0.0763
Robust 250	0.0099	0.26	3.9	39.0	-1.16	7.35	0.26	0.32	-0.02	-0.08	-0.0033
Robust 1000	0.0099	0.26	4.0	40.6	-1.19	7.39	0.20	0.23	-0.02	-0.07	-0.0033
Robust Exact	0.0099	0.26	4.1	41.5	-0.16	7.42	0.16	0.16	-0.02	-0.07	-0.0033
VGSV model; $p = 0.1895$; Robust Strategy: $\overline{H}^{III}_{DT}$, $K_1 = 1.0445$, $K_2 = 0.979$, $K_3 = 0.8845$, $K_4 = 0.811$											
No Hedging	0.0675	0.33	-2.3	6.3	-0.81	0.19	0.81	0.81	-0.00	-0.22	0.0000
Δ 250	0.0580	0.22	-1.4	20.7	-5.42	4.04	0.70	1.01	0.03	-0.31	-0.0108
Δ 1000	0.0565	0.22	-0.8	64.1	-7.83	11.24	0.68	0.99	0.02	-31.47	-0.0107
$\Delta/\mathcal{V}$ 250	0.0347	0.19	-1.4	30.1	-5.41	4.04	0.64	0.95	0.01	-0.32	0.0127
$\Delta/\mathcal{V}$ 1000	0.0300	0.19	-0.7	101.0	-7.81	11.14	0.64	0.95	-0.00	-30.35	0.0127
$\Delta+$ 250	0.0615	0.22	-1.5	19.4	-5.39	4.24	0.70	1.01	0.03	-0.29	-0.0078
$\Delta+$ 1000	0.0612	0.22	-0.8	55.6	-6.33	11.06	0.69	0.99	0.03	-1.62	-0.0078
$\Delta/\mathcal{V}+$ 250	0.0453	0.19	-1.2	26.2	-5.51	4.23	0.60	0.91	0.02	-0.35	0.0256
$\Delta/\mathcal{V}+$ 1000	0.0447	0.19	-0.2	87.2	-6.25	11.04	0.58	0.88	0.02	-1.38	0.0267
Robust 250	0.0599	0.23	2.9	24.8	-1.23	5.53	0.19	0.27	0.04	0.04	0.0196
Robust 1000	0.0599	0.23	3.0	26.2	-0.81	5.53	0.12	0.17	0.04	0.04	0.0198
Robust Exact	0.0599	0.23	3.0	26.6	-0.13	5.53	0.12	0.12	0.04	0.04	0.0198

Table 2: Simulation results (cf. Section 2.4) for the short position in the considered digital double no-touch barrier option.

Strategy	Mean	SD	Skew	Kurt	Min	Max	VaR	CVaR	EUM	EUH	Adj
BS model; $p = 0.0295$; Robust Strategy: $\overline{H}^I_{DNT}$											
No Hedging	0.0000	0.17	-5.5	31.4	-0.97	0.03	0.97	0.97	-0.02	-0.12	0.0005
Δ 250	-0.0085	0.05	-2.8	70.9	-1.17	1.55	0.21	0.32	-0.01	-0.02	0.0005
Δ 1000	-0.0161	0.04	-5.1	63.5	-1.22	0.88	0.19	0.27	-0.02	-0.04	0.0005
$\Delta/\mathcal{V}$ 250	-0.0357	0.06	-3.6	43.8	-1.21	1.41	0.28	0.39	-0.04	-0.08	0.0005
$\Delta/\mathcal{V}$ 1000	-0.0673	0.08	-2.8	14.5	-1.44	0.69	0.38	0.47	-0.07	-0.16	0.0005
$\Delta+$ 250	-0.0045	0.06	-2.2	47.8	-1.58	1.52	0.25	0.36	-0.01	-0.02	0.0007
$\Delta+$ 1000	-0.0051	0.05	-2.7	44.0	-1.07	1.18	0.19	0.28	-0.01	-0.01	0.0008
$\Delta/\mathcal{V}+$ 250	-0.0330	0.07	-3.5	36.5	-1.41	1.43	0.32	0.44	-0.04	-0.08	0.0005
$\Delta/\mathcal{V}+$ 1000	-0.0580	0.08	-2.9	16.4	-1.30	0.99	0.38	0.47	-0.06	-0.14	0.0005
Robust 250	-0.0097	0.49	0.4	1.2	-0.41	0.59	0.41	0.41	-0.13	-0.49	0.0005
Robust 1000	-0.0097	0.49	0.4	1.2	-0.41	0.59	0.41	0.41	-0.13	-0.49	0.0005
Robust Exact	-0.0097	0.49	0.4	1.2	-0.41	0.59	0.41	0.41	-0.13	-0.49	0.0005
BS LV model; $p = 0.0295$; Robust Strategy: $\overline{H}^I_{DNT}$											
No Hedging	-0.2125	0.43	-1.2	2.4	-0.97	0.03	0.97	0.97	-0.37	-1.40	0.0009
Δ 250	-0.2314	0.26	-1.2	3.7	-2.53	0.99	0.96	1.16	-0.31	-0.89	0.0007
Δ 1000	-0.2483	0.28	-1.1	3.2	-2.84	1.14	0.97	1.14	-0.34	-0.98	0.0006
$\Delta/\mathcal{V}$ 250	-0.1851	0.26	-1.1	3.8	-2.51	1.06	0.92	1.12	-0.25	-0.71	0.0009
$\Delta/\mathcal{V}$ 1000	-0.2265	0.29	-1.0	3.1	-2.80	1.17	0.98	1.16	-0.31	-0.92	0.0009
$\Delta+$ 250	-0.2214	0.27	-1.2	3.8	-2.34	0.92	0.98	1.16	-0.30	-0.86	0.0006
$\Delta+$ 1000	-0.2224	0.26	-1.2	3.6	-2.33	1.45	0.94	1.11	-0.30	-0.84	0.0005
$\Delta/\mathcal{V}+$ 250	-0.1773	0.27	-1.2	3.9	-2.50	1.02	0.93	1.12	-0.24	-0.70	0.0009
$\Delta/\mathcal{V}+$ 1000	-0.2034	0.28	-1.0	3.3	-2.44	1.45	0.95	1.12	-0.28	-0.80	0.0009
Robust 250	-0.0434	0.48	0.5	1.3	-0.41	0.59	0.41	0.41	-0.16	-0.55	0.0003
Robust 1000	-0.0434	0.48	0.5	1.3	-0.41	0.59	0.41	0.41	-0.16	-0.55	0.0003
Robust Exact	-0.0434	0.48	0.5	1.3	-0.41	0.59	0.41	0.41	-0.16	-0.55	0.0003
Bates model; $p = 0.0228$; Robust Strategy: $\overline{H}^I_{DNT}$											
No Hedging	-0.0920	0.32	-2.4	6.8	-0.98	0.02	0.98	0.98	-0.17	-0.66	0.0000
Δ 250	-0.1020	0.29	-1.2	20.3	-5.50	8.95	1.10	1.34	-0.17	-1.22	-0.0004
Δ 1000	-0.1102	0.29	-1.4	20.2	-5.46	8.35	1.11	1.36	-0.18	-1.11	-0.0006
$\Delta/\mathcal{V}$ 250	-0.1307	0.26	-0.9	26.1	-5.77	8.40	1.04	1.31	-0.19	-1.65	-0.0574
$\Delta/\mathcal{V}$ 1000	-0.1623	0.27	-1.2	24.6	-5.78	7.78	1.10	1.38	-0.23	-1.55	-0.0579
$\Delta+$ 250	-0.0975	0.29	-1.3	18.2	-5.31	8.98	1.09	1.31	-0.16	-0.86	-0.0003
$\Delta+$ 1000	-0.0983	0.29	-1.3	18.7	-4.76	8.66	1.08	1.30	-0.16	-0.81	-0.0004
$\Delta/\mathcal{V}+$ 250	-0.1270	0.26	-1.0	20.8	-5.62	7.73	1.03	1.28	-0.18	-1.19	-0.0574
$\Delta/\mathcal{V}+$ 1000	-0.1507	0.27	-1.2	22.1	-5.34	8.14	1.07	1.33	-0.21	-1.18	-0.0579
Robust 250	-0.1022	0.47	0.7	1.5	-0.43	0.57	0.43	0.43	-0.22	-0.69	-0.0036
Robust 1000	-0.1022	0.47	0.7	1.5	-0.43	0.57	0.43	0.43	-0.22	-0.69	-0.0036
Robust Exact	-0.1022	0.47	0.7	1.5	-0.43	0.57	0.43	0.43	-0.22	-0.69	-0.0036
VGSV model; $p = 0.0208$; Robust Strategy: $\overline{H}^I_{DNT}$											
No Hedging	-0.1087	0.34	-2.2	5.9	-0.98	0.02	0.98	0.98	-0.20	-0.75	0.0000
Δ 250	-0.1161	0.31	-1.5	11.9	-5.77	4.28	1.21	1.44	-0.19	-1.26	0.0149
Δ 1000	-0.1178	0.31	-1.3	18.9	-5.87	8.71	1.21	1.46	-0.20	-1.56	0.0149
$\Delta/\mathcal{V}$ 250	-0.1373	0.27	-1.6	15.0	-5.79	4.19	1.15	1.39	-0.20	-1.26	-0.0092
$\Delta/\mathcal{V}$ 1000	-0.1418	0.27	-1.4	26.4	-5.90	8.80	1.15	1.42	-0.21	-1.57	-0.0093
$\Delta+$ 250	-0.1140	0.31	-1.5	11.2	-5.73	4.23	1.17	1.38	-0.19	-1.16	0.0118
$\Delta+$ 1000	-0.1145	0.30	-1.5	11.8	-5.52	4.95	1.17	1.39	-0.19	-1.07	0.0120
$\Delta/\mathcal{V}+$ 250	-0.1355	0.27	-1.5	13.6	-5.72	3.97	1.12	1.34	-0.20	-1.13	-0.0091
$\Delta/\mathcal{V}+$ 1000	-0.1387	0.27	-1.6	14.8	-5.47	5.05	1.12	1.35	-0.20	-1.06	-0.0091
Robust 250	-0.1189	0.46	0.8	1.7	-0.43	0.57	0.43	0.43	-0.23	-0.72	0.0003
Robust 1000	-0.1189	0.46	0.8	1.7	-0.43	0.57	0.43	0.43	-0.23	-0.72	0.0003
Robust Exact	-0.1189	0.46	0.8	1.7	-0.43	0.57	0.43	0.43	-0.23	-0.72	0.0003

Table 3: Simulation results (cf. Section 2.4) for the long position in the considered digital double touch barrier option.

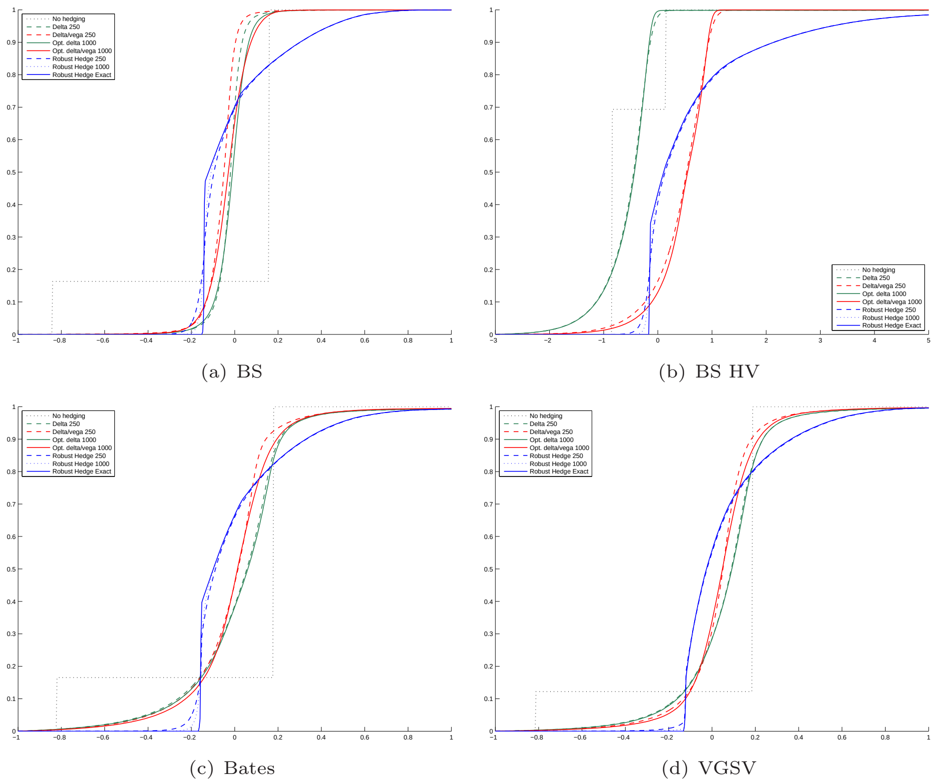
Strategy	Mean	SD	Skew	Kurt	Min	Max	VaR	CVaR	EUM	EUH	Adj
BS model; $p = 0.3896$; Robust Strategy: $\underline{H}^I_{DT}$, $K_1 = 1.1525$, $K_2 = 0.7385$											
No Hedging	-0.0000	0.49	0.5	1.2	-0.39	0.61	0.39	0.39	-0.11	-0.44	0.0031
Δ 250	-0.0230	0.08	-0.6	32.2	-2.54	2.01	0.25	0.38	-0.03	-0.06	0.0017
Δ 1000	-0.0425	0.05	-3.4	79.9	-2.14	1.53	0.20	0.30	-0.04	-0.10	0.0019
$\Delta/\mathcal{V}$ 250	-0.0628	0.08	-0.8	28.9	-2.57	1.95	0.30	0.43	-0.07	-0.15	0.0017
$\Delta/\mathcal{V}$ 1000	-0.1145	0.08	-1.6	19.8	-2.27	1.48	0.33	0.43	-0.12	-0.27	0.0019
$\Delta+$ 250	-0.0110	0.10	-0.1	17.2	-2.49	2.17	0.29	0.41	-0.02	-0.05	0.0016
$\Delta+$ 1000	-0.0120	0.08	-0.5	16.6	-1.49	1.66	0.24	0.33	-0.02	-0.04	0.0019
$\Delta/\mathcal{V}+$ 250	-0.0272	0.10	-0.2	16.7	-2.66	1.99	0.31	0.43	-0.03	-0.08	0.0025
$\Delta/\mathcal{V}+$ 1000	-0.0291	0.08	-0.6	16.6	-1.79	1.55	0.25	0.34	-0.03	-0.07	0.0027
Robust 250	-0.0297	0.32	1.0	2.8	-0.45	1.99	0.35	0.36	-0.08	-0.25	0.0014
Robust 1000	-0.0297	0.32	1.0	2.8	-0.38	1.94	0.32	0.33	-0.08	-0.25	0.0015
Robust Exact	-0.0297	0.32	1.0	2.8	-0.30	1.94	0.30	0.30	-0.08	-0.25	0.0015
BS LV model; $p = 0.3896$; Robust Strategy: $\underline{H}^I_{DT}$, $K_1 = 1.1525$, $K_2 = 0.7385$											
No Hedging	-0.2437	0.35	2.0	5.1	-0.39	0.61	0.39	0.39	-0.34	-0.90	0.0012
Δ 250	-0.2664	0.14	-1.9	11.1	-1.84	0.81	0.77	0.96	-0.32	-0.79	0.0013
Δ 1000	-0.2859	0.14	-1.8	10.1	-2.06	0.82	0.78	0.95	-0.35	-0.87	0.0013
$\Delta/\mathcal{V}$ 250	-0.0964	0.13	-2.3	14.4	-1.63	1.01	0.58	0.78	-0.11	-0.27	0.0005
$\Delta/\mathcal{V}$ 1000	-0.1303	0.14	-2.1	12.3	-1.90	0.98	0.62	0.79	-0.15	-0.36	0.0005
$\Delta+$ 250	-0.2529	0.16	-1.6	8.7	-1.77	0.79	0.79	0.97	-0.30	-0.76	0.0013
$\Delta+$ 1000	-0.2536	0.15	-1.6	8.8	-1.82	0.85	0.76	0.93	-0.30	-0.75	0.0012
$\Delta/\mathcal{V}+$ 250	-0.0720	0.14	-1.8	10.1	-1.58	0.98	0.58	0.76	-0.09	-0.21	0.0019
$\Delta/\mathcal{V}+$ 1000	-0.0728	0.14	-1.8	10.5	-1.77	1.07	0.55	0.72	-0.09	-0.21	0.0020
Robust 250	-0.1299	0.26	1.7	5.3	-0.40	0.79	0.33	0.34	-0.17	-0.43	0.0011
Robust 1000	-0.1299	0.26	1.7	5.3	-0.35	0.74	0.31	0.32	-0.17	-0.43	0.0011
Robust Exact	-0.1299	0.26	1.7	5.3	-0.30	0.70	0.30	0.30	-0.17	-0.43	0.0011
Bates model; $p = 0.4083$; Robust Strategy: $\underline{H}^I_{DT}$, $K_1 = 1.153$, $K_2 = 0.689$											
No Hedging	-0.0820	0.47	0.7	1.5	-0.41	0.59	0.41	0.41	-0.19	-0.62	0.0000
Δ 250	-0.1007	0.29	0.8	14.2	-3.87	8.60	0.79	1.02	-0.15	-0.47	-0.0000
Δ 1000	-0.1156	0.28	0.8	17.3	-3.54	9.33	0.80	1.03	-0.17	-0.49	-0.0001
$\Delta/\mathcal{V}$ 250	-0.1320	0.26	0.4	17.6	-3.96	7.97	0.85	1.09	-0.18	-0.54	-0.0973
$\Delta/\mathcal{V}$ 1000	-0.1682	0.25	0.2	21.1	-3.88	8.52	0.89	1.12	-0.22	-0.64	-0.0975
$\Delta+$ 250	-0.0915	0.29	0.9	13.7	-3.96	8.62	0.78	0.99	-0.14	-0.44	-0.0000
$\Delta+$ 1000	-0.0926	0.28	0.8	10.7	-3.38	5.50	0.77	0.98	-0.14	-0.42	-0.0001
$\Delta/\mathcal{V}+$ 250	-0.1070	0.26	0.6	16.8	-4.02	8.06	0.80	1.02	-0.15	-0.47	-0.0908
$\Delta/\mathcal{V}+$ 1000	-0.1087	0.25	0.4	12.9	-3.97	5.28	0.79	1.00	-0.15	-0.44	-0.0916
Robust 250	-0.1108	0.32	1.2	4.2	-1.31	3.61	0.61	0.70	-0.17	-0.46	-0.0051
Robust 1000	-0.1108	0.31	1.2	4.2	-1.31	3.61	0.61	0.70	-0.17	-0.46	-0.0051
Robust Exact	-0.1108	0.31	1.2	4.2	-1.31	3.63	0.61	0.69	-0.17	-0.46	-0.0051
VGSV model; $p = 0.4145$; Robust Strategy: $\underline{H}^I_{DT}$, $K_1 = 1.149$, $K_2 = 0.693$											
No Hedging	-0.1540	0.44	1.1	2.2	-0.41	0.59	0.41	0.41	-0.26	-0.77	0.0000
Δ 250	-0.1667	0.27	0.2	11.9	-5.88	3.63	0.91	1.17	-0.23	-1.42	0.0091
Δ 1000	-0.1685	0.27	0.5	28.8	-6.00	10.97	0.91	1.17	-0.23	-1.91	0.0093
$\Delta/\mathcal{V}$ 250	-0.1928	0.24	-0.1	17.4	-5.98	3.43	0.89	1.16	-0.25	-1.60	-0.0258
$\Delta/\mathcal{V}$ 1000	-0.1982	0.23	0.3	48.8	-6.11	10.89	0.89	1.16	-0.26	-2.17	-0.0257
$\Delta+$ 250	-0.1625	0.27	0.4	10.5	-4.84	3.87	0.89	1.11	-0.22	-0.78	0.0048
$\Delta+$ 1000	-0.1629	0.27	0.7	27.6	-5.13	11.23	0.89	1.11	-0.22	-0.89	0.0049
$\Delta/\mathcal{V}+$ 250	-0.1783	0.24	0.1	13.9	-4.81	3.61	0.86	1.09	-0.23	-0.78	-0.0296
$\Delta/\mathcal{V}+$ 1000	-0.1790	0.23	0.6	43.4	-4.96	11.08	0.85	1.08	-0.23	-0.86	-0.0307
Robust 250	-0.1828	0.28	1.4	4.8	-1.10	3.17	0.56	0.62	-0.24	-0.63	-0.0158
Robust 1000	-0.1828	0.28	1.4	4.8	-1.11	3.17	0.55	0.61	-0.24	-0.63	-0.0157
Robust Exact	-0.1828	0.28	1.4	4.8	-1.11	3.17	0.55	0.61	-0.24	-0.63	-0.0158

Table 4: Simulation results (cf. Section 2.4) for the long position in the considered digital double no-touch barrier option.

Strategy	Mean	SD	Skew	Kurt	Min	Max	VaR	CVaR	EUM	EUH	Adj
BS model; $p = 0.6722$; Robust Strategy: $\underline{H}^{II}_{DNT}$, $K_1 = 0.7615$, $K_2 = 0.997$											
No Hedging	-0.0000	0.47	-0.7	1.5	-0.67	0.33	0.67	0.67	-0.13	-0.61	0.0002
Δ 250	-0.0277	0.08	-1.6	46.1	-3.02	1.64	0.27	0.43	-0.03	-0.07	-0.0008
Δ 1000	-0.0533	0.05	-3.1	122.3	-2.05	2.60	0.22	0.34	-0.06	-0.12	-0.0007
$\Delta/\mathcal{V}$ 250	-0.0761	0.08	-2.3	45.0	-3.06	1.60	0.34	0.51	-0.08	-0.19	-0.0008
$\Delta/\mathcal{V}$ 1000	-0.1412	0.08	-2.5	30.3	-2.31	2.42	0.43	0.55	-0.16	-0.35	-0.0007
$\Delta+$ 250	-0.0133	0.10	-0.8	19.1	-3.04	1.64	0.31	0.46	-0.02	-0.05	-0.0009
$\Delta+$ 1000	-0.0151	0.08	-1.0	17.0	-1.75	1.54	0.24	0.36	-0.02	-0.05	-0.0008
$\Delta/\mathcal{V}+$ 250	-0.0662	0.10	-1.5	25.5	-3.02	1.55	0.36	0.53	-0.07	-0.17	-0.0009
$\Delta/\mathcal{V}+$ 1000	-0.1120	0.09	-1.7	14.2	-1.85	1.21	0.41	0.54	-0.12	-0.28	-0.0008
Robust 250	-0.0070	0.23	5.0	44.7	-0.58	5.55	0.23	0.27	-0.03	-0.08	-0.0009
Robust 1000	-0.0070	0.23	5.2	46.8	-0.31	5.67	0.16	0.18	-0.03	-0.08	-0.0009
Robust Exact	-0.0070	0.23	5.2	47.5	-0.09	5.62	0.09	0.09	-0.03	-0.08	-0.0008
BS HV model; $p = 0.6722$; Robust Strategy: $\underline{H}^{II}_{DNT}$, $K_1 = 0.7615$, $K_2 = 0.997$											
No Hedging	-0.6660	0.08	12.3	151.5	-0.67	0.33	0.67	0.67	-0.95	-2.82	-0.0003
Δ 250	-0.6857	0.66	-1.9	8.0	-7.66	0.52	3.16	3.66	-1.88	-55.33	-0.0001
Δ 1000	-0.7027	0.68	-2.0	8.1	-7.64	0.25	3.26	3.75	-2.01	-68.52	0.0003
$\Delta/\mathcal{V}$ 250	0.7148	0.61	-1.2	5.3	-5.57	1.99	1.29	1.68	0.37	-0.15	-0.0003
$\Delta/\mathcal{V}$ 1000	0.6123	0.68	-1.5	6.1	-5.45	1.71	1.72	2.16	0.24	-1.50	-0.0002
$\Delta+$ 250	-0.6824	0.66	-1.9	8.1	-7.66	0.54	3.14	3.66	-1.87	-57.69	-0.0002
$\Delta+$ 1000	-0.6871	0.67	-2.0	8.2	-7.64	0.24	3.18	3.69	-1.91	-62.45	0.0001
$\Delta/\mathcal{V}+$ 250	0.7163	0.61	-1.2	5.3	-5.77	2.01	1.26	1.67	0.37	-0.19	-0.0004
$\Delta/\mathcal{V}+$ 1000	0.6219	0.67	-1.4	6.0	-5.13	1.69	1.65	2.11	0.26	-1.22	-0.0003
Robust 250	1.8606	3.97	3.7	26.1	-1.70	87.92	0.62	0.76	0.28	0.20	0.0109
Robust 1000	1.8606	3.97	3.7	26.1	-0.92	88.21	0.35	0.42	0.29	0.27	0.0109
Robust Exact	1.8606	3.97	3.7	26.1	-0.09	88.21	0.08	0.08	0.29	0.28	0.0112
Bates model; $p = 0.6466$; Robust Strategy: $\underline{H}^{II}_{DNT}$, $K_1 = 0.809$, $K_2 = 1.0065$											
No Hedging	0.0057	0.48	-0.6	1.4	-0.65	0.35	0.65	0.65	-0.12	-0.59	0.0000
Δ 250	-0.0169	0.39	-1.1	9.2	-5.53	5.69	1.28	1.67	-0.12	-1.59	-0.0001
Δ 1000	-0.0366	0.39	-1.1	10.5	-6.03	7.53	1.30	1.71	-0.15	-2.88	-0.0003
$\Delta/\mathcal{V}$ 250	-0.0575	0.35	-1.0	11.7	-5.61	5.72	1.19	1.59	-0.14	-1.43	0.1116
$\Delta/\mathcal{V}$ 1000	-0.1052	0.35	-1.1	13.6	-6.17	7.49	1.28	1.67	-0.20	-2.98	0.1118
$\Delta+$ 250	-0.0063	0.39	-1.1	8.8	-5.56	5.68	1.27	1.67	-0.11	-1.34	-0.0001
$\Delta+$ 1000	-0.0082	0.39	-1.0	9.9	-5.89	7.60	1.26	1.66	-0.11	-1.63	-0.0003
$\Delta/\mathcal{V}+$ 250	-0.0502	0.35	-1.0	11.5	-5.60	5.65	1.19	1.58	-0.13	-1.25	0.1118
$\Delta/\mathcal{V}+$ 1000	-0.0837	0.35	-1.0	12.9	-6.02	7.15	1.23	1.63	-0.17	-1.88	0.1119
Robust 250	-0.0019	0.46	9.0	145.1	-2.08	17.03	0.32	0.40	-0.05	-0.16	-0.0057
Robust 1000	-0.0019	0.45	9.1	147.6	-1.06	17.03	0.24	0.27	-0.05	-0.16	-0.0057
Robust Exact	-0.0019	0.45	9.1	147.9	-0.16	16.88	0.16	0.16	-0.05	-0.16	-0.0058
VGSV model; $p = 0.6377$; Robust Strategy: $\underline{H}^{II}_{DNT}$, $K_1 = 0.806$, $K_2 = 1.0075$											
No Hedging	0.0428	0.47	-0.8	1.6	-0.64	0.36	0.64	0.64	-0.08	-0.47	0.0000
Δ 250	0.0280	0.35	-0.6	10.4	-5.75	6.39	1.04	1.40	-0.05	-1.22	-0.0484
Δ 1000	0.0250	0.35	-0.6	11.2	-5.65	5.91	1.04	1.40	-0.05	-1.03	-0.0487
$\Delta/\mathcal{V}$ 250	-0.0049	0.30	-0.5	15.1	-5.89	6.26	0.92	1.27	-0.06	-1.34	-0.0155
$\Delta/\mathcal{V}$ 1000	-0.0120	0.30	-0.5	16.5	-5.80	5.76	0.93	1.28	-0.07	-1.14	-0.0157
$\Delta+$ 250	0.0326	0.35	-0.7	9.4	-5.02	6.10	1.04	1.39	-0.04	-0.52	-0.0432
$\Delta+$ 1000	0.0319	0.34	-0.6	10.3	-4.44	7.65	1.03	1.37	-0.04	-0.44	-0.0433
$\Delta/\mathcal{V}+$ 250	-0.0016	0.29	-0.5	13.6	-4.40	6.14	0.91	1.25	-0.05	-0.38	-0.0093
$\Delta/\mathcal{V}+$ 1000	-0.0068	0.29	-0.5	16.1	-4.56	7.80	0.92	1.25	-0.06	-0.45	-0.0094
Robust 250	0.0354	0.43	8.2	112.1	-1.74	11.39	0.21	0.35	-0.01	-0.08	-0.0048
Robust 1000	0.0354	0.43	8.3	113.9	-1.03	11.39	0.14	0.21	-0.01	-0.07	-0.0047
Robust Exact	0.0354	0.43	8.3	114.6	-0.14	11.39	0.14	0.14	-0.01	-0.07	-0.0047

Table 5: Additional simulation results (cf. Section 2.4) for all the position in the considered digital double touch barrier options under BS HV/LV models.

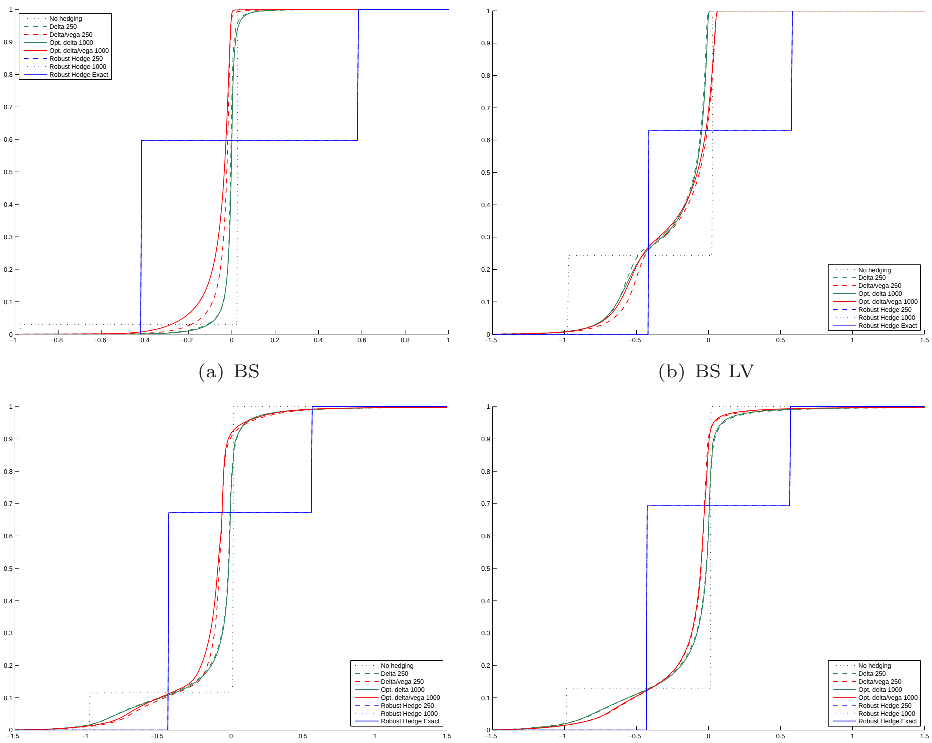
Strategy	Mean	SD	Skew	Kurt	Min	Max	VaR	CVaR	EUM	EUH	Adj
Short DT; BS LV model; $p = 0.1644$; Robust Strategy: $\overline{H}^{III}_{DT}$, $K_1 = 1.0425$, $K_2 = 0.9805$, $K_3 = 0.8795$, $K_4 = 0.819$											
No Hedging	0.1447	0.14	-7.0	49.4	-0.84	0.16	0.84	0.84	0.12	0.19	-0.0002
Δ 250	0.1319	0.08	3.8	28.5	-0.49	1.60	-0.05	-0.03	0.12	0.22	-0.0002
Δ 1000	0.1204	0.07	3.7	27.4	-0.16	1.57	-0.05	-0.04	0.11	0.21	-0.0002
$\Delta/\mathcal{V}$ 250	-0.0714	0.07	4.2	35.6	-0.69	1.40	0.14	0.15	-0.08	-0.16	0.0011
$\Delta/\mathcal{V}$ 1000	-0.0930	0.07	3.8	32.7	-0.41	1.34	0.16	0.17	-0.10	-0.21	0.0010
$\Delta+$ 250	0.1404	0.09	3.2	21.6	-0.48	1.53	-0.02	0.01	0.13	0.23	-0.0002
$\Delta+$ 1000	0.1402	0.09	3.1	21.1	-0.46	1.56	-0.03	-0.01	0.13	0.23	-0.0002
$\Delta/\mathcal{V}+$ 250	-0.0562	0.11	1.5	11.0	-0.87	1.37	0.27	0.29	-0.06	-0.14	-0.0015
$\Delta/\mathcal{V}+$ 1000	-0.0566	0.11	1.2	9.3	-0.69	1.35	0.27	0.29	-0.06	-0.14	-0.0012
Robust 250	-0.0261	0.17	2.0	6.7	-0.33	0.91	0.18	0.20	-0.04	-0.10	0.0000
Robust 1000	-0.0261	0.17	2.0	6.8	-0.26	0.88	0.16	0.17	-0.04	-0.10	0.0000
Robust Exact	-0.0261	0.17	2.0	6.8	-0.14	0.86	0.14	0.14	-0.04	-0.10	0.0000
Short DNT; BS HV model; $p = 0.0295$; Robust Strategy: $\overline{H}^{I}_{DNT}$											
No Hedging	0.0295	0.00	1.0	1.0	0.03	0.03	-0.03	-0.03	0.03	0.06	0.0000
Δ 250	0.0286	0.03	2.3	13.7	-0.13	0.64	0.01	0.02	0.03	0.05	0.0001
Δ 1000	0.0278	0.03	2.7	17.8	-0.11	0.77	0.00	0.01	0.03	0.05	0.0001
$\Delta/\mathcal{V}$ 250	-0.4240	0.03	0.3	3.8	-0.59	-0.06	0.48	0.48	-0.53	-1.34	0.0000
$\Delta/\mathcal{V}$ 1000	-0.4372	0.02	-0.2	2.4	-0.60	-0.23	0.49	0.50	-0.55	-1.40	0.0000
$\Delta+$ 250	0.0289	0.03	2.5	14.8	-0.06	0.68	0.01	0.02	0.03	0.05	0.0000
$\Delta+$ 1000	0.0287	0.03	2.8	18.0	-0.03	0.78	0.00	0.01	0.03	0.05	0.0001
$\Delta/\mathcal{V}+$ 250	-0.4239	0.03	0.4	3.7	-0.56	-0.07	0.48	0.49	-0.53	-1.34	0.0000
$\Delta/\mathcal{V}+$ 1000	-0.4367	0.03	-0.2	2.7	-0.63	-0.24	0.50	0.51	-0.55	-1.40	0.0000
Robust 250	-0.2650	0.35	2.0	4.9	-0.41	0.59	0.41	0.41	-0.37	-0.99	0.0002
Robust 1000	-0.2650	0.35	2.0	4.9	-0.41	0.59	0.41	0.41	-0.37	-0.99	0.0002
Robust Exact	-0.2650	0.35	2.0	4.9	-0.41	0.59	0.41	0.41	-0.37	-0.99	0.0002
Long DT; BS HV model; $p = 0.3896$; Robust Strategy: $\underline{H}^{I}_{DT}$, $K_1 = 1.1525$, $K_2 = 0.7385$											
No Hedging	0.4003	0.41	-1.4	3.0	-0.39	0.61	0.39	0.39	0.26	0.31	0.0021
Δ 250	0.3867	0.41	2.0	9.5	-2.26	5.98	0.17	0.24	0.27	0.42	0.0036
Δ 1000	0.3761	0.39	2.1	9.9	-1.44	5.12	0.06	0.10	0.27	0.42	0.0037
$\Delta/\mathcal{V}$ 250	-0.8327	0.38	1.7	8.2	-3.04	4.51	1.39	1.46	-1.44	-5.52	0.0020
$\Delta/\mathcal{V}$ 1000	-0.9272	0.31	2.1	9.8	-2.30	3.30	1.29	1.33	-1.63	-6.33	0.0021
$\Delta+$ 250	0.3899	0.42	2.0	9.6	-2.48	5.81	0.17	0.23	0.27	0.42	0.0037
$\Delta+$ 1000	0.3877	0.41	2.2	10.0	-1.22	5.72	0.06	0.09	0.28	0.43	0.0037
$\Delta/\mathcal{V}+$ 250	-0.7829	0.41	1.5	7.0	-3.16	4.60	1.37	1.44	-1.35	-5.10	0.0020
$\Delta/\mathcal{V}+$ 1000	-0.7925	0.39	1.6	7.0	-2.28	3.58	1.25	1.28	-1.35	-5.06	0.0021
Robust 250	0.0816	0.72	4.8	45.4	-0.90	21.23	0.51	0.56	-0.06	-0.27	0.0023
Robust 1000	0.0816	0.71	4.9	46.2	-0.59	21.23	0.41	0.43	-0.05	-0.25	0.0024
Robust Exact	0.0816	0.71	4.9	46.5	-0.32	21.23	0.31	0.31	-0.05	-0.25	0.0024
Long DNT; BS LV model; $p = 0.6722$; Robust Strategy: $\underline{H}^{II}_{DNT}$, $K_1 = 0.7615$, $K_2 = 0.997$											
No Hedging	0.2330	0.29	-2.8	8.6	-0.67	0.33	0.67	0.67	0.16	0.17	0.0000
Δ 250	0.2131	0.10	2.8	18.7	-0.72	1.96	-0.04	-0.01	0.19	0.34	0.0001
Δ 1000	0.1946	0.09	2.6	16.4	-1.20	2.00	-0.05	-0.03	0.17	0.31	0.0000
$\Delta/\mathcal{V}$ 250	-0.0718	0.11	3.1	19.3	-1.06	1.69	0.23	0.26	-0.08	-0.18	0.0001
$\Delta/\mathcal{V}$ 1000	-0.1091	0.09	2.7	17.6	-1.57	1.65	0.28	0.31	-0.12	-0.26	0.0000
$\Delta+$ 250	0.2256	0.12	2.3	13.7	-0.66	1.92	-0.02	0.01	0.20	0.35	-0.0001
$\Delta+$ 1000	0.2250	0.11	2.2	12.5	-0.71	1.68	-0.03	-0.01	0.20	0.35	0.0000
$\Delta/\mathcal{V}+$ 250	-0.0633	0.12	2.9	17.3	-0.98	1.41	0.24	0.27	-0.07	-0.16	-0.0001
$\Delta/\mathcal{V}+$ 1000	-0.0872	0.11	2.6	16.3	-1.04	1.60	0.27	0.30	-0.10	-0.21	-0.0000
Robust 250	-0.0237	0.16	3.2	13.2	-0.33	1.93	0.14	0.17	-0.03	-0.09	0.0001
Robust 1000	-0.0237	0.16	3.2	13.4	-0.22	2.05	0.11	0.13	-0.03	-0.09	0.0001
Robust Exact	-0.0237	0.16	3.2	13.4	-0.09	2.07	0.09	0.09	-0.03	-0.09	0.0001



(a) BS (b) BS HV

(c) Bates (d) VGSV

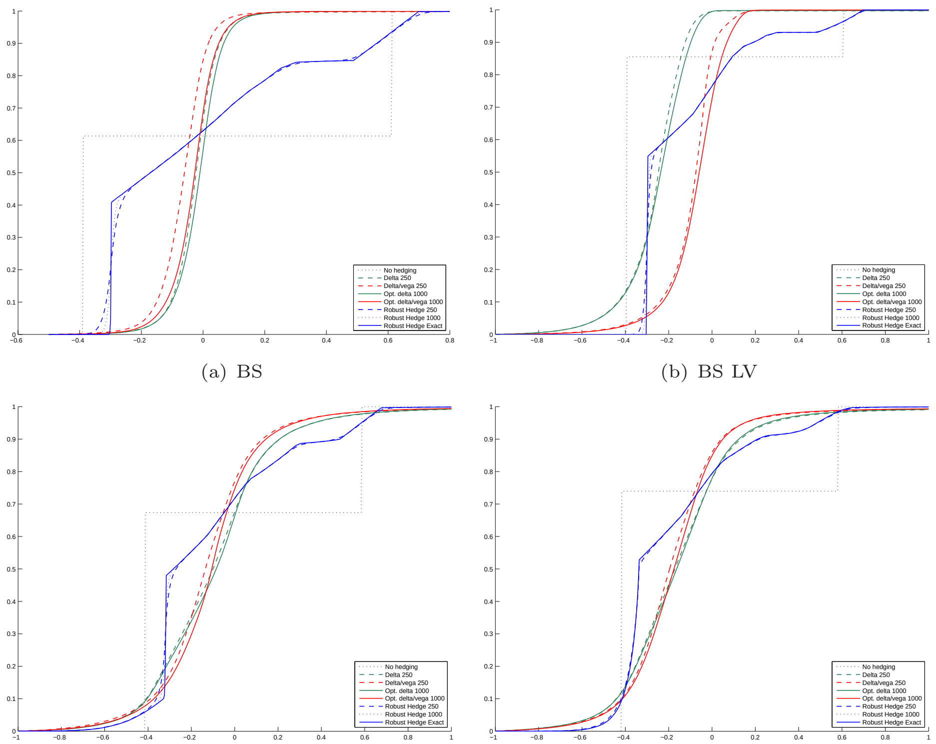
Fig. 1: Cumulative distribution function of hedging errors achieved on the short position in the digital double touch barrier option under each one of the considered models for the underlying process.



(a) BS (b) BS LV

(c) Bates (d) VGSV

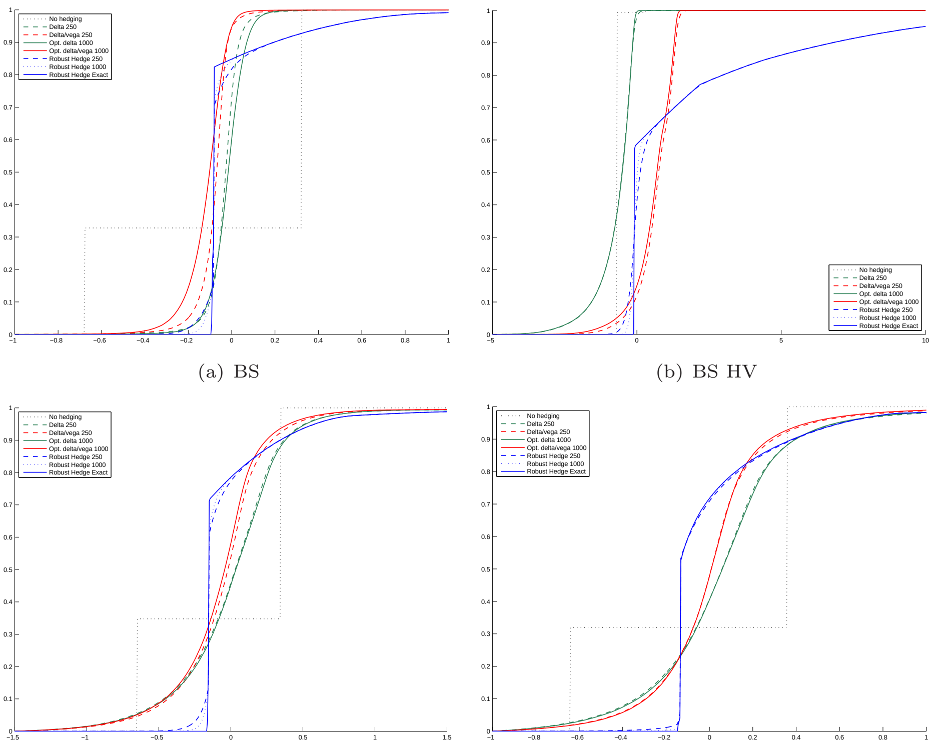
Fig. 2: Cumulative distribution function of hedging errors achieved on the short position in the digital double no-touch barrier option under each one of the considered models for the underlying process.



(a) BS (b) BS LV

(c) Bates (d) VGSV

Fig. 3: Cumulative distribution function of hedging errors achieved on the long position in the digital double touch barrier option under each one of the considered models for the underlying process.



(a) BS (b) BS HV

(c) Bates (d) VGSV

Fig. 4: Cumulative distribution function of hedging errors achieved on the long position in the digital double no-touch barrier option under each one of the considered models for the underlying process.

Appendix B. Summary of numerical methods

B.1. *Calibration*

As mentioned in Section 2.1, the parameters of the considered models were based on real-life global spot foreign exchange interbank market data observed on 14 January 2010 for the AUD/USD foreign exchange spot rate. We calibrated the models to bid and ask implied volatility quotes on standard European options on AUD/USD for different maturities and spot delta points, without premium adjustment. We excluded from the data set the market quotes for options with maturity of one day as their prices are potentially subject to microstructure effects that would only create unwanted noise in our calibration procedure.

The calibration was performed by minimising the weighted sum of squared pricing errors. The pricing error was taken to equal the difference between the model option price and the market ask price if the model option price was higher than the market ask price, to the difference between the model option price and the market bid price if the model option price was lower than the market bid price, and to zero if the model option price was within the market bid and ask prices. For each maturity considered, we used market call option prices for strikes higher than the at-the-money strike and market put option prices for strikes lower than or equal to the at-the-money strike.

To avoid putting excessive weight on near-the-money options relative to out-of-the-money ones and on long-term options relative to short-term ones, we used the weighting scheme suggested by [23], whereby an optimal weighting matrix is estimated using the variance of the normalized option prices. The normalized option price is defined as being equal to the option price divided by the spot price in percentage points. The variance of the normalized option prices at each level of moneyness and maturity is estimated using a nonparametric regression and the weight for the pricing error at that moneyness and maturity is given by the reciprocal of that estimated variance. As both the VGSV and Bates models may be regarded as time-changed Lévy processes (see [8]) and the stochastic time parameters have little influence on short-dated option prices relative to the Lévy parameters, we split the overall calibration procedure into two steps: we first ran the calibration routine on all model parameters against the short-dated maturities in order to arrive at better estimates for the Lévy process parameters; we then ran the calibration routine again against all the maturities in the data set, only this time we set the Lévy process parameters to the values estimated in the first calibration run.

The Black-Scholes model is calibrated to the ATM mid implied volatilities derived from the bid and ask market quotes for options of maturity equal to that of the digital double barrier option (one year).

B.2. *Computing prices and Greeks*

In our simulations we needed to compute the prices, deltas and vegas of the digital double barrier options under the BS model. To this end we implemented a Crank-Nicolson finite difference scheme (see [13]). We used a finite difference grid with 1,000 uniform time steps and restricted the grid in the spatial dimension to the region $S \in [\underline{b}, \overline{b}]$ where we concentrated via coordinate transformation the number of spatial points towards the barrier levels (see [34] for a description on how well chosen coordinate transformations can generally improve the stability, accuracy and solver convergence rate of finite difference calculations) and where the number of spatial steps was chosen so that the spatial step size in the original coordinate be 5E-5 $\times\, S_0$. Restricting the spatial region was motivated by the observation that in the region $S \in (0, \underline{b}]$ a digital double touch option is equivalent to a digital up-and-in single barrier option with barrier at $\overline{b}$ and a digital no-touch option is knocked out, and in the region $S \in [\overline{b}, \infty)$ a digital double touch option is equivalent to a digital down-and-in single barrier option with barrier at $\underline{b}$ and a digital no-touch option is knocked out. Analytical formulae for the pricing of digital single barrier options are known (see [32]) and we derived the formulae for the delta and vega of those options by direct differentiation of the price with respect to the relevant variable. In solving the Black Scholes PDE, we may thus apply the finite difference scheme to the region $S \in [\underline{b}, \overline{b}]$ and use the aforementioned equivalences and analytical formulae to compute the relevant values on the boundaries $S = \underline{b}$ and $S = \overline{b}$ of the finite difference grid as well as the price, delta and vega of the digital double barrier option outside the region $S \in [\underline{b}, \overline{b}]$. In order to concentrate the spatial grid towards the barriers, we used the one-dimensional and time-independent coordinate transformation suggested in [34] summarised as follows. Denote ξ the new spatial coordinate and write $S = S(\xi)$ as the equation defining the original space coordinate in terms of the new one. We used a uniformly spaced grid in the coordinate ξ such that $\xi_{n+1} - \xi_n = \Delta\xi$ is constant. We then have that $\Delta S(\xi) = J(\xi)\Delta\xi$ where $J(\xi) = \mathrm{d}S(\xi)/\mathrm{d}\xi$ is the global Jacobian of the transformation. As suggested in [34], we defined $J(\xi)$ as follows:

$$J(\xi) = A\left[\sum_{k=1}^{n}(\alpha_k^2 + (S(\xi) - B_k)^2)^{-1}\right]^{-1/2},$$

where we set $\alpha_1 = \alpha_2 = (\overline{b} - \underline{b})/50$. We used a standard ODE integrator to integrate $J(\xi)$ numerically to yield the transformation $S(\xi)$ by setting $S(\xi = 0) = \underline{b}$ as an initial condition. The normalization constant A was found by adjusting A iteratively until the second boundary condition $S(\xi = 1) = \overline{b}$ was met (this is guaranteed to happen because $S(\xi = 1)$ is monotically increasing with A). This finite difference scheme generates option values on the entire grid and we computed the deltas at each grid node by direct discretisation at negligible additional computational cost. In order to avoid accuracy and computational speed issues, we did not use the same type of direct discretisation for the vegas and instead we computed them by solving the PDE governing the vega dynamics using the same type of finite difference scheme used to solve the PDE governing the price dynamics.

In order to compute the prices of market call and put options under the Bates and VGSV models, we used the COS pricing method (see [16]), which is based on Fourier-cosine series expansions and is computationally fast and accurate. We applied the COS pricing method with 5,000 grid points and used the truncation ranges suggested in [16] with $L = 30$ (using the notation in [16]).

With respect to the simulation of sample price paths followed by the underlying asset under the Bates model, we used the quadratic exponential (QE) scheme proposed by [1] to simulate a process driven by the Heston stochastic volatility model ([19]) and added a jump term and an additional jump-related martingale correction (the jumps are independent from all other stochastic variables). Our scheme under the Bates model may then be expressed as follows:

$$\begin{aligned}\log S(t+\Delta t) = \log S(t) + K_0^* + K_1 v(t) + K_2 v(t+\Delta t) + \sqrt{K_3 v(t) + K_4 v(t+\Delta t)}\,Z \\ + L\left(\log(1+\mu_j) - \frac{1}{2}\sigma_j^2\right) + \sigma_j\sqrt{L}Z_j - \mu_j\lambda\Delta t, \end{aligned} \qquad \text{(B.1)}$$

where the first line of (B.1) is the expression of a QE scheme for a Heston process and the second line is what is added to that scheme to simulate the jumps. Z and Z_j are standard Gaussian random variables and L is a Poisson random variable with mean parameter $\lambda\Delta t$. Z, L and Z_j are independent of each other and of all other random variables in the model. The term $-\mu_j\lambda\Delta t$ in (B.1) is the martingale correction related to the presence of jumps. The expressions for K_0^*, K_1, K_2, K_3 and K_4 are lengthy and we refer the reader to [1] for their definitions in the case of the QE scheme for martingale processes.

In order to simulate sample paths of the underlying asset price under the VGSV model, we first need to simulate the rate of time change $y(t)$ in (2.5). We did so by sampling from the exact transition law of the CIR process, as described in

[18]. The distribution of $y(t)$ given $y(s)$, for $s < t$, is a noncentral chi-square distribution (see [12]), up to a scale factor, and is given by

$$\mathbb{P}(y(t) \leq x | y(s)) = F_{\chi'^2}\left(\frac{x\eta(s,t)}{e^{-\kappa(t-s)}}; d, y(s)\eta(s,t)\right), \tag{B.2}$$

where

$$d = \frac{4\kappa\theta}{\zeta^2}, \quad \eta(s,t) = \frac{4\kappa e^{-\kappa(t-s)}}{\zeta^2(1-e^{-\kappa(t-s)})}, \tag{B.3}$$

and $F_{\chi'^2}(z,\nu,\lambda)$ denotes the cumulative distribution function for a non-central chi-square distribution with ν degrees of freedom and non-centrality parameter λ, which is given by

$$F_{\chi'^2}(z,\nu,\lambda) = e^{-\lambda/2}\sum_{j=0}^{\infty}\frac{(\lambda/2)^j}{j!2^{\nu/2+j}\Gamma(\nu/2+j)}\int_0^z z^{\nu/2+j-1}e^{-x/2}\mathrm{d}x, \tag{B.4}$$

where $\Gamma(x)$ denotes the gamma function. (B.4) indicates that for any $\nu > 0$ a noncentral chi-square random variable may be represented as an ordinary chi-square random variable with a random number of degrees of freedom. Following a scheme suggested by [33], we use this property to sample from a noncentral chi-square random variable by first generating a Poisson random variable L and then, conditional on L, sampling a chi-square random variable with $\nu + 2L$ degrees of freedom.
Now, to simulate the VGSV process from time $t = 0$ to time $t = T$ for some $T > 0$, we considered the time discretizations $\mathcal{T}_1 = \{t_i = iT/n, i = 0, 1, \ldots, n\}$ for some $n \in \mathbb{N}^*$ and $\mathcal{T}_2 = \{t_j = jT/nm, j = 0, 1, \ldots, nm\}$ for some $m \in \mathbb{N}^*$. In our application, we chose $n = 200,000$ and m=1. $t_i \in \mathcal{T}_1$ is a time at which the VGSV process is computed, whilst $t_j \in \mathcal{T}_2$ is a time at which the operational time is computed. We simulated the rate of time change $y(t)$ at each time $t_j \in \mathcal{T}_2$ following the method described above, starting from $y(0) = 1$. We then computed the operational time $Y(t) = \int_0^t y(s)\mathrm{d}s$ at each time $t_i \in \mathcal{T}_1$ by approximating the integration of the process y from 0 to t_i as follows

$$Y(t_i) = \int_0^{t_i} y(s)\mathrm{d}s \approx \sum_{j=0}^{im} y_{t_j}. \tag{B.5}$$

We then used the fact that the VG process X may be expressed as the difference between two independent gamma processes (see [25]) to write the value of the VGSV process at time $t_i \in \mathcal{T}_1$ is $X(Y(t_i)$ as follows:

$$X(Y(t_i)) = \gamma_p(Y(t_i); \mu_p, \nu_p) - \gamma_n(Y(t_i); \mu_n, \nu_n), \tag{B.6}$$

where $\mu_p = C/M$, $\nu_p = C/M^2$, $\mu_n = C/G$ and $\nu_n = C/G^2$. We thus simulated the VG process at time $Y(t_i)$ by simulating the two independent gamma processes γ_p and γ_n at time $Y(t_i)$ and by computing their difference. The gamma process $\gamma := \{\gamma(t;\mu,\nu) : t \geq 0\}$ with mean rate μ and variance rate ν is a continuous-time process with stationary, independent gamma increments such that for any $h > 0$

$$\gamma(t+h;\mu,\nu) - \gamma(t;\mu,\nu) \sim \mathcal{G}\left(\frac{\mu^2 h}{\nu}, \frac{\nu}{\mu}\right), \tag{B.7}$$

where $\mathcal{G}(\alpha,\beta)$ denotes the gamma distribution with density $f(x) = \frac{x^{\alpha-1}e^{-x/\beta}}{\beta^{\alpha}\Gamma(\alpha)}$. Noting that $\mu_p^2/\nu_p = \mu_n^2/\nu_n = C$, $\nu_p/\mu_p = 1/M$ and $\nu_n/\mu_n = 1/G$, we have

$$\gamma_p(t+\Delta t;\mu_p,\nu_p) - \gamma_p(t;\mu_p,\nu_p) \sim \mathcal{G}\left(C\Delta t, \frac{1}{M}\right), \tag{B.8}$$

$$\gamma_n(t+\Delta t;\mu_n,\nu_n) - \gamma_n(t;\mu_n,\nu_n) \sim \mathcal{G}\left(C\Delta t, \frac{1}{G}\right). \tag{B.9}$$

By setting $t = t_i$ and $\Delta t = Y(t_{i+1}) - Y(t_i)$ in (B.8) and (B.9) one obtains the equations required to compute the increments of the time-changed VG process.

B.3. *Mean adjustment of the hedging errors*

When analysing the performance of hedging strategies Hull and White [24] suggested to mean-adjust them i.e. to add a constant to the whole vector of hedging errors of any given strategy so that the final mean is equal to the theoretical mean. This control variate approach, subsequently used e.g. by Davis, Schachermayer and Tompkins [14], makes it possible to decouple the object of interest (say influence of transaction costs) from the noise of Monte Carlo simulations. Since the MC noise could affect differently each strategy and *a priori* impact fine relative comparison between them, we decided to adopt this approach in this paper and proceeded as follows:

- For the BS, BSHV and BSLV models we computed the theoretical mean of each hedge using the finite difference scheme for the digital double barrier option and explicit formulae for European options and then adjusted the empirical mean of the hedging strategy (without transaction costs) to that value.
- For Bates an VGSV models we only compute the digital double barrier option price using MC. Hence we assumed that the *no hedging* strategy has the appropriate mean and adjusted all other strategies to this value.

For each strategy the size of the mean adjustment is reported in the column (Adj) in Tables 1–5. *A posteriori*, we observe that these are very small numbers, mainly of order of 10^{-3}. Our results and conclusions stay the same with or without the mean adjustment. We link this to the fact that we are able to run 200000 MC paths as opposed to e.g. 1000 in [14].

References

[1] L. Andersen. Efficient simulation of the Heston stochastic volatility model. http://ssrn.com/abstract=946405, 2006.
[2] P. Artzner, F. Delbaen, J-M. Eber, and D. Heath. Coherent measures of risk. *Mathematical Finance*, 9(3):203–228, 1999.
[3] D.S. Bates. Jumps and stochastic volatility: Exchange rate processes implicit in Deutsche Mark options. *The Review of Financial Studies*, 9(1):69–107, 1996.
[4] F. Black and M. Scholes. The pricing of options and corporate liabilities. *Journal of political economy*, 81(3):637, 1973.
[5] H. Brown, D. Hobson, and L. C. G. Rogers. Robust hedging of barrier options. *Mathematical Finance*, 11(3):285–314, 2001.
[6] P. Carr, H. Geman, D.B. Madan, and M. Yor. The Fine Structure of Asset Returns: An Empirical Investigation. *The Journal of Business*, 75(2):305–332, 2002.
[7] P. Carr, H. Geman, D.B. Madan, and M. Yor. Stochastic volatility for Lévy processes. *Mathematical Finance*, 13(3):345–382, 2003.
[8] P. Carr and L. Wu. Time-changed Lévy processes and option pricing. *Journal of Financial Economics*, 71(1):113–142, 2004.
[9] L. Clewlow and S.D. Hodges. Optimal delta-hedging under transactions costs. *Journal of Economic Dynamics and Control*, 21(8-9):1353 – 1376, 1997.
[10] A.M.G. Cox and J. Obłój. Robust hedging of double touch barrier options. *SIAM J. on Financial Mathematics*, 2(1):141–182, 2011.
[11] A.M.G. Cox and J. Obłój. Robust pricing and hedging of double no-touch options. *Finance and Stochastics*, 2011. to appear.
[12] J.C. Cox, J.E. Ingersoll Jr, and S.A. Ross. A theory of the term structure of interest rates. *Econometrica*, 53(2):385–407, 1985.
[13] J. Crank and P. Nicolson. A practical method for numerical evaluation of solutions of partial differential equations of the heat-conduction type. *Advances in Computational Mathematics*, 6(1):207–226, 1996.
[14] M. H. A. Davis, W. Schachermayer, and R. G. Tompkins. Pricing, no-arbitrage bounds and robust hedging of instalment options. *Quantitative Finance*, 1(6):597–610, 2001.
[15] N. El Karoui, M. Jeanblanc-Picqué, and S.E. Shreve. Robustness of the Black and Scholes formula. *Mathematical Finance*, 8(2):93–126, 1998.
[16] F. Fang and C.W. Oosterlee. A novel pricing method for European options based on Fourier-cosine series expansions. *SIAM Journal on Scientific Computing*, 31(2):826–848, 2008.
[17] H. Föllmer and A. Schied. *Stochastic finance*, volume 27 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, extended edition, 2004. An introduction in discrete time.
[18] P. Glasserman. *Monte Carlo methods in financial engineering*. Springer Verlag, 2003.
[19] S.L. Heston. A closed–form solution for options with stochastic volatility with applications to bond and currency options. *The Review of Financial Studies*, 6(2):327–343, 1993.
[20] D. Hobson. The Skorokhod Embedding Problem and Model-Independent Bounds for Option Prices. In R.A. Carmona, E. Çinlar, I. Ekeland, E. Jouini, J.A. Scheinkman, and N. Touzi, editors, *Paris-Princeton Lectures on Mathematical Finance 2010*, volume 2003 of *Lecture Notes in Math.*, pages 267–318. Springer, 2010.
[21] D. G. Hobson. Robust hedging of the lookback option. *Finance and Stochastics*, 2(4):329–347, 1998.
[22] S.D. Hodges and A. Neuberger. Optimal replication of contingent claims under trans-

action costs. *Review of Futures Markets*, 8(2):222–239, 1989.
[23] J. Huang and L. Wu. Specification analysis of option pricing models based on time–changed Lévy processes. *Journal of Finance*, pages 1405–1439, 2004.
[24] J. Hull and A. White. The use of the control variate the use of the control variate technique in option pricing. *Journal of Financial and Quantitative Analysis*, 23:237–251, 1988.
[25] D.B. Madan, P.P. Carr, and E.C. Chang. The variance gamma process and option pricing. *Review of Finance*, 2(1):79, 1998.
[26] D.B. Madan and E. Seneta. The variance gamma (VG) model for share market returns. *The Journal of Business*, 63(4):511–524, 1990.
[27] L. Martellini and P. Priaulet. Competing methods for option hedging in the presence of transaction costs. *The Journal of Derivatives*, 9(3):26–38, 2002.
[28] B. Mohamed. Simulations of transaction costs and optimal rehedging. *Applied Mathematical Finance*, 1(1):49–62, 1994.
[29] Michael Monoyios. Optimal hedging and parameter uncertainty. *IMA J Management Math*, 18(4):331–351, 2007.
[30] J. Obłój. The Skorokhod embedding problem and its offspring. *Probability Surveys*, 1:321–392, 2004.
[31] J. Obłój. Skorokhod Embedding. In R. Cont, editor, *Encyclopedia of Quantitative Finance*, pages 1653–1657. Wiley, 2010.
[32] M. Rubinstein and E. Reiner. Unscrambling the binary code. *Risk*, 4(9):75–83, 1991.
[33] L.O. Scott. Simulating a multi–factor term structure model over relatively long discrete time periods. In *Proceedings of the IAFE First Annual Computational Finance Conference*, 1996.
[34] D. Tavella and C. Randall. *Pricing financial instruments: the finite difference method.* John Wiley & Sons New York, 2000.
[35] F. Ulmer. Performance of robust model-free hedging via Skorokhod embeddings of digital double barrier options. Master's thesis, Mathematical Institute, University of Oxford, 2010.
[36] A.E. Whalley and P. Wilmott. An asymptotic analysis of an optimal hedging model for option pricing with transaction costs. *Mathematical Finance*, 7(3):307–324, 1997.
[37] V. Zakamouline. Optimal Hedging of Option Portfolios with Transaction Costs. http://ssrn.com/abstract=938934, 2006.