American Options Under Stochastic Volatility:
Parameter Estimation and Pricing Efficiency

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Abstract

The stochastic volatility model of Heston (1993) is widely popular for its ability to capture many stylized facts of asset returns and for its resulting closed-form expressions for European option prices. However, its parameter estimation is challenging, and so is its application to the pricing of American options. In this paper, we present evidence that American option prices are insensitive to the accuracy of spot and long–term volatility estimates in the Heston (1993) model, for which drastically different parameter values can be derived. Our results derive from a new accurate pricing technique that we provide and which is based on a well-developed and efficient procedure for the constant volatility model of Black and Scholes. In addition, through an out–of–sample validation based on S&P 100 data, we also show that our method generates prices close to market values. In essence, our approach is predicated upon the classical Chernoff concentration bounds and the robustness of the Black-Scholes formula relative to misspecified stochastic volatility as shown by El Karoui et al. (1998).

Introduction: The Heston Model

Following Heston (1993), an asset price S_t and its instantaneous return variance v_t follow the bi-variate diffusion:

$$\begin{aligned} dS_t &= (r-q)S_t dt + \sqrt{v_t} S_t dZ_{1t} \\ dv_t &= \kappa^*(\theta^* - v_t)dt + \eta\sqrt{v_t}\left(\rho dZ_1(t) + \sqrt{1-\rho^2} dZ_2(t)\right) \end{aligned}$$

where Z_1 and Z_2 are two independent standard Brownian motions defined over an associated filtered probability space, with ρ being the correlation between the innovations affecting the asset price and its volatility. In addition to the risk–free rate of return r and the dividend rate q, the model involves the parameters

$$\kappa^* = \kappa + \lambda, \quad \text{and} \quad \theta^* = \kappa\theta/(\kappa+\lambda),$$

where the model parameters are defined as:

κ : is the instant volatility rate of mean reversion,
θ : the long-term mean volatility,
η : the volatility of volatility,
λ : the market price of volatility risk,

and $2\kappa\theta \geq \eta^2$ in order to ensure that $v_t \geq 0$ for all t (Feller condition.)

- Heston (1993) addresses empirical issues raised about the classical Black-Scholes-Merton model
  - non-normality of return distributions,
  - correlation of periodic returns,
  - volatility clustering
- It results in analytic (nearly closed-form) pricing formula for European-style options.
- However,
  - the model parameters are very sensitive to the estimation method employed
  - the corresponding American option pricing is significantly more challenging.

American Put Option Pricing Under Heston's SV Model: Price Decomposition Formula

Let $P_A(S,v,t)$ be the American put price when the stock price is S, the variance of its return is v, and τ units of time are left to maturity. Then, adapting the PDE approach developed by Chiarella et al. (2010), we obtain the following pricing decomposition formula for an American put:

$$\begin{aligned} P_A(S,v,\tau) &= Ke^{-r\tau}\bar{P}_2(S,v,\tau;K,0) - Se^{-q\tau}\bar{P}_1(S,v,\tau;K,0) \\ &+ \int_0^\tau \int_0^\infty rKe^{-r(\tau-\xi)}\bar{P}_2(S,v,\tau-\xi;w,b(w,\xi))dwd\xi \\ &- \int_0^\tau \int_0^\infty qSe^{-q(\tau-\xi)}\bar{P}_1(S,v,\tau-\xi;w,b(w,\xi))dwd\xi, \end{aligned} \tag{1}$$

where, $b(v,t)$ is the optimal exercise boundary delineating the no–exercise region $\{(S,v,\tau): S > b(v,t)\}$ and, for $j = 1, 2$,

$$\bar{P}_j(S,v,\tau;\alpha,\psi) \equiv \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left(\frac{e^{-i\phi\ln\alpha}}{i\phi} f_j(S,v,\tau;\phi,\psi)\right) d\phi \tag{2}$$

and

$$\begin{aligned} f_j(S,v,\tau;\phi,\psi) &\equiv \exp\{B_j(\phi,\psi,\tau) + D_j(\phi,\psi,\tau)v + i\phi\ln S\} \\ B_j(\phi,\psi,\tau) &\equiv i\phi(r-q)\tau + \frac{\alpha}{\eta^2}\left\{(\Theta_j + \Omega_j)\tau - 2\ln\left(\frac{1-Q_je^{\Omega_j\tau}}{1-Q_j}\right)\right\} \\ D_j(\phi,\psi,\tau) &\equiv i\psi + \frac{(\Theta_j - \eta^2 i\psi + \Omega_j)}{\eta^2}\left(\frac{1-e^{\Omega_j\tau}}{1-Q_je^{\Omega_j\tau}}\right) \end{aligned} \tag{3}$$

with $Q_j \equiv (\Theta_j - \eta^2 i\psi + \Omega_j) \,/\, (\Theta_j - \eta^2 i\psi - \Omega_j)$, $\Theta_1 \equiv \Theta(i-\phi)$, $\Theta_2 \equiv \Theta(-\phi)$, $\Omega_1 \equiv \Omega(i-\phi)$, and $\Omega_2 \equiv \Omega(-\phi)$. We should note here that the first line in the expression (1) above is the price of the corresponding European put and the remaining two lines capture the early exercise premium, which requires the determination of the **early exercise boundary** $b(v,t)$, where v is the instantaneous variance at time t. This boundary is a surface separating in the (S,v,t)–space the optimal exercise region, where the put value is its payoff, and the continuation region, where the American put option value satisfies the same PDE as the European put.

Given the decomposition formula (1), $b(v,t)$ solves the integral equation

$$\begin{aligned} K - b(v,\tau) &= Ke^{-r\tau}\bar{P}_2(S,v,\tau;K,0) - b(v,\tau)e^{-q\tau}\bar{P}_1(S,v,\tau;K,0) \\ &+ \int_0^\tau \int_0^\infty rKe^{-r(\tau-\xi)}\bar{P}_2(b(v,\tau),v,\tau-\xi;w,b(w,\xi))dwd\xi \\ &- \int_0^\tau \int_0^\infty qb(v,\tau)e^{-q(\tau-\xi)}\bar{P}_1(b(v,\tau),v,\tau-\xi;w,b(w,\xi))dwd\xi, \end{aligned} \tag{4}$$

The above equation requires the **(numerical) evaluation of triple integrals**, which is very burdensome. Based on the empirical evidence in Broadie et al. (2000) suggesting an **approximate linear relationship** between $\ln b(v,t)$ and v, namely

$$\ln b(v,t) \approx b_0(t) + vb_1(t), \tag{5}$$

where $b_0(t)$ and $b_1(t)$ are deterministic functions of t, **Adolfsson et al. (2013)** obtain a decomposition formula for the price of an American call option in the context of Heston's stochastic volatility model. As a result, they manage to **reduce the integration dimensionality to two** but their approach still requires solving numerically for the roots of two-dimensional non–linear systems, which are prone to numerical instability.

We go one step further in dimension reduction to solve the integral equation by approximating the optimal exercise surface with one that is volatility invariant; i.e., by setting $b_1 \equiv 0$. This is a clearly a faster method than that of Adolfsson et al. (2013). We thus have for the American put:

$$\begin{aligned} K - e^{b_0(\tau)} &\approx Ke^{-r\tau}\bar{P}_2(S,v,\tau;K,0) - Se^{-q\tau}\bar{P}_1(S,v,\tau;K,0) \\ &+ \int_0^\tau rKe^{-r(\tau-\xi)}\bar{P}_2(S,v,\tau-\xi;e^{b_0(\xi)},0)d\xi \\ &- \int_0^\tau qSe^{-q(\tau-\xi)}\bar{P}_1(S,v,\tau-\xi;e^{b_0(\xi)},0)d\xi, \end{aligned} \tag{6}$$

The latter is now within the realm of the classical Black–Scholes model, to which we apply the spline method of AitSahlia and Lai (2001).

Results

American put price comparison: strike $K = 10$, maturity $T = .25$, spot price S_0 and two spot volatility values v_0. Parameters for Heston's stochastic volatility model: $\kappa = 5.00, \theta = 0.16, \eta = 0.9, \rho = 0.1, \lambda = 0, r = 0.1, q = 0.0$. CV-Decomp refers to our constant volatility – decomposition formula approach with $\sigma^2 = \theta$ using the 3–point method of AitSahlia and Lai (2001). Entries for this approach are the average of the option prices obtained with 3–, 5–, 10–, 25– and 50-piece approximate exercise boundaries (with their standard deviations in parentheses). Ikonen and Toivanen (2007) use implicit finite-difference schemes with various mesh sizes, the results of which are averaged (and SD provided in parentheses). Chockalingam and Muthuraman (2011) use a sequence of fixed–boundary European option prices that converge to the American option price (with their average and SD listed below.)

Method	v_0	S_0				
		8	9	10	11	12
CV-Decomp (Proposed Method)	0.0625	1.9486	1.0832	0.5189	0.2212	0.0891
		(0.0023)	(0.0021)	(0.0043)	(0.0016)	(0.0002)
	0.25	2.0867	1.3345	0.8001	0.4568	0.2528
		(0.0042)	(0.0011)	(0.0020)	(0.0013)	(0.0004)
Chockalingam and Muthuraman (2011)	0.0625	2.0000	1.1030	0.5120	0.2101	0.0823
		(0.0000)	(0.0054)	(0.0105)	(0.0041)	(0.0010)
	0.25	2.0752	1.3270	0.7880	0.4420	0.2393
		(0.0041)	(0.0088)	(0.0108)	(0.0085)	(0.0046)
Ikonen and Toivanen (2007)	0.0625	2.0001	1.1046	0.5129	0.2099	0.0820
		(0.0002)	(0.0031)	(0.0094)	(0.0045)	(0.0006)
	0.25	2.0747	1.3257	0.7858	0.4401	0.2385
		(0.0048)	(0.0110)	(0.0143)	(0.0114)	(0.0059)

Remarks:

- Approximating the constant-volatility free boundary for the CV-Decomp with other estimates for the constant volatility, such as the the spot volatility and $\theta + \eta\sqrt{\theta}$ yielded similar accuracy. ($\theta - \eta\sqrt{\theta}$ was not used as it is negative.)
- In addition to having comparable accuracy to the PDE–based alternatives above, the CV-Decomp is much faster.
- Using S&P 100 index options data, for which σ in the constant boundary approximation was set at spot volatility, long–term average, the CV-Decomp generated prices that had an average out–of-sample relative error of 2%, where the constant boundary approximation based on the long–term average dominated that which was based on spot volatility.

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