The Left-Curtain martingale coupling and the American Put in the presence of atoms

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We consider the problem of pricing an American put, given the prices of European puts, in a two-period problem.

Let μ_1, μ_2 be two integrable probability measures on $\mathbb{R}$.

Definition

Let $\mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = \{\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2\})$ be a filtered probability space. We say $X = (\bar{\mu}, X_1, X_2)$ is a $(\mathcal{S}, \mu_1, \mu_2)$ consistent stochastic process and we write $X \in \mathcal{M}(\mathcal{S}, \mu_1, \mu_2)$ if

1. *X is a $\mathcal{S}$-martingale*
2. *$\mathcal{L}(X_1) = \mu_1$ and $\mathcal{L}(X_2) = \mu_2$*

We say $(\mathcal{S}, X)$ is a (μ_1, μ_2)-consistent model if $\mathcal{S}$ is a filtered probability space and X is a $(\mathcal{S}, \mu_1, \mu_2)$ consistent stochastic process.

- American put pays:

$$\begin{aligned} &(K_1 - x_1)^+ \quad \text{if exercised at time-1} \\ &(K_2 - x_2)^+ \quad \text{if exercised at time-2} \end{aligned}$$

- Write $\psi \in \mathcal{H}$, if ψ is convex and $(K_2 - z)^+ \leq \psi(z)$
- For $\psi \in \mathcal{H}$, define $\phi, \theta : \mathbb{R} \to \mathbb{R}$ by

$$\phi(z) = \Big((K_1 - z)^+ - \psi(z)\Big)^+$$

$$\theta(z) = -\psi'(z)$$

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&\le \Big((K_1 - x_1)^+ - \psi(x_1) \Big)^+ + \psi(x_1) \\
&= \phi(x_1) + \psi(x_1) \qquad \Big(\psi(x_2) - \psi(x_1) - \psi'(x_1)(x_2 - x_1) \ge 0 \Big) \\
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- Let $(\mathcal{S}, X)$ be a (μ_1, μ_2)-consistent model
- For any $\mathcal{S}$-stopping time τ taking values in $\{1, 2\}$ we have

$$\mathbb{E}^{\mathcal{S}}[(K_\tau - X_\tau)^+] \leq \mathbb{E}^{\mu_1}[\phi(X_1)] + \mathbb{E}^{\mu_2}[\phi(X_2)]$$

- Therefore

$$\sup_{\tau \in \mathcal{T}_{1,2}} \mathbb{E}^{\mathcal{S}}[(K_\tau - X_\tau)^+] \leq \mathbb{E}^{\mu_1}[\phi(X_1)] + \mathbb{E}^{\mu_2}[\phi(X_2)]$$

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- Let (X_1, X_2) be the co-ordinate process so that

$$X_1(\omega) = \omega_1 \quad \text{and} \quad X_2(\omega) = \omega_2$$

where $\Omega^* = \mathbb{R} \times \mathbb{R} = \{\omega = (\omega_1, \omega_2)\}$ is equipped with the Borel-sigma algebra $\mathcal{F}^* = \mathcal{B}(\Omega^*)$.

- Let $\mathcal{F}_0^* = \{\emptyset, \Omega\}, \mathcal{F}_1^* = \sigma(X_1), \mathcal{F}_2^* = \sigma(X_1, X_2)$.
- Let $\Pi(\mu_1, \mu_2)$ be the set of martingale couplings of μ_1 and μ_2: the set of probability measures π on $\mathbb{R}^2$, with a first marginal μ_1 and a second marginal μ_2 (so that $X_1 \sim \mu_1$ and $X_2 \sim \mu_2$) and $\mathbb{E}^\pi[X_2|X_1] = X_1$, μ_1-a.s.
- For $\pi \in \Pi(\mu_1, \mu_2)$ let $\mathbb{P}_\pi(X_1 \in dx_1, X_2 \in dx_2) = \pi(dx_1, dx_2)$
- Set $\mathcal{S}_\pi^* = (\Omega^*, \mathcal{F}^*, \mathbb{F}^* := (\mathcal{F}_0^*, \mathcal{F}_1^*, \mathcal{F}_2^*), \mathbb{P}_\pi)$.
- Then $(\mathcal{S}_\pi^*, X = (\bar{\mu}, X_1, X_2))$ is a (μ_1, μ_2)-consistent model.

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- Then the Modified Primal problem is to find

$$\mathcal{P}^* := \sup_{\pi \in \Pi(\mu_1,\mu_2)} \sup_{\tau \in \mathcal{T}_{1,2}} \mathbb{E}^{\mathcal{S}^*_\pi}[(K_\tau - X_\tau)^+]$$

- We have

$$\mathcal{P}^* \leq \mathcal{P} \leq \mathcal{D}$$

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Martingale Optimal Transport (and Friends), Oxford 2017

Theorem

Suppose μ_1 is continuous.

$$\mathcal{P}^* = \mathcal{P} = \mathcal{D}.$$

The Left-curtain coupling

Theorem (Beiglböck and Juillet ['16], Henry-Labordère and Touzi ['16])

Suppose $\mu_1 \leq_{cx} \mu_2$. Then there exists a unique left-monotone martingale coupling $\pi_{lc} \in \Pi(\mu_1, \mu_2)$. If μ_1 is continuous, then there exists a Borel set $S \subseteq \mathbb{R} \times \mathbb{R}$ and two measurable functions $f, g : S \to \mathbb{R}$:

- *π_{lc} is concentrated on the graphs of f and g*
- *for all $x \in \mathbb{R}$, $f(x) \leq x \leq g(x)$*
- *for all $x < x' \in \mathbb{R}$, $g(x) \leq g(x')$ and $f(x') \notin (f(x), g(x))$*

The Dispersion Assumption

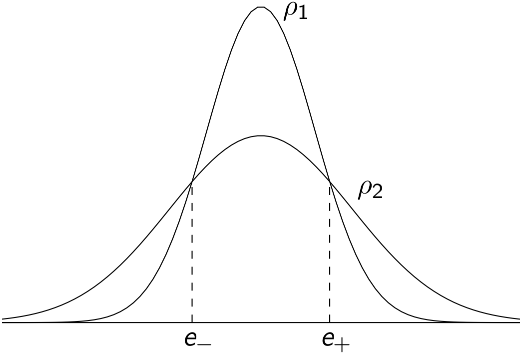

Figure: The Dispersion assumption: μ_1 has density ρ_1 and μ_2 has density ρ_2. We have $\rho_1 > \rho_2$ on (e_-, e_+).

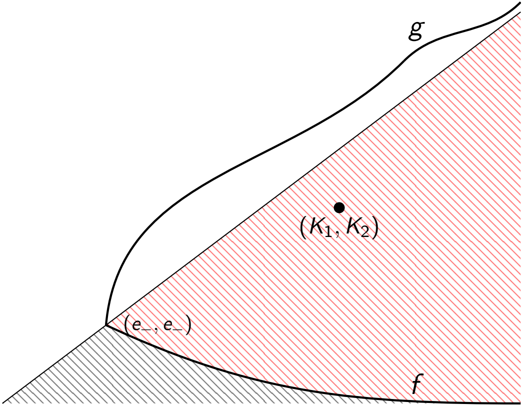

Figure: Sketch of functions f and g under DA, with shading of regions: $\{(k_1, k_2) : k_2 \leq f(k_1) \leq k_1\}$ and $\{(k_1, k_2) : e_- \leq k_1,\ f(k_1) \leq k_2 \leq k_1\}$.

Suppose that $e_- < K_1$ and $f(K_1) < K_2$

For $e_- < x$, let $\psi_x : \mathbb{R} \to \mathbb{R}$ be given by

$$\psi_x(z) = \begin{cases} (K_2 - z)^+ & z \leq f(x) \\ (K_2 - f(x))^+ - \frac{(K_2 - f(x))^+ - (K_1 - x)^+}{x - f(x)}(z - f(x)) & f(x) < z \leq x \\ (K_1 - x)^+ - \frac{(K_1 - x)^+ - (K_2 - g(x))^+}{g(x) - x}(z - x) & x < z \leq g(x) \\ (K_2 - z)^+ & z > g(x) \end{cases}$$

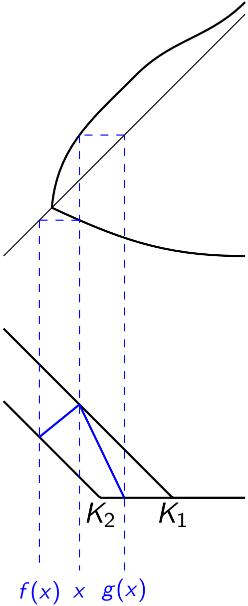

Figure: ψ_x is concave on $(f(x), g(x))$ for $x \leq g^{-1}(K_1)$.

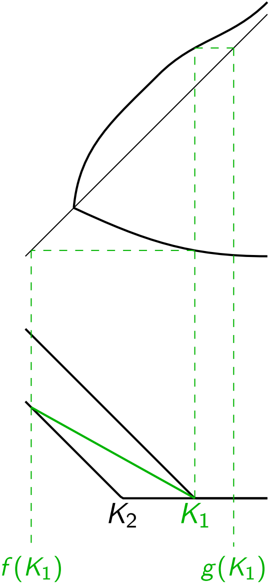

Figure: ψ_{K_1} is convex on $(f(K_1), g(K_1))$.

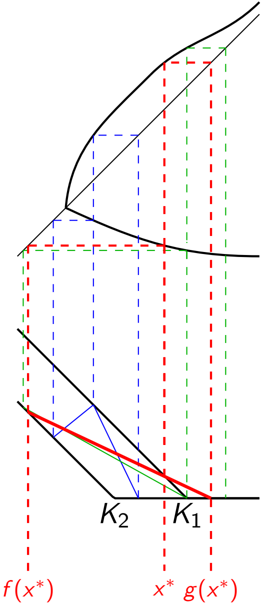

Figure: There exists $x^* \in (g^{-1}(K_1), K_1)$ s.t. ψ_{x^*} is linear on $(f(x^*), g(x^*))$.

- We have a candidate model: $\mathcal{S}^*_{\pi_{lc}}$.
- We have a candidate superhedge: $\psi_{x^*} \in \mathcal{H}$.
- We also have a candidate stopping strategy:

$$\tau^* = I_{\{X_1 \leq x^*\}} + 2I_{\{X_1 > x^*\}}$$

Exercise: show that

$$\mathbb{E}^{\mathcal{S}^*_{\pi_{lc}}}[(K_{\tau^*} - X_{\tau^*})^+]$$
$$= \int \Big((K_1 - x_1)^+ - \psi_{x^*}(x_1)\Big)^+ \mu_1(dx_1) + \int \psi_{x^*}(x_2)\mu_2(dx_2)$$
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What if we allow atoms in μ_1 ?

- $\mathcal{P}$rimal. $\{\mathcal{S}^*_\pi : \pi \in \Pi(\mu_1, \mu_2)\}$ is too small. If at time-1 we end up at the atom of μ_1, sometimes we want to stop and sometimes to continue, i.e. some mass can "stop" at time-1 and the rest to "continue" until time-2. But this information is not included in $\sigma(X_1)$: a corresponding stopping strategy will not be an $\mathcal{F}^*$-stopping time.
- $\mathcal{D}$ual. f and g are not well-defined on the atoms of μ_1.

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How to find optimal x^*?

Example: $\mathcal{P}^* < \mathcal{P}$.

- Let $\bar{\mu} = 1$ and take $\mu_1 = \delta_{\bar{\mu}}$ and $\mu_2 = U[0,2]$.
- Consider $K_1 = \frac{5}{4}$ and $K_2 = \bar{\mu} = 1$.
- Then $\bar{\mu} = X_0 = X_1$ and thus $\{\emptyset, \Omega\} = \mathcal{F}_0^* = \mathcal{F}_1^*$.
- Then

$$\begin{aligned}
\mathcal{P}^* &= \sup_{\pi \in \Pi(\mu_1,\mu_2)} \sup_{\tau \in \mathcal{T}_{1,2}} \mathbb{E}^{\mathcal{S}_\pi^*}[(K_\tau - X_\tau)^+] \\
&= \sup_{\pi \in \Pi(\mu_1,\mu_2)} \max\Big\{\mathbb{E}^{\mu_1}[(K_1 - X_1)^+], \mathbb{E}^{\mu_2}[(K_2 - X_2)^+\Big\} \\
&= \max\{\mathbb{E}^{\mu_1}[(K_1 - X_1)^+], \mathbb{E}^{\mu_2}[(K_2 - X_2)^+] \\
&= (K_1 - \bar{\mu})^+ \vee \int_0^2 \frac{(K_2 - z)^+}{2} dz \\
&= \frac{1}{4} \vee \int_0^1 \frac{1-z}{2} dz = \frac{1}{4}
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\mathcal{P}^* &= \sup_{\pi \in \Pi(\mu_1,\mu_2)} \sup_{\tau \in \mathcal{T}_{1,2}} \mathbb{E}^{\mathcal{S}_\pi^*}[(K_\tau - X_\tau)^+] \\
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&= \frac{1}{4} \vee \int_0^1 \frac{1-z}{2} dz = \frac{1}{4}
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Example: $\mathcal{P}^* < \mathcal{P}$.

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Consider $\Omega = (0,1) \times (0,1)$, $\mathcal{F} = \mathcal{B}(\Omega)$, $\mathbb{P} = \text{Leb}(\Omega)$.
Let U and V be independent $U(0,1)$ random variables.
Set $\mathbb{F} = (\mathcal{F}_0 = \{\emptyset, \Omega\}, \mathcal{F}_1 = \sigma(U), \mathcal{F}_2 = \sigma(U,V)\})$.
Let $G = G_{\mu_1}$ denote the quantile function of μ_1.

Theorem

There exist functions $R, S : (0,1) \mapsto \mathbb{R}$ satisfying

$$R(u) \le G(u) \le S(u); \qquad S \text{ is increasing;} \\ \text{for } 0 < u < v < 1,\ R(v) \notin (R(u), S(u)). \tag{1}$$

such that if we define $X(u,v) = X(u) = G(u)$ and $Y(u,v) \in \{R(u), S(u)\}$ by $Y(u,v) = G(u)$ on $G(u) = S(u)$ and

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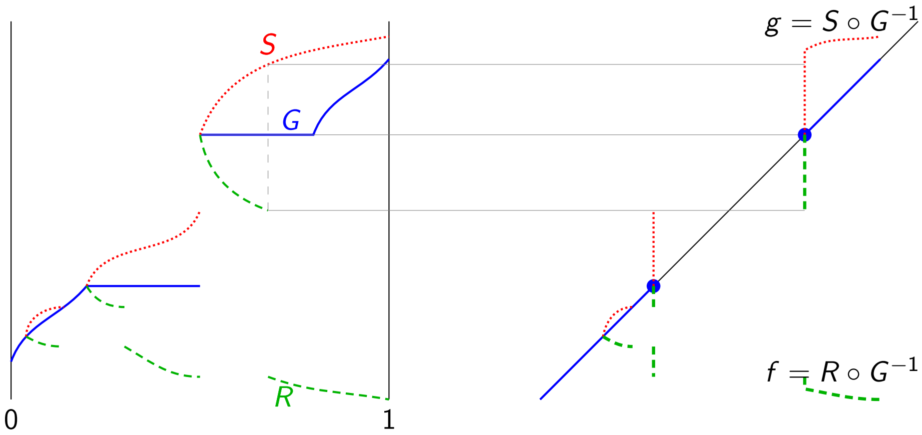
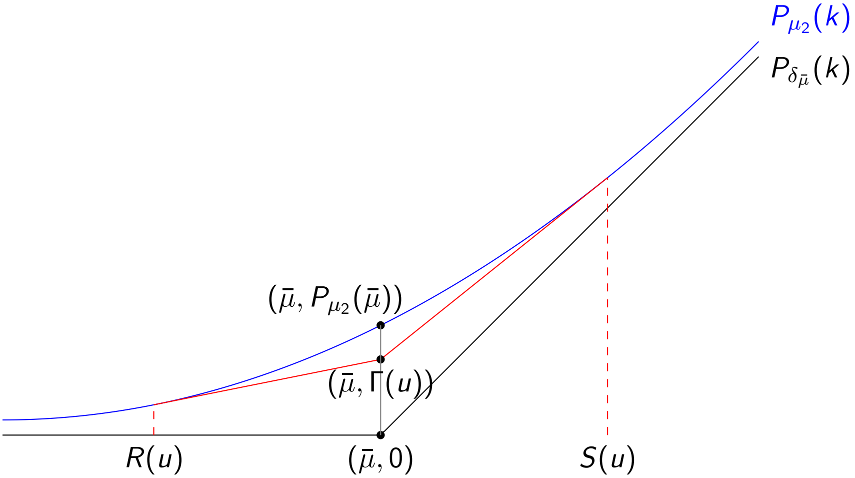


Figure: Sketch of R, G, S and the corresponding f and g. On the atoms of μ, G is flat, and f and g are multi-valued, but R and S remain well-defined.

- $P_{\mu_2}(k) := \int (k-z)^+ \mu_2(dz)$ and $P_{\delta_{\bar{\mu}}}(k) := (k-\bar{\mu})^+$.
- Let $\Gamma : [0,1] \to [0, P_{\mu_2}(\bar{\mu})]$ be continuous, decreasing, $\Gamma(0) = P_{\mu_2}(\bar{\mu})$, $\Gamma(1) = 0$ AND WELL-CHOSEN.



(R, S) are monotonic functions with

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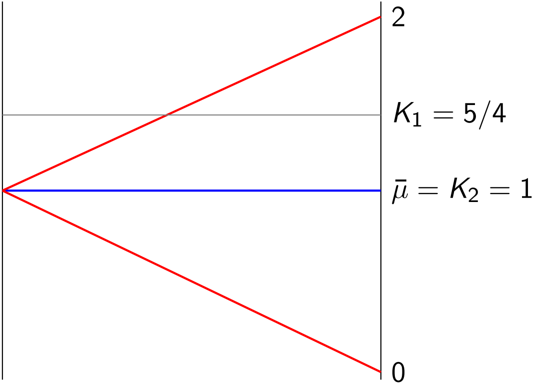
$S(u) = 1 + u$

$G(u) = 1$

$R(u) = 1 - u$



The value $A(u)$ of the American put under the stopping rule

$$\tau_u = I_{\{U \le u\}} + 2I_{\{U > u\}}$$

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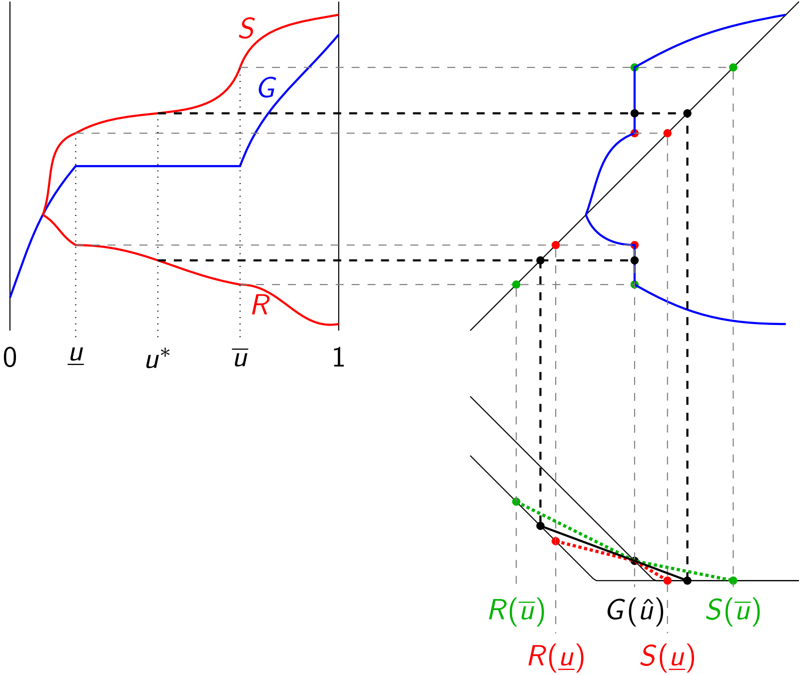
S
G
R
0
$\underline{u}$
u^*
$\overline{u}$
1
$R(\overline{u})$
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