A Left-Monotone Solution to the Peacock Problem

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Robust Finance and Martingale Optimal Transport

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Finitely many time steps $T = \{1, ..., n\}$, underlying $S = (S_t)_{t \in T}$ and derivative Φ. $(G_\alpha)_{\alpha \in \mathcal{A}}$ family of derivatives with (market) prices $(p_\alpha)_{\alpha \in \mathcal{A}}$.

$$\underline{P} = \inf_{\substack{\gamma \textit{ martingale measure,} \\ \forall \alpha: \mathbb{E}_\gamma[G_\alpha(S)] = p_\alpha}} \mathbb{E}_\gamma \left[\Phi(S_1, ..., S_n)\right]$$

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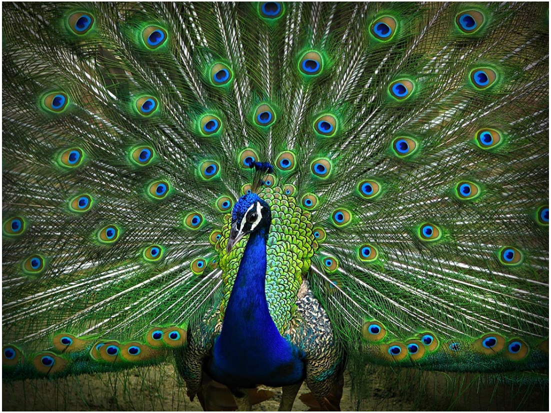

The Peacock Problem

Let $T = [0, 1]$ and $(\mu_t)_{t \in T}$ be a family of probability measures on $\mathbb{R}$.

Definition

The family $(\mu_t)_{t \in T}$ is called a **peacock**[a] if $t \mapsto \int_{\mathbb{R}} \varphi \, \mathrm{d}\mu_t$ is increasing for all convex functions $\varphi : \mathbb{R} \to \mathbb{R}$.

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There exists a (cadlag) martingale with one-dimensional marginals $(\mu_t)_{t \in T}$ if and only if $(\mu_t)_{t \in T}$ is a (right-continuous) peacock.

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- Hirsch-Profeta-Roynette-Yor ['11], Lowther ['08], Hobson ['17], ...
- Henry-Labordere-Tan-Touzi ['16] and Juillet ['18]

An MOT Approach to the Peacock Problem

Choose a sequence $(R_n)_{n\in\mathbb{N}}$ of finitely many time points in T.

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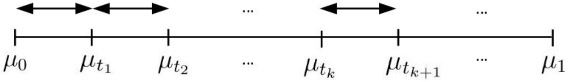
  Construct a sequence $(\pi_n)_{n\in\mathbb{N}}$ of discrete time martingal couplings by joining the Left-Curtain couplings of two consecutive marginals:



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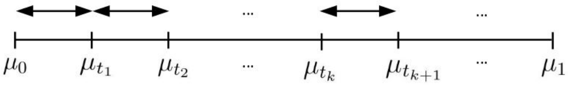

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  Construct a sequence $(\pi_n)_{n\in\mathbb{N}}$ of discrete time martingal couplings by extending the Left-Curtain Coupling over all marginals:

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(c.f. Nutz-Stebegg-Tan ['17], Beiglböck-Cox-Huesmann ['17])

The Left-Curtain Coupling

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(i) For all $c \in C^{1,2}(\mathbb{R}^2)$ with $\partial_{xyy} c < 0$ (SMCF) it holds

$$\mathbb{E}_\pi[c(X,Y)] = \inf \left\{ \mathbb{E}_\gamma[c(X,Y)] : \gamma \textit{ mart. cpl. of } \mu_0 \textit{ and } \mu_1 \right\}.$$

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(ii) For all $a \in \mathbb{R}$ it holds

$$\pi(X \leq a, Y \in \cdot) = \mathcal{S}^{\mu_1}(\mu_{0|(-\infty,a]}),$$

i.e. $\pi(X \leq a, Y \in \cdot)$ is 'the most concentrated submeasure of μ_1 that is in convex order greater than $\mu_{0|(-\infty|a)}$'

The Shadow of ν in μ

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Let μ and ν be finite Borel measures on $\mathbb{R}$.

- **convex order**: $\nu \leq_c \mu$ if $\int_{\mathbb{R}} \varphi \,\mathrm{d}\mu \leq \int_{\mathbb{R}} \varphi \,\mathrm{d}\nu$ for all convex φ.
- **positive order**: $\nu \leq_+ \mu$ if $\int_{\mathbb{R}} \varphi \,\mathrm{d}\mu \leq \int_{\mathbb{R}} \varphi \,\mathrm{d}\nu$ for all $\varphi \geq 0$.
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Definition

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It is denoted by $\mathcal{S}^\mu(\nu)$.

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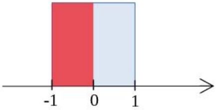
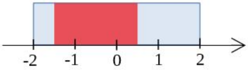
Example: $\mu_0 = \mathrm{Unif}_{[-1,1]}$, $\mu_1 = \mathrm{Unif}_{[-2,2]}$, $\nu = \mu_{0|(-\infty,a])}$





(Generalized) Shadow

Definition

Let $(\mu_t)_{t\in T}$ be a peacock and $\nu \leq_{c,+} \mu_t$ for all $t \in T$. The (generalized) **shadow** of ν in $(\mu_t)_{t\in T}$ is defined as

$$\mathrm{Csup}\left\{\mathcal{S}^{\mu_{r_n}}(\mathcal{S}^{\mu_{r_{n-1}}}(\ \ldots\ \mathcal{S}^{\mu_{r_1}}(\nu)))\ |n \in \mathbb{N}, r_1, \ldots, r_n \in T\right\}.$$

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Theorem

Let $(\mu_t)_{t \in T}$ be a peacock, $\nu \leq_{c,+} \mu_t$ for all $t \in T$ and suppose that T has a maximal element t_{max}. It holds

$$\mathcal{S}^{(\mu_t)_{t \in T}}(\nu) = \mathrm{Cinf}\left\{\eta_{t_{max}} \,|\, \exists (\eta_t)_{t \in T} \forall s \leq t : \nu \leq_c \eta_s \leq_c \eta_t \leq_+ \mu_t\right\}.$$

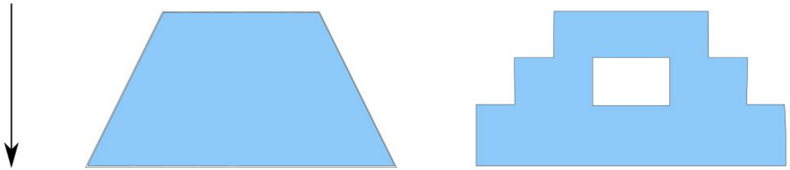
Moreover, the infimum is attained, i.e. we have $\mathcal{S}^{(\mu_t)_{t \in T}}(\nu) \leq_+ \mu_{t_{max}}$.

Example of a Shadow

Define $(\mu_t)_{t\in[0,1]}$ and $(\mu'_t)_{t\in[0,1]}$ as

$$\mu_t = \mathrm{Unif}_{[-t,t]} \qquad \mu'_t = \begin{cases} \mathrm{Unif}_{[-1,1]} & , t < 0.3 \\ \frac{1}{2}\mathrm{Unif}_{[-\frac{3}{2},-\frac{1}{2}]} + \frac{1}{2}\mathrm{Unif}_{[\frac{1}{2},\frac{3}{2}]} & , t \in [0.3, 0.6) \\ \mathrm{Unif}_{[-2,2]} & , t \geq 0.6 \end{cases}$$

and $\nu = \mu_{0|(-\infty,0]} = \frac{1}{2}\mathrm{Unif}_{[-1,0]}$.



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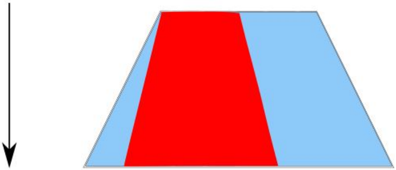

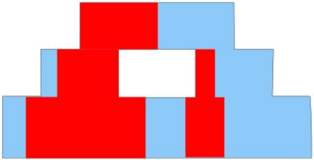

First Result: Existence

Theorem

Let $(\mu_t)_{t\in[0,1]}$ *be a right-continuous peacock with* $\mu_0(\{x\}) = 0$ *for all* $x \in \mathbb{R}$*. There* **exists** *a solution* π *to the corresponding peacock problem with the following two properties:*

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(i) *π is a simultaneous minimizer for all SMCF c, i.e. it holds*

$$\mathbb{E}_\pi [c(S_0, S_t)] = \inf \{\mathbb{E}_\gamma [c(S_0, S_t)] : \gamma \text{ solution}\}$$

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- We get sequence $(\pi_n)_{n\in\mathbb{N}}$ of solutions corresponding to $(\mu_t^n)_{t\in[0,1]}$ satisfying (i) and (ii) w.r.t. $(\mu_t^n)_{t\in[0,1]}$.

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- There exists a convergent subsequence and the limit is a solution corresponding to $(\mu_t)_{t\in[0,1]}$ (similar to Beiglöck-Huesmann-Stebegg ['16]).
- The properties are (i) and (ii) are stable under this topology.

Second Result: Equivalence

Theorem

Let $(\mu_t)_{t\in[0,1]}$ be a right-continuous peacock with $\mu_0(\{x\}) = 0$ for all $x \in \mathbb{R}$. For all solutions π to the corresponding peacock problem the following two properties are **equivalent***:*

(i) *π is a simultaneous minimizer for all SMCF c, i.e. it holds*

$$\mathbb{E}_\pi \left[c(S_0, S_t)\right] = \inf \left\{\mathbb{E}_\gamma \left[c(S_0, S_t)\right] : \gamma \in \Pi_M((\mu_t)_{t\in[0,1]})\right\}$$

for all c satisfying appropriate conditions and all $t \in [0,1]$.

(ii) *π is left-monotone, i.e. it holds*

$$\mathrm{Law}_\pi(S_0 \leq a, S_t \in \cdot) = \mathcal{S}^{(\mu_s)_{s\in[0,1]}}(\mu_{0|(-\infty,a]})$$

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- The potential function of a finite measure μ on $\mathbb{R}$ with finite first moment is defined as $U(x) = \int_{\mathbb{R}} |y - x| \,\mathrm{d}\mu(y)$ and characterizes μ uniquely.
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and this is the potential function of $\mathcal{S}^{(\mu_s)_{s\in[0,1]}}(\mu_{0|(-\infty,a]})$.

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  for all SMCF c and $t \in [0,1]$.

Non-Obstructing Marginals

Definition

A peacock $(\mu_t)_{t\in[0,1]}$ is called non-obstructed if

$$\mathcal{S}^{(\mu_s)_{s\in[0,t]}}(\mu_{0|(-\infty,a]}) = \mathcal{S}^{\mu_t}(\mu_{0|(-\infty,a]})$$

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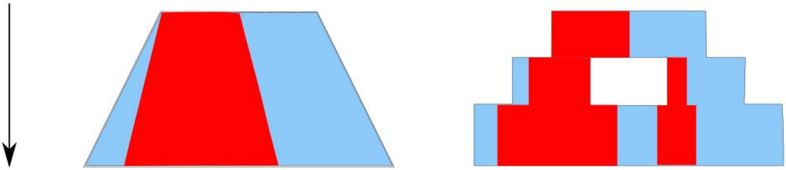

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Theorem

Let $(\mu_t)_{t\in[0,1]}$ be a non-obstructed right-continuous peacock. There exists a **unique** *left-monotone solution to the corresponding peacock problem and $(S_0, S_t)_{t\in[0,1]}$ is a* **Markov** *process under π.*

Non-Obstructing Marginals

Lemma

Let $(\mu_t)_{t\in[0,1]}$ *be a right-continuous peacock with* $\mu_0(\{x\}) = 0$ *for all* $x \in \mathbb{R}$*. If there exists a family of intervals* $(E_t)_{t\in[0,1]}$ *in* $\mathbb{R}$ *such that*

(i) $\mathrm{supp}(\mu_t) \subset E_t$ *and*

(ii) $\mathrm{supp}((\mu_u - \mu_t)^+) \subset E_t^c$

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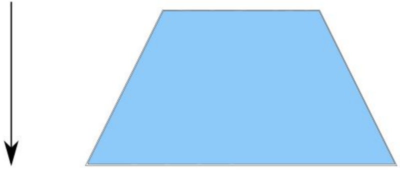

Comparison

Let $\mu_t = \mathrm{Unif}_{[-t,t]}$.

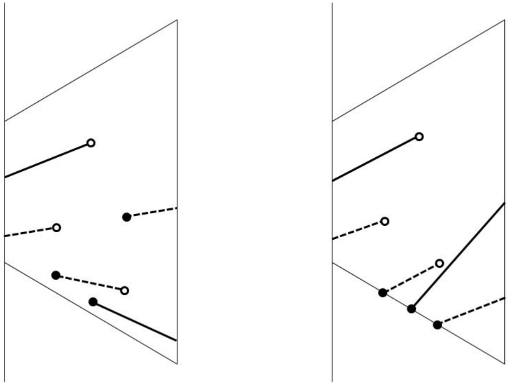

Figure: Two sample paths of the left-monotone martingale measure (left) and the limit-curtain martingale measure (right).

Thank you for your attention!