Good Deal Hedging and Valuation under Combined Uncertainty about Drift & Volatility

Dirk Becherer

Humboldt-Universität zu Berlin

joint work with Klébert Kentia

Robust Techniques in Quantitative Finance 2018

Oxford, 3rd September

Motivating example

- How to hedge and value a **put option** $X = (\mathcal{K} - H_T)^+$ on a **non-traded asset**

$$dH_t = H_t\big(\gamma dt + \beta(\rho dB_t^1 + \sqrt{1-\rho^2}dB_t^2)\big)$$

if there is only a **correlated** tradable asset

$$dS_t = S_t\sigma^S(\xi dt + dB_t^1)$$

available for **partial hedging** : ?

- non-perfect correlation: $-1 < \rho < 1$.

Motivating example

- How to hedge and value a **put option** $X = (\mathcal{K} - H_T)^+$ on a **non-traded asset**

$$dH_t = H_t\big(\gamma dt + \beta(\rho dB_t^1 + \sqrt{1-\rho^2}dB_t^2)\big)$$

if there is only a **correlated** tradable asset

$$dS_t = S_t\sigma^S(\xi dt + dB_t^1)$$

available for **partial hedging** : ?

- non-perfect correlation: $-1 < \rho < 1$.
- $\rightsquigarrow$ Superreplication (a.s.-hedging) is prohibitively expensive! Hedge would be extreme but trivial (don't trade even if $\rho = 99\%$)! **Uncertainty on drift&volatilitiy** matters for any alternative...

Aspect I: incomplete vs complete

- Complete market: unique price obtained by replication

$$X = \underbrace{E^Q[X]}_{\text{replication cost}} + \underbrace{\int_0^T \phi_t dS_t}_{\text{gain/loss from trading}} \quad \text{a.s.,} \quad \text{for } \mathcal{M}^e(S) = \{Q\}.$$

- Incomplete market: infinitely many martingale measures $\mathcal{M}^e(S)$, upper NA-bound is seller's super-replicating price

$$\sup_{Q \in \mathcal{M}^e(S)} E^Q[X] = \inf \left\{ m : \exists \phi \text{ s.t. } m + \int_0^T \phi_t dS_t \geq X \ P\text{-a.s.} \right\}.$$

Aspect I: incomplete vs complete

- Complete market: unique price obtained by replication

$$X = \underbrace{E^Q[X]}_{\text{replication cost}} + \underbrace{\int_0^T \phi_t dS_t}_{\text{gain/loss from trading}} \quad \text{a.s.,} \quad \text{for } \mathcal{M}^e(S) = \{Q\}.$$

- Incomplete market: infinitely many martingale measures $\mathcal{M}^e(S)$, upper NA-bound is seller's super-replicating price

$$\sup_{Q \in \mathcal{M}^e(S)} E^Q[X] = \inf \left\{ m : \exists \phi \text{ s.t. } m + \int_0^T \phi_t dS_t \geq X \ P\text{-a.s.} \right\}.$$

- **But:** Superreplication is too expensive, a.s.-hedge is too extreme !
- **Aim:** less expensive valuations, less extreme than a.s.-hedging. How? Exclude not just arbitrage but also "too good deals"
- $\rightsquigarrow$ hedging error distribution matters, hence **uncertainty** on P!

Ansatz for no-good-deal valuation and hedging

- **Valuation:** Use only subset $\mathcal{Q}^{\mathrm{ngd}} \subset \mathcal{M}^e$ with *economic meaning*
  - Choose $\mathcal{Q}^{\mathrm{ngd}}$ such that **"too good deals" are excluded** for any market extension $\big(S_t, E_t^Q[X]\big)$ by derivatives priced with Q.
  - Define upper and lower good-deal valuation bounds

$$\pi_t^u(X) := \operatorname*{esssup}_{Q\in\mathcal{Q}^{\mathrm{ngd}}} E_t^Q[X], \quad \pi_t^l(X) := \operatorname*{essinf}_{Q\in\mathcal{Q}^{\mathrm{ngd}}} E_t^Q[X] = -\pi^u(-X).$$

Ansatz for no-good-deal valuation and hedging

- **Valuation:** Use only subset $\mathcal{Q}^{\text{ngd}} \subset \mathcal{M}^e$ with *economic meaning*
  - Choose $\mathcal{Q}^{\text{ngd}}$ such that **"too good deals" are excluded** for any market extension $\big(S_t, E_t^Q[X]\big)$ by derivatives priced with Q.
  - Define upper and lower good-deal valuation bounds

$$\pi_t^u(X) := \operatorname*{esssup}_{Q \in \mathcal{Q}^{\text{ngd}}} E_t^Q[X], \quad \pi_t^l(X) := \operatorname*{essinf}_{Q \in \mathcal{Q}^{\text{ngd}}} E_t^Q[X] = -\pi^u(-X).$$

- **Hedging Strategy: minimize a-priori coherent risk measure ρ** over all strategies ϕ for optimal risk sharing with the market:

$$\pi_t^u(X) = \rho_t\left(X - \int_t^T \bar{\phi}_s dS_s\right) = \operatorname*{essinf}_{\phi \in \Phi} \rho_t\left(X - \int_t^T \phi_s dS_s\right) \quad \forall t \in [0, T].$$

- Solution for dominated uncertainty (about drifts): via standard BSDE, g-expectations: [B., Kentia, MMOR 2017]

Aspect II: non-dominated vs dominated uncertainty

- For super-replication price v only non-dominated uncertainty matters, but not drift (or market price of risk) uncertainty:
- Example:
  Black-Scholes-type model $dS_t = S_t\sigma_t(\xi_t dt + dW_t)$
- Black-Scholes pricing pde :

$$\sigma^2 s^2 v_{ss} + v_t = 0$$

Aspect II: non-dominated vs dominated uncertainty

- For super-replication price v only non-dominated uncertainty matters, but not drift (or market price of risk) uncertainty:
- Example:
  Black-Scholes-type model $dS_t = S_t\sigma_t(\xi_t dt + dW_t)$
  with **(non-dominated) volatility uncertainty**

$$0 < \underline{\sigma}^2 \le \sigma_t(\omega)^2 \le \overline{\sigma}^2$$

- Black-Scholes-Barenblatt pricing pde (fully non-linear):

$$(\overline{\sigma}^2 1_{\{v_{ss}\ge 0\}} + \underline{\sigma}^2 1_{\{v_{ss}<0\}}) s^2 v_{ss} + v_t = 0$$

- more general path-dependent claims, non-Makovian setup...
  $\rightsquigarrow$ **2nd order BSDE** (or: G-expectation) in BSDE-setup!
- For any given $\sigma > 0$ model is complete here. But **what to do in generically incomplete models?** Superreplication is too expensive!

Outline

1. Idea of the good-deal approach in the absence of ambiguity
2. Valuation and hedging under ambiguity about drift and volatility
   - Framework for combined drift and volatility uncertainty
   - Robust valuation and hedging via 2BSDEs
   - Some examples

Outline

1. Idea of the good-deal approach in the absence of ambiguity

2. Valuation and hedging under ambiguity about drift and volatility
   - Framework for combined drift and volatility uncertainty
   - Robust valuation and hedging via 2BSDEs
   - Some examples

Financial market and no-good-deal pricing measures

- Fixed $(\Omega, \mathcal{F}, \mathbb{F}, P)$, W n-dimensional P-Wiener process, $\mathbb{F} = \overline{\mathbb{F}^{W}}^{P}$.
- Prices $dS_t = \mathrm{diag}(S_t)\sigma_t(\xi_t dt + dW_t)$ of d traded assets, with market price of risk $\xi \in \mathrm{Im}\,\sigma^{\mathrm{tr}}$, volatility matrix σ of full rank $d < n$.
- Good-deal pricing and a-priori valuation measures

$$\mathcal{Q}^{\mathrm{ngd}}(P) := \left\{ Q \in \mathcal{M}^e(P) : dQ/dP = \mathcal{E}\left(\lambda \cdot W\right), \text{ and } |\lambda| \le h \right\}$$

$$\mathcal{P}^{\mathrm{ngd}}(P) := \left\{ Q \sim P : dQ/dP = \mathcal{E}\left(\lambda \cdot W\right), \text{ with } |\lambda| \le h \right\}$$

- Pricing with $\mathcal{Q}^{\mathrm{ngd}}(P)$ imposes bounds on **instantaneous Sharpe ratios** $SR_t := \frac{\text{mean excess return(t)}}{\text{standard deviation (t)}} \le h_t$, or on P-expected **optimal growth rates** of returns in any market extension $(S, E_t^Q[X])$.
- **Hedging** strategy $\bar{\phi}^P \in \Phi(P)$ **minimizes coherent risk** measure :

$$\pi_t^{u,P}(X) = \rho_t^P\left(X - \int_t^T \bar{\phi}_s^{\mathrm{tr}} d\widehat{W}_s\right) = \operatorname{essinf}_{\phi \in \Phi(P)} \rho_t^P\left(X - \int_t^T \phi_s^{\mathrm{tr}} d\widehat{W}_s\right)$$

for a-priori risk $\rho_t^P(X) := \operatorname{esssup}_{Q \in \mathcal{P}^{\mathrm{ngd}}(P)} E_t^Q[X]$.

Financial market and no-good-deal pricing measures

- Prices $dS_t = \mathrm{diag}(S_t)\sigma_t(\xi_t dt + dW_t)$ of d traded assets
- Good-deal pricing and a-priori valuation measures

$$\mathcal{Q}^{\mathrm{ngd}}(P) := \Big\{ Q \in \mathcal{M}^e(P) \,:\, dQ/dP = \mathcal{E}\left(\lambda \cdot W\right),\ \text{and } |\lambda| \le h \Big\}$$

$$\mathcal{P}^{\mathrm{ngd}}(P) := \Big\{ Q \sim P \,:\, dQ/dP = \mathcal{E}\left(\lambda \cdot W\right),\ \text{with } |\lambda| \le h \Big\}$$

- Pricing with $\mathcal{Q}^{\mathrm{ngd}}(P)$ imposes bounds on **instantaneous Sharpe ratios** $SR_t := \frac{\text{mean excess return(t)}}{\text{standard deviation (t)}} \le h_t$, or on P-expected **optimal growth rates** of returns in any market extension $(S, E_t^Q[X])$.
- **Hedging** strategy $\bar{\phi}^P \in \Phi(P)$ **minimizes coherent risk** measure :

$$\pi_t^{u,P}(X) = \rho_t^P\Big(X - \int_t^T \bar{\phi}_s^{\mathrm{tr}} d\widehat{W}_s\Big) = \operatorname{essinf}_{\phi\in\Phi(P)}\ \rho_t^P\Big(X - \int_t^T \phi_s^{\mathrm{tr}} d\widehat{W}_s\Big)$$

for a-priori risk $\rho_t^P(X) := \operatorname{esssup}_{Q\in\mathcal{P}^{\mathrm{ngd}}(P)}\ E_t^Q[X]$.

- Hedging to acceptability, use market to minimize risk

Hedging error and its supermartingale characterization

- For strategy $\phi \in \Phi(P)$, its tracking (or hedging) error is

$$L_t^{\phi}(X) := \underbrace{\pi_t^{u,P}(X) - \pi_0^{u,P}(X)}_{\text{variation of monetary capital requirement}} - \underbrace{\left(\int_0^t \phi_s^{\mathrm{tr}}\, d\widehat{W}_s\right)}_{\text{P\&L from dyn.trading}}$$

- Tracking error $L^{\bar{\phi}}$ of the good-deal hedging strategy $\bar{\phi}^P$ is Q-supermartingale for any $Q \in \mathcal{P}^{\mathrm{ngd}}(P)$.
(sufficient and necessary condition for optimality)

- Hedging strategy $\bar{\phi}^P$ is at least mean-self-financing under any a-priori valuation measure $Q \in \mathcal{P}^{\mathrm{ngd}}(P)$.

Outline

1. Idea of the good-deal approach in the absence of ambiguity

2. Valuation and hedging under ambiguity about drift and volatility
   - Framework for combined drift and volatility uncertainty
   - Robust valuation and hedging via 2BSDEs
   - Some examples

Knightean Uncertainty

- **Problem:**
  market prices of risk ξ (drift) and volatilities σ are not known:
  we do not know the objective real world measure P precisely...
- There is ambiguity $\rightsquigarrow$ Knightian model uncertainty.

Knightean Uncertainty

- **Problem:**
  market prices of risk ξ (drift) and volatilities σ are not known:
  we do not know the objective real world measure P precisely...
- There is ambiguity $\rightsquigarrow$ Knightian model uncertainty.
- Instead of single reference measure P, consider set $\mathcal{R}$ of plausible reference priors, capturing ambiguity about both drift and volatility.

Knightean Uncertainty

- **Problem:**
  market prices of risk ξ (drift) and volatilities σ are not known: we do not know the objective real world measure P precisely...
- There is ambiguity $\rightsquigarrow$ Knightian model uncertainty.
- Instead of single reference measure P, consider set $\mathcal{R}$ of plausible reference priors, capturing ambiguity about both drift and volatility.
- **Aim:** Robust good-deal valuation and hedging w.r.t. uncertainty about ξ and σ.

2BSDE setup on canonical space

- $\Omega := C_0([0,T],\mathbb{R}^n)$, B coordinate process, P_0 Wiener measure, $\mathbb{F} := (\mathcal{F}_t)_{t\le T}$ filtration generated by B, $\mathbb{F}^+$ right limit of $\mathbb{F}$.
- $\overline{\mathcal{P}}_S$: collection of all local martingale measures P_α for B with

$$P_\alpha := P_0 \circ (X^\alpha)^{-1}, \text{ where } X^\alpha_t := \int_0^t \alpha_s^{\frac{1}{2}} dB_s, \quad P_0\text{-a.s. } t \in [0,T],$$

with positive-definite-valued α, $\mathbb{F}$-prog. meas., $\int_0^T |\alpha_s|\, ds < \infty$, P_0-a.s..

- Karandikar '95: The stochastic integral $\int_0^\cdot B_s dB_s^{\mathrm{tr}}$ can be defined ω-wise such that it coincides with Itô integral P-a.s. for all $P \in \overline{\mathcal{P}}_S$.
- quadratic variation $\langle B\rangle_t$ and density $\widehat{a}_t := \frac{d\langle B\rangle_t}{dt}$ are defined ω-wise.

2BSDE setup: non-dominated, quasi-sure analysis

- **Note:** The probability measures in $\overline{\mathcal{P}}_S$ can be mutually singular.
- For fixed $\underline{a}, \overline{a} \in \mathbb{S}_n^{>0}$ (set of positive-definite matrices), consider subclass $\mathcal{P} \subset \overline{\mathcal{P}}_S$ defined by

$$\mathcal{P} := \left\{ P \in \overline{\mathcal{P}}_S : \ \underline{a} \le \widehat{a} \le \overline{a}, \ P \otimes dt\text{-a.s.} \right\}.$$

- $\underline{a} \le \widehat{a} \le \overline{a} \quad \rightsquigarrow$ confidence set for **uncertainty about volatility**.
- **Def**: A property holds $\mathcal{Q}$-quasi-everywhere (shortly $\mathcal{Q}$-q.e.) for family $\mathcal{Q}$ of measures if it holds outside a set which is Q-negligible $\forall Q \in \mathcal{Q}$.

Financial market with uncertainty

Risky asset prices $S = (S^i)_{i=1}^d$ modeled as

$$dS_t = \mathrm{diag}(S_t)\,(b\,dt + \sigma dB_t),\ \mathcal{P}\text{-q.s.}$$

with $\sigma \in \mathbb{R}^{d\times n}$ ($d < n$) and $\sigma \widehat{a}^{\frac{1}{2}}$ $\mathcal{P} \otimes dt$-q.e. of maximal rank d.

- $W^P := \int_0^\cdot \widehat{a}_s^{-\frac{1}{2}} dB_s$ is a P-Brownian motion, for each $P \in \mathcal{P}$.
- **Volatility uncertainty:** $\sigma\widehat{a}^{\frac{1}{2}}$ plays the role of volatility matrix for stock prices S under each $P \in \mathcal{P}$, with $dB_t = \widehat{a}_t^{\frac{1}{2}} dW_t^P$.
- $\sigma\widehat{a}^{\frac{1}{2}}$ of maximal rank $d < n \rightsquigarrow$ incomplete market under any $P \in \mathcal{P}$.
- (Minimal) market prices of risk for any model $P \in \mathcal{P}$ given by

$$\widehat{\xi}_t := \widehat{a}_t^{\frac{1}{2}} \sigma^{\mathrm{tr}} (\sigma \widehat{a}_t \sigma^{\mathrm{tr}})^{-1} b,$$

multiple non-dominated probability reference priors

- uncertain market prices of risk (drifts) and uncertain volatilities

$$\xi_t^{P,\theta} = \widehat{\xi}_t + \theta_t \quad \text{and} \quad \sigma_t^{P,\theta} = \sigma \widehat{a}_t^{\frac{1}{2}},$$

for θ_t in some ellipsoidal confidence region $\Theta_t \subset \mathrm{Im}\left(\sigma_t \widehat{a}_t^{\frac{1}{2}}\right)^{\mathrm{tr}}$, $t \leq T$.

- reference priors with drift & volatility uncertainty

$$\mathcal{R} := \left\{ Q = Q^{P,\theta} \;\middle|\; Q \sim P,\ \frac{dQ}{dP} = {}^{(P)}\mathcal{E}(\theta \cdot W^P),\ \text{for } P \in \mathcal{P},\ \theta_t \in \Theta_t\ \forall t \right\}.$$

- set of no-good-deal pricing measures for each prior $Q^{P,\theta} \in \mathcal{R}$:

$$\mathcal{Q}^{\mathrm{ngd}}(Q^{P,\theta}) = \left\{ Q \in \mathcal{M}^e(P) \;\middle|\; \frac{dQ}{dP} = {}^{(P)}\mathcal{E}(\lambda \cdot W^P),\ |\lambda + \theta| \leq h \right\}.$$

Uncertainty: Worst-case bounds and a-priori risk measure

- For each $Q^{P,\theta} \in \mathcal{R}$, consider the sets $\mathcal{Q}^{\mathrm{ngd}}(Q^{P,\theta})$ and $\mathcal{P}^{\mathrm{ngd}}(Q^{P,\theta})$.
- For model $Q^{P,\theta}$, good-deal bound and risk measure are

$$\pi_t^{u,P,\theta}(X) = \operatorname*{esssup}^{P}_{Q\in\mathcal{Q}^{\mathrm{ngd}}(Q^{P,\theta})} E_t^Q[X] \quad , \quad \rho_t^{P,\theta}(X) = \operatorname*{esssup}^{P}_{Q\in\mathcal{P}^{\mathrm{ngd}}(Q^{P,\theta})} E_t^Q[X], \;\; P\text{-a.s..}$$

- Robust worst-case good-deal bound under uncertainty:

$$\pi_t^u(X) := \operatorname*{esssup}^{P}_{P'\in\mathcal{P}(t+,P)} \; \operatorname*{esssup}^{P}_{\theta\in\Theta} \pi_t^{u,P',\theta}(X), \;\; P\text{-a.s.}, \quad \forall P \in \mathcal{P}$$

where $\mathcal{P}(t+,P) = \{P' \in \mathcal{P} : P' = P \,\mathrm{on}\, \mathcal{F}_t^+\}$.

- a-priori risk measure to be minimized by dynamic hedging:

$$\rho_t(X) := \operatorname*{esssup}^{P}_{P'\in\mathcal{P}(t+,P)} \; \operatorname*{esssup}^{P}_{\theta\in\Theta} \rho_t^{P',\theta}(X), \;\; P\text{-a.s.}, \quad \forall P \in \mathcal{P}.$$

2BSDE formulation and wellposedness

For a measurable generator $F : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}_n^{>0} \to \mathbb{R}$, Lipschitz in (y, z), denote $\widehat{F}_t(y, z) := F_t(B_{\cdot \wedge t}, y, z, \widehat{a}_t)$.

- For $\mathcal{F}_T$-measurable X, define second-order BSDE (2BSDE)

$$Y_t = X - \int_t^T \widehat{F}_s(Y_s, \widehat{a}_s^{\frac{1}{2}} Z_s)ds - \int_t^T Z_s dB_s + K_T^P - K_t^P, \ \mathcal{P}\text{-q.s.}$$

- Solution triple $(Y, Z, (K^P)_{P \in \mathcal{P}})$, satisfies minimum condition

$$K_t^P = \underset{P' \in \mathcal{P}(t+, P)}{\operatorname{essinf}^P} E_t^{P'}[K_T^{P'}], \ P\text{-a.s.}, \ t \in [0, T], \ \forall P \in \mathcal{P}.$$

- Wellposedness under suitable measurability and integrability properties on X and F (Possamai/Tan/Zhou, AAP 2018), convexity and continuity **not** required.

Solution: Good-deal Valuation via 2BSDEs

- For each $a \in \mathbb{S}_n^{>0}$ and $t \in [0, T]$, consider the orthogonal projections

$$\Pi_t^a : \mathbb{R}^n \to \mathrm{Im}\left(\sigma_t a^{1/2}\right)^{\mathrm{tr}} \quad \text{and} \quad \Pi_t^{\perp,a} : \mathbb{R}^n \to \mathrm{Ker}\left(\sigma_t\, a^{1/2}\right).$$

- Let F be the generator function defined by

$$F_t(z,a) := \inf_{\theta \in \Theta} \left(\widehat{\xi}_t^{\,\mathrm{tr}} \Pi_t^a(z) - \sqrt{h_t^2 - |\widehat{\xi}_t + \theta_t|^2}\,\left|\Pi_t^{\perp,a}(z)\right| \right)$$

- For suitable X, there exists a unique (in a suitable space) solution $(Y, Z, (K^P)_{P \in \mathcal{P}})$ to the 2BSDE with parameters (F, X).
- Worst-case good-deal bound process is given by

$$\pi_t^u(X) = Y_t,\ \ P\text{-a.s.},\ \forall P \in \mathcal{P}$$

.

Solution: Good-deal Hedging via 2BSDEs

- trading strategy ϕ (as amount of wealth invested in S)
  $\rightsquigarrow$ has wealth process

$$V^{\phi} = V_0 + \int_0^{\cdot} \phi_s^{\mathrm{tr}} \big(\widehat{a}_s^{\frac{1}{2}} \xi_s ds + dB_s \big), \quad \text{with } \phi \in \operatorname{Im} \sigma^{\mathrm{tr}}.$$

- Hedging problem under drift and vol. uncertainty: Find $\bar{\phi} \in \Phi$ s.t.

$$\pi_t^u(X) = \rho_t \Big(X - \big(V_T^{\bar{\phi}} - V_t^{\bar{\phi}} \big) \Big) = \operatorname*{essinf}_{\phi \in \Phi} \ \rho_t \Big(X - \big(V_T^{\phi} - V_t^{\phi} \big) \Big).$$

Good-deal hedging via 2BSDEs (cont.)

Denote $\widehat{\Pi}_t := \Pi_{\mathrm{Im}\,(\sigma_t \widehat{a}^{1/2})^{\mathrm{tr}}}$ and $\widehat{\Pi}_t^{\perp} := \Pi_{\mathrm{Ker}\,(\sigma_t \widehat{a}^{1/2})}$, and recall $(Y, Z, (K^P)_{P\in\mathcal{P}})$ solution to the 2BSDE for $\pi_{\cdot}^u(X)$.

- For some worst-case $\bar{\theta} \in \Theta$, the **hedging strategy** is given by

$$\widehat{a}_t^{1/2}\bar{\phi}_t(X) = \underbrace{\widehat{\Pi}_t\big(\widehat{a}_t^{1/2} Z_t\big)}_{\text{Non-speculative component}} + \underbrace{\frac{\big|\widehat{\Pi}_t^{\perp}\big(\widehat{a}_t^{1/2} Z_t\big)\big|}{\sqrt{h_t^2 - \big|\xi_t + \bar{\theta}_t\big|^2}}\big(\xi_t + \bar{\theta}_t\big)}_{\text{Speculative component}}.$$

- **Robust hedging** error $L^{\bar{\phi}} = \pi_{\cdot}^u(X) - \pi_0^u(X) - (V_T^{\bar{\phi}} - V_{\cdot}^{\bar{\phi}})$ is a Q-supermartingale for any $Q \in \mathcal{P}^{\mathrm{ngd}}(Q^{P,\theta})$ for all $P \in \mathcal{P}, \theta \in \Theta$

  $\rightsquigarrow$ is at least mean-self.financing wrt. drift & vola uncertainty

Instructive examples

Re-visit our example of hedging a put on a correlated non-traded asset:

1. Only drift uncertainty about drift (market price of risk)

Instructive examples

Re-visit our example of hedging a put on a correlated non-traded asset:

1. Only drift uncertainty about drift (market price of risk)
   $\rightsquigarrow$ closed-form solution for good deal valuation and hedge: constant worst-case parameters, modified Black-Scholes formula (B.,Klebert Kentia, MMOR 2017+)

Instructive examples

Re-visit our example of hedging a put on a correlated non-traded asset:

1. Only drift uncertainty about drift (market price of risk) $\rightsquigarrow$ closed-form solution for good deal valuation and hedge: constant worst-case parameters, modified Black-Scholes formula (B.,Klebert Kentia, MMOR 2017+)
2. Only volatility uncertainty

Instructive examples

Re-visit our example of hedging a put on a correlated non-traded asset:

1. Only drift uncertainty about drift (market price of risk) $\rightsquigarrow$ closed-form solution for good deal valuation and hedge: constant worst-case parameters, modified Black-Scholes formula (B.,Klebert Kentia, MMOR 2017+)
2. Only volatility uncertainty $\rightsquigarrow$ closed form solution: constant worst-case parameters, modified Black-Scholes formula (B., Klebert Kentia, PUQR 2018)

Instructive examples

Re-visit our example of hedging a put on a correlated non-traded asset:

1. Only drift uncertainty about drift (market price of risk) $\rightsquigarrow$ closed-form solution for good deal valuation and hedge: constant worst-case parameters, modified Black-Scholes formula (B.,Klebert Kentia, MMOR 2017+)
2. Only volatility uncertainty $\rightsquigarrow$ closed form solution: constant worst-case parameters, modified Black-Scholes formula (B., Klebert Kentia, PUQR 2018)
3. Combined uncertainty about volatility and drift

Instructive examples

Re-visit our example of hedging a put on a correlated non-traded asset:

1. Only drift uncertainty about drift (market price of risk) $\rightsquigarrow$ closed-form solution for good deal valuation and hedge: constant worst-case parameters, modified Black-Scholes formula (B.,Klebert Kentia, MMOR 2017+)
2. Only volatility uncertainty $\rightsquigarrow$ closed form solution: constant worst-case parameters, modified Black-Scholes formula (B., Klebert Kentia, PUQR 2018)
3. Combined uncertainty about volatility and drift $\rightsquigarrow$ no closed-form solution, more complex stochastic optimization problem over non-rectangular set of controls without trivial solution already in simple Markovian case (B., Klebert Kentia, PUQR 2018)

One particular Example with an explicit solution:

- Market with two Black-Scholes-type assets ($d = 1,\ n = 2$):

$$dS_t = S_t \sigma^S dB^1_t, \quad dH_t = H_t\big(\gamma dt + \beta(\rho dB^1_t + \sqrt{1-\rho^2} dB^2_t)\big),\ \mathcal{P}\text{-q.s.},$$

  for $\rho \in [-1, 1]$, $\widehat{a} = (\widehat{a}^{ij})_{i,j=1,2} \in [\underline{a}, \overline{a}]$: uncertainty about volatility but not about drift, market price of risk $\xi = 0$.

- **Put** option $X = (\mathcal{K} - H_T)^+$ **on non-traded asset** H.
- Valuation at maximum vol. level $\overline{a}$, for worst-case model $P_{\overline{a}} \in \mathcal{P}$:

$$\pi^u_t(X) = \pi^{u,P_{\overline{a}}}_t(X) = C{*}\text{Black-Scholes-Put-price}\Big(\text{spot } H_t,\ \text{strike } \frac{\mathcal{K}}{C},\ \text{vol. } \bar{\beta}\Big),$$

  for some $C, \bar{\beta} \in (0, \infty)$.

- Hedging strategy: $\bar{\phi}_t(X) = L_t\, H_t\Big(\rho + \frac{\widehat{a}^{12}_t}{\widehat{a}^{11}_t}\sqrt{1-\rho^2}, 0\Big)^{\text{tr}}$ for some $L_t < 0$.
- Incompleteness ($|\rho| \neq 1$) $\rightsquigarrow$ Good-deal hedging $\neq$ Super-replication

Example: a No-Speculation Result for Hedging

Large drift but no volatility uncertainty $\rightsquigarrow$ no speculation in the hedge, but Föllmer/Schweizer risk-minimization:

- **Def:** Strategy $\hat{\phi}$ is risk-minimizing for claim X under $Q \in \mathcal{M}^e$ if

$$\hat{\phi} = \operatorname*{argmin}_{\phi \in \Phi} E_t^Q \left[\left(L_T^{\phi} - L_t^{\phi} \right)^2 \right], \quad \forall t \leq T.$$

- Let (Y, Z) be unique solution to following (classical) BSDE under P_0:

$$-dY_t = \left(-\xi_t^{\mathrm{tr}} \Pi_t(Z_t) + h_t \left| \Pi_t^{\perp}(Z_t) \right| \right) dt - Z_t^{\mathrm{tr}} dB_t, \quad Y_T = X.$$

- Have $\mathcal{R} \cap \mathcal{M}^e \neq \emptyset$ **iff** $\pi_\cdot^u(X) = Y$ and $\bar{\phi} = \Pi(Z)\ \forall X$.
  $\rightsquigarrow$ No speculative hedging component if drift uncertainty is large.

- Robust good-deal hedging strategy $\bar{\phi}$ is then risk-minimizing under worst-case measure $\bar{Q}(X) \in \mathcal{Q}^{\mathrm{ngd}}(P^{\bar{\theta}})$ w.r.t. worst-case model $P^{\bar{\theta}}$.

Conclusions

Valuation and hedging under combined drift and volatility uncertainty.

1. Good-deal approach yields less expensive valuations and less extreme hedges than quasi-sure superreplication.
2. Hedging strategies are at-least-mean-self-financing, uniformly over all a-priori valuation measures wrt. all (uncertain, ambiguous) reference probability priors.
3. Valuations and Hedges fully characterized by the solution to a 2nd-order BSDEs for general claims (measurable, no continuity conds). By building on 2BSDE theory from D.Possamai,X.Tan,C.Zhou (AAP,2018) for 2BSDEs whose generator is not continuous or convex.
4. Combined uncertainty about drift and volatility is more complex but matters !

t h a n k   y o u

1) D.B., Klebert Kentia: Good deal hedging and valuation under combined uncertainty about drift and volatility.
Probability, Uncertainty and Quantitative Risk 2:13, 2018
https://doi.org/10.1186/s41546-017-0024-5

2) D.B., Klebert Kentia: Hedging under generalized good-deal bounds and model uncertainty.
Mathematical Methods of Operations Research 86(1), 2017+
http://dx.doi.org/10.1007/s00186-017-0588-y
(on dominated uncertainty, many (semi-)closed-form examples, no-speculation result)