Parameter Uncertainty in the Kalman–Bucy Filter

Andrew Allan
Joint work with Samuel Cohen

Oxford University

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Suppose we have a hidden Markov process $X$, that affects another process $Y$ which we can see. Stochastic filtering is the problem of estimating the current value of $X$, from our observations of $Y$. That is, we are interested in the distribution of $X_t|Y_t$, where $Y_t=\sigma(Y_s: 0 \leq s \leq t)$ is all the information we have about $Y$ up to time $t$. As in any Bayesian framework, we recursively update this distribution as we make new observations.
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That is, we assume the signal $X$ and observation $Y$ satisfy

$$
\begin{align*}
\mathrm{d}X_t &= \alpha_t X_t \, \mathrm{d}t + \sqrt{\beta_t} \, \mathrm{d}B_t, \\
\mathrm{d}Y_t &= c_t X_t \, \mathrm{d}t + \mathrm{d}W_t,
\end{align*}
$$

with the initial conditions $Y_0 = 0$ and $X_0 \sim \mathcal{N}(\mu_0, \sigma_0^2)$. 
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In this case the posterior distribution of $X_t$ is Gaussian, i.e. $X_t|Y_t \sim N(q_t, R_t)$. 
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$$dX_t = \alpha_t X_t \, dt + \sqrt{\beta_t} \, dB_t,$$
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with the initial conditions $Y_0 = 0$ and $X_0 \sim N(\mu_0, \sigma_0^2)$.

In this case the posterior distribution of $X_t$ is Gaussian, i.e. $X_t|Y_t \sim N(q_t, R_t)$. Moreover, the conditional mean $q_t$ and variance $R_t$ satisfy the equations

$$dq_t = \alpha_t q_t \, dt + c_t R_t (dY_t - c_t q_t \, dt),$$
$$\frac{dR_t}{dt} = \beta_t + 2\alpha_t R_t - c_t^2 R_t^2.$$

This is the standard Kalman–Bucy filter.
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In practice these parameters must be estimated from data, which introduces statistical uncertainty.

This is problematic, as stochastic filters are generally sensitive to perturbations in the parameters.

Thus, our goal is to construct filters which are robust to parameter uncertainty.
Filtering
Nonlinear expectations
Change of variables
A control problem
Numerical example
Future extensions

\[ X_t \]
\[ \mathbb{E}^{\text{true}}[X_t] \]
\[ \mathbb{E}^{\text{est}}[X_t] \]

<table>
<thead>
<tr>
<th></th>
<th>( \alpha )</th>
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We could consider the (essential) supremum over all choices of parameters:

\[
\text{ess sup}_{\alpha,\beta,\mu_0,\sigma_0} \mathbb{E}^{\alpha,\beta,\mu_0,\sigma_0}[\varphi(X_t) \mid \mathcal{Y}_t],
\]

which corresponds to taking a **sublinear expectation** of \( \varphi(X_t) \).
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We can calculate estimates $E^{\alpha, \beta, \mu_0, \sigma_0}[\varphi(X_t) \mid Y_t]$ for any choice of parameters, but we don’t know which choice is the correct one.

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This is rather pessimistic, as we consider all possible models equally, including very implausible ones.

We propose that one can do better by using a convex expectation, which amounts to introducing a penalty term:

$$\mathcal{E}(\varphi(X_t) \mid Y_t) = \text{ess sup}_{\alpha, \beta, \mu_0, \sigma_0} \left\{ E^{\alpha, \beta, \mu_0, \sigma_0}[\varphi(X_t) \mid Y_t] - \text{‘penalty’}^{\alpha, \beta, \mu_0, \sigma_0} \right\}.$$
More precisely, we define

\[
\mathcal{E}(\varphi(X_t) \mid \mathcal{Y}_t) = \text{ess sup}_{\alpha, \beta, \mu_0, \sigma_0} \left\{ \mathbb{E}^{\alpha, \beta, \mu_0, \sigma_0}[\varphi(X_t) \mid \mathcal{Y}_t] - \left( \frac{1}{k_1} \left( \int_0^t \gamma(s, \alpha_s, \beta_s) \, ds + \kappa_0(\mu_0, \sigma_0^2) \right) \right)^{k_2} \right\},
\]

where \(\gamma\) and \(\kappa_0\) are penalty functions, and the parameters \(k_1\) and \(k_2\) can be chosen depending on how averse we are to uncertainty. The functions \(\gamma, \kappa_0\) are nonnegative, and equal to zero for our a\priori estimate of the parameters. In particular, the standard Kalman–Bucy filter is recovered in the limit as \(k_1 \to 0\).
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In filtering we perform estimation based on observations of $Y$. Therefore, we actually want to work with a fixed realisation $y$ of $Y$. 

Rearranging things a bit, for any such realisation $y$, we have that

$\mathbb{E}(\phi(X_t) | y) = \sup_{(\mu,\sigma^2) \in \mathbb{R} \times (0,\infty)} \{ \mathbb{E}_{\mu,\sigma^2}[\phi(X_t) | y] - \left(\frac{1}{k^2} \kappa_t(\mu,\sigma^2 | y)\right)^2 \}$

where $X_t \sim \mathcal{N}(\mu,\sigma^2)$ under $\mathbb{E}_{\mu,\sigma^2}$, and

$\kappa_t(\mu,\sigma^2 | y) = \inf \{ \int_0^t \gamma(s,\alpha_s,\beta_s) \, ds + \kappa_0(\mu_0,\sigma_0^2) \ | \ alpha, beta, \mu_0, \sigma_0 \}$

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\kappa_t(\mu, \sigma^2 \mid y) = \inf \left\{ \int_0^t \gamma(s, \alpha_s, \beta_s) \, ds + \kappa_0(\mu_0, \sigma_0^2) \mid \alpha, \beta, \mu_0, \sigma_0 \text{ such that } (q(y)_t, R_t) = (\mu, \sigma^2) \right\}.
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where \( q \) and \( R \) satisfy

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We recognise this as an **optimal control problem**, where \( \kappa \) is the value function, \( q, R \) are the state trajectories, and \( \alpha, \beta \) are the controls.
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Note that this control problem is posed ‘backwards in time’.
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Moreover, since \( y \) is a realisation of the stochastic process \( Y \), this is actually a pathwise stochastic control problem.
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where \( q \) and \( R \) satisfy
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Note that this control problem is posed ‘backwards in time’.

Moreover, since \( y \) is a realisation of the stochastic process \( Y \), this is actually a \textbf{pathwise stochastic} control problem. In particular, \( \kappa \) has ‘Brownian-like’ regularity in time.
We transformation the state trajectories via

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where the function \( f \) is given by

\[ f(x, t, a, b) = \left( -(x_1 + \eta_t)(a + bx_2), \right. \]
\[ \left. -bx_2^2 - 2ax_2 + c_t^2 \right). \]
We also transform the value function via

\[ v(x, t) = \kappa_t(\mu, \sigma^2 | y), \]

where

\[ x = \left( \frac{\mu}{\sigma^2} - \eta t, \frac{1}{\sigma^2} \right). \]
We thus obtain a deterministic control problem:

\[
\nu(x, t) = \inf_{\alpha, \beta} \left\{ \int_0^t \gamma(s, \alpha_s, \beta_s) \, ds + \nu_0(w(0)) \right\},
\]

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$$v(x, t) = \inf_{\alpha, \beta} \left\{ \int_0^t \gamma(s, \alpha_s, \beta_s) \, ds + v_0(w(0)) \right\},$$

where

$$\frac{dw}{ds}(s) = f(w(s), s, \alpha_s, \beta_s).$$

The problem is still somewhat degenerate, as the ODE satisfied by $w$ is not Lipschitz. We must therefore work to avoid ‘blow ups’ in the state trajectories.
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where

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\frac{d\nu}{ds}(s) = f(w(s), s, \alpha_s, \beta_s).
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The problem is still somewhat degenerate, as the ODE satisfied by \( w \) is not Lipschitz. We must therefore work to avoid ‘blow ups’ in the state trajectories.

Moreover, the value function itself blows up at the boundary of its domain, as extreme values of the posterior mean and variance are considered to be very implausible.
Nonetheless, we establish a dynamic programming principle, and can then prove:
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**Theorem**

The value function \( v \) is the unique viscosity solution of the Hamilton–Jacobi–Bellman (HJB) equation

\[
\frac{\partial v}{\partial t}(x, t) + \sup_{a, b} \{f(x, t, a, b) \cdot \nabla v(x, t) - \gamma(t, a, b)\} = 0
\]

which blows up at the boundary.
Consider again the parameters

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We simulate the signal $X$ using the true parameters, and simulate the observation $Y$ using $c = 1$.

We use the penalty functions given by

\[
\gamma(t, a, b) = 5(a - \alpha^*)^2 + 10(b - \beta^*)^2, \\
v_0(x_1, x_2) = 15(x_1 - x_1^*)^2 + 15(x_2 - x_2^*)^2,
\]

where $x_1^* = \frac{\mu_0^*}{(\sigma_0^*)^2}$ and $x_2^* = \frac{1}{(\sigma_0^*)^2}$.
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<td>$\sigma_0^* = 1$</td>
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We simulate the signal $X$ using the true parameters, and simulate the observation $Y$ using $c = 1$.

We use the penalty functions given by

$$
\gamma(t, a, b) = 5(a - \alpha^*)^2 + 10(b - \beta^*)^2,
$$

$$
\nu_0(x_1, x_2) = 15(x_1 - x_1^*)^2 + 15(x_2 - x_2^*)^2,
$$

where $x_1^* = \frac{\mu_0^*}{(\sigma_0^*)^2}$ and $x_2^* = \frac{1}{(\sigma_0^*)^2}$.

We then numerically solve the HJB equation to find the value function $\nu$. 
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Change of variables
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Having solved for $v$, we can reverse the change of variables we did earlier to obtain $κ$. Recalling the relation

$$E(\phi(X_t)|y) = \sup_{(\mu,\sigma) \in \mathbb{R} \times (0,\infty)} \left\{ E_{\mu,\sigma}^2[\phi(X_t)|y] - \left(\frac{1}{k_1} \kappa_t(\mu,\sigma)^2|y)\right)^{k_2}\right\},$$

we are now in a position to calculate robust estimates of the signal. In the following we take $k_1 = 10$ and $k_2 = 5$. 
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\[
\mathcal{E}(\varphi(X_t) \mid y) = \sup_{(\mu, \sigma) \in \mathbb{R} \times (0, \infty)} \left\{ \mathbb{E}^{\mu, \sigma^2} [\varphi(X_t) \mid y] - \left( \frac{1}{k_1} \kappa_t(\mu, \sigma^2 \mid y) \right)^{k_2} \right\},
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In the following we take \( k_1 = 10 \) and \( k_2 = 5 \).
Here the robust estimate is calculated as

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\[ \arg \min_{\xi} E \left( \left( (X_t - 1)^+ - \xi \right)^2 \mid \mathcal{Y}_t \right). \]
We also calculate the ‘upper’ and ‘lower’ expectations of the signal, i.e. \( \mathcal{E}(X_t | \mathcal{Y}_t) \) and \( -\mathcal{E}(-X_t | \mathcal{Y}_t) \) respectively.
Finally, we calculate upper and lower expectations of \((X_t - 1)^+\).
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• considering uncertainty in the parameter \( c \),
• including cases with correlation between the signal noise and observation noise,
• allowing the nonlinear expectation term to 'learn' from new observations, and update the penalty terms accordingly.
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Recall the filter dynamics

\begin{align*}
\, dq_t &= \alpha_t q_t \, dt + c_t R_t (dY_t - c_t q_t \, dt), \\
\frac{dR_t}{dt} &= \beta_t + 2\alpha_t R_t - c_t^2 R_t^2.
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Allowing either correlated noise or ‘learning’ leads to the same phenomenon.
The end