On the Martingale Selection Problem and its applications

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based on joint works with M. Šikíc and E. Bayraktar

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Outline

- The martingale selection problem
  - Characterization of the existence of solutions;
- Applications to arbitrage theory
  - Frictionless models;
  - Proportional transaction costs;
  - Illiquidity models.
- Applications to superhedging duality
- Conclusions

THE MARTINGALE SELECTION PROBLEM

The Problem

Let $V := (V_t)_{t\in\mathcal{I}}$ be a collection of $\mathcal{F}_t$-measurable random sets, with $\mathcal{I} := \{0, \dots, T\}$.

Problem

Find a process $\xi := (\xi_t)_{t\in\mathcal{I}}$ and a probability measure $\mathbb{Q}$ such that:

- $\xi_t \in V_t$ *for any* $t \in \mathcal{I}$;
- *for any* $0 \leq t \leq T - 1$,

$$\mathbb{E}[\xi_{t+1} - \xi_t \mid \mathcal{F}_t] = 0 \quad \mathbb{Q} - a.s.$$

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- $\mathbb{Q}$ *has finite support;*
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The Problem

Let $V := (V_t)_{t\in\mathcal{I}}$ and $C := (C_t)_{t\in\mathcal{I}}$ be collections of $\mathcal{F}_t$-measurable random sets, with $\mathcal{I} := \{0, \dots, T\}$.

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- *$\xi_t \in V_t$ for any $t \in \mathcal{I}$;*
- *for any $0 \le t \le T-1$,*

$$\mathbb{E}[\xi_{t+1} - \xi_t \mid \mathcal{F}_t] \in C_t^\circ \quad \mathbb{Q} - a.s.$$

Here, C_t° denotes the polar set of C_t, i.e.,

$$C_t^\circ := \{y \in \mathbb{R}^d \mid \langle c, y\rangle \le 0, \ \forall c \in C_t\};$$

The program

Aim: Give a geometric characterization for the existence of solutions to the $\textsc{Msp}$ (V, C);

Approach: We follow dynamic programming ideas to identify the subsets of V that can support a martingale process;

Example: One period deterministic $\textsc{Msp}$: $V_0 = [1, 3]$, $V_1 = [2, 4]$. The interval $[1, 2)$ cannot support any martingale process.

Meta-result: $\textsc{Msp}$ is solvable if and only if a certain collection $(W_t)_{t \in \mathcal{I}}$ with $W_t \subset V_t$ for any $t \in \mathcal{I}$ is non-empty.

The approach

Consider a martingale selection problem (V, C). Recall that we want the martingale property:

$$\mathbb{E}[\xi_{t+1} - \xi_t \mid \mathcal{F}_t] \in C_t^\circ \quad \Leftrightarrow \quad \xi_t = \mathbb{E}[\xi_{t+1} \mid \mathcal{F}_t] - \eta_t.$$

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Backward construction

- Start with $W_T := V_T$;
- for $t = T-1, \ldots, 0$, set

$$W_t := V_t \cap \left(W_{t+1}^\sharp - C_t^\circ\right).$$

where

$$W_{t+1}^\sharp(\omega) = \left\{\mathbb{E}_\mathbb{Q}[\xi|\mathcal{F}_t](\omega) \mid \mathbb{Q} \in \mathcal{P},\ \xi \in W_{t+1}\ \mathbb{Q}\text{-a.s.}\right\}$$

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Backward construction (B., Šikíc ('18))

- Start with $W_T := V_T$;
- for $t = T-1, \ldots, 0$, set

$$W_t := V_t \cap \left(W_{t+1}^\flat - C_t^\circ \right).$$

where

$$W_{t+1}^\flat(\omega) = \left\{ y \in \mathbb{R}^d \,\middle|\, \forall \bar{\omega} \in \Omega \; \exists \, \mathbb{Q} \in \mathcal{P}(\bar{\omega}), \xi \in W_{t+1} \; \mathbb{Q}\text{-a.s.} \right. \\ \left. : \mathbb{E}_{\mathbb{Q}}[\xi | \mathcal{F}_t](\omega) = y \right\}$$

The main result

The (robust) MSP (V, C) is *solvable* if $\forall \bar{\omega} \in \Omega$ there exist $(\xi, \mathbb{Q})$ such that

- $\xi_t \in V_t$ for $t \in \mathcal{I}$;
- $\bar{\omega} \in \operatorname{supp} \mathbb{Q}$;
- $\mathbb{E}_Q[\xi_{t+1} - \xi_t | \mathcal{F}_t] \in C_t^{\circ}$ $\quad Q$-a.s., for all $0 \leq t \leq T-1$,

Theorem (B., Šikíc ('18))

The martingale selection problem (V, C) *is solvable if and only if* $W_t(\omega) \neq \varnothing$ *for all* $t \in \mathcal{I}$ *and* $\omega \in \Omega$.

APPLICATIONS TO NO ARBITRAGE THEORY

We want to obtain a version of the Fundamental Theorem of Asset Pricing in a general framework.

Theorem (Meta-FTAP)

No arbitrage $\iff$ *exists a consistent price system.*

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This can be translated into

Theorem (Meta-Theorem)

No arbitrage $\iff$ *a suitable* MSP (V, C) *is solvable.*

In particular, the set of consistent price system is given by the set of "martingales" living in V.

FRICTIONLESS MARKETS

Frictionless markets

Model: price process $(S_t)_{t\in\mathcal{I}}$, set of constraints $(C_t)_{t\in\mathcal{I}}$;

NA: $(H\cdot S)_T \geq 0$ with H admissible $\Rightarrow (H\cdot S)_T = 0$;

MSP: $V_t = \text{cone}(S_t),\ C_t$ given.

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Theorem

NA *if and only if for every* $\omega\in\Omega$, *there exists* $\mathbb{Q}$ *such that*

- $\omega \in \operatorname{supp}\mathbb{Q}$;
- $E_{\mathbb{Q}}[S_{t+1} - S_t|\mathcal{F}_t] \in C_t^{\circ}, \quad \mathbb{Q}$*-a.s.*.

Frictionless markets

Proof: consider the $\textsc{Msp}$,

$$V_t = \text{cone}(S_t), \qquad C_t \text{ given.}$$

We have two cases:

Solvable: martingale measures exist $\Rightarrow \mathrm{NA}$;

Non Solvable: For some $\bar{\omega} \in \Omega$ and $t \in \mathcal{I}$, we have

$$W_t(\bar{\omega}) = V_t \cap (W^{\flat}_{t+1} - C^{\circ}_t) = \varnothing.$$

One can show that a separator of the two sets defines an arbitrage strategy.

PROPORTIONAL TRANSACTION COSTS

The currency market model of Kabanov

Model: solvency cone process $(K_t)_{t\in\mathcal{I}}$, set of constraints $(C_t)_{t\in\mathcal{I}}$;

NA: $h_T \in \mathcal{L}(\mathcal{F}_T;\mathbb{R}^d_+)$ admissible $\Rightarrow h_T = 0$;

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Theorem

"NA" if and only if for every $\omega\in\Omega$, there exists $(\xi,\mathbb{Q})$ such that

- $\xi_t \in \mathrm{ri}(K_t^*)$ *for every* $t\in\mathcal{I}$;
- $\omega \in \mathrm{supp}\,\mathbb{Q}$;
- $E_{\mathbb{Q}}[\xi_{t+1}-\xi_t|\mathcal{F}_t] \in C_t^\circ$, $\quad\mathbb{Q}$-*a.s.*.

ILLIQUIDITY MODELS

The illiquidity model of Pennanen '11

Model: cost process $(S_t)_{t\in\mathcal{I}}$, set of constraints $(C_t)_{t\in\mathcal{I}}$;

- Example: $S_t(\omega,x) = s_t(\omega)\varphi(x)$ where φ is the "cost of illiquidity" (see Çetin and Rogers '07);
- Value function: $\mathcal{V}_T(H) = -\sum_{t=0}^{T} S_t(\omega, H_t - H_{t-1})$.

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NA: $\mathcal{V}_T(H) \geq 0$ with H admissible $\Rightarrow \mathcal{V}_T(H) = 0$;

- Note that scalability matters!

The illiquidity model of Pennanen

MSP: $V_t = \text{ri}\, \partial S_t^\infty(\cdot, 0)$, C_t given,

$$S_t^\infty(\cdot, x) := \sup_{\alpha>0} \frac{S_t(\cdot, \alpha x)}{\alpha}.$$

Interpretation: worst sublinear model compatible with S.

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Theorem

No scalable arbitrage if and only if for every $\omega \in \Omega$, there exists $(\xi, \mathbb{Q})$ such that

- $\xi_t(\omega) \in \text{ri}\,\partial S^\infty(\omega, 0)$ *for every* $t \in \mathcal{I}$ *and* $\omega \in \Omega$*;*
- $\omega \in \text{supp}\,\mathbb{Q}$*;*
- $E_{\mathbb{Q}}[\xi_{t+1} - \xi_t | \mathcal{F}_t] \in C_t^\circ$, $\quad \mathbb{Q}$*-a.s..*

SUPERHEDGING DUALITY

The randomization approach

The idea: Model the transaction costs as an extra state variable (see Bouchard, Deng, Tan, *Math. Fin.*, '18).

1. Define the frictionless process

$$\hat{S}_t(\omega,\theta) = \Pi_{K_t^*}([S_t^1(\omega)\theta^1,\ldots,S_t^d(\omega)\theta^d]$$

   for a Borel-measurable selector S_t of K_t^*;
2. Show that the primal problems coincides for the original and the extended market;
3. Show the same for the dual;
4. Use frictionless results for showing the duality.

The randomization approach

The assumptions of Bouchard et al.:

1. $\mathcal{P}$-q.s. framework of Bouchard and Nutz;
2. $K_t^* \cap \partial\mathbb{R}^d_+ = \{0\}$;
3. Uniform bound on transaction costs;
4. No Arbitrage of the second kind:

$$\xi \in K_{t+1} \ \mathcal{P}\text{-q.s.} \quad \Rightarrow \quad \xi \in K_t \ \mathcal{P}\text{-q.s.}$$

equivalently: every process taking values in $(K_t^*)_{t\in\mathcal{I}}$ can be extended as a martingale.

Our contribution

Denote by $\mathcal{S}^0$ the class of strictly consistent price systems. Moreover,

$$\pi_K(G) := \inf\left\{y \in \mathbb{R} \mid \exists H \text{ adm. s.t. } y\mathbf{1}_d + H_T - G \in K_T, \quad \mathcal{P}\text{-q.s.}\right\}.$$

Theorem (Bayraktar, B. ('18+))

Assume "NA" for $(K, \mathcal{P})$. For any Borel-measurable random vector G,

$$\pi_K(G) = \sup_{(Z,\mathbb{Q}) \in \mathcal{S}^0} \mathbb{E}^{\mathbb{Q}}[G \cdot Z_T].$$

Moreover, the superhedging price is attained when $\pi_K(G) < \infty$.

Our contribution

Main difficulty: The collection of sets

$$\tilde{K}_t^* := K_t^* \cap K_{t+1}^\sharp$$

is not Borel measurable so one cannot replicate the construction of Bouchard et al.;

The idea: Choose an appropriate family of priors $\hat{\mathcal{P}}$ such that

$$\hat{P}\left(\hat{S}_t \in \mathrm{int}\left(\tilde{K}_t^*\right),\ \forall t \in \mathcal{I}\right) = 1,\ \forall\ \hat{P} \in \hat{\mathcal{P}};$$

Generalization: A duality result can be derived for the case of portfolio constraints (with $\mathcal{S}^0$ suitably replaced).

CONCLUSIONS

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- We have considered a martingale selection problem in absence of a reference probability and characterize existence of solutions;
- We provided a FTAP for various market models, from frictionless to general illiquidity frameworks;
- We showed how one can derive the superhedging duality, via randomization, through the solutions of the MSP.

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- We have considered a martingale selection problem in absence of a reference probability and characterize existence of solutions;
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Thank you for your kind attention.