Convergence to the Mean Field Game Limit: A Case Study

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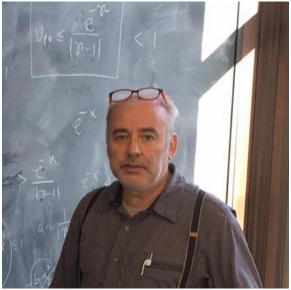
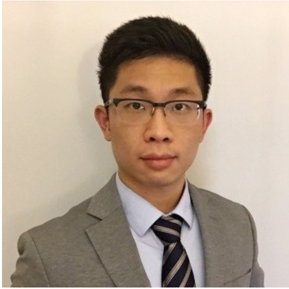
Joint Work with



Jaime San Martin



Xiaowei Tan

Outline

Mean Field Games

- Introduced by Lasry–Lions and Huang–Malhamé–Caines, 2006
- Nash equilibria for $n \to \infty$ players
- Interaction through empirical distribution of the private states

Games of Optimal Stopping (Timing):

- Agents aim to stop optimally
- Interaction through proportion of players that have already stopped
- Guiding idea: bank-run models as in Diamond–Dybvig
- Bertucci, Carmona–Delarue–Lacker, N.

Connecting Mean Field Game and n-Player Game

Convergence Backward:

- Mean field equilibria induce approximate n-player Nash equilibria.
  Huang–Malhame–Caines, Lacker, Carmona–Delarue–Lacker, Cecchin–Fischer, Campi–Fischer, ...
- Unclear if these are close/similar to actual equilibria

Convergence Forward:

- Under monotonicity condition, in particular uniqueness, all (closed-loop) n-player equilibria converge to mean field equilibrium.
  Cardaliaguet–Delarue–Lasry–Lions
- In general, n-player equilibria converge to weak mean field equilibria. Includes mixtures. Lacker, Carmona–Delarue–Lacker, Fischer, Lacker, ...
- Here: if there are multiple mean field equilibria, are they limits of n-player equilibria? Are they "justified" by the n-player game?

Notion of Equilibrium

Agent space $(I, \mathcal{I}, \lambda)$, either $I = \{1, \dots, n\}$ or $I = [0, 1]$, λ uniform

- Each agent i solves an optimal stopping problem: τ^i
- Compute proportion $\rho_t^{-i} = \lambda\{j \neq i : \tau^j \leq t\}$ of other agents that have stopped
- Optimal stopping problem depends on ρ_t^{-i}: fixed point
- An Nash equilibrium consists of $\rho_t = \lambda\{i : \tau^i \leq t\}$ and $(\tau^i)_{i \in I}$

The Single-Agent Problem

Optimal stopping problem:

$$\sup_{\tau \in \mathcal{T}} E\left[e^{r\tau}\mathbf{1}_{\{\theta>\tau\}\cup\{\theta=\infty\}}\right].$$

- r is an interest rate
- θ is the default of the bank
- θ comes as a surprise, but has an observed subjective intensity γ^i
- First jump of a Cox process: $\theta = \inf\{t : \int_0^t \gamma_s^i\, ds = \text{Exp}\,(1)\}$.

Specification in this Talk

- Intensities

$$\gamma_t^i = Y_t^i + c\rho_t^{-i}, \quad \rho_t^{-i} = \lambda\{j \neq i : i : \tau^j \leq t\}$$

- Y_t^i are i.i.d., increasing, right-continuous processes
- $F_t(y) := P\{Y_t^i \leq y\}$ the continuous c.d.f. at time t
- Solution of single-agent problem:

$$\tau^i = \inf\{t : \; Y_t^i + c\rho_t^{-i} \geq r\} \quad \text{(assume } < \infty\text{)}$$

- Unique e.g. if Y^i is strictly increasing
- Assume all agents use this stopping rule

Multiplicity of Equilibria:

- If everybody stops, you also want to stop (and vice versa)
- "Strategic complementarity"
- Multiple equilibria arise naturally as agents coordinate

Outline

Equilibria of the n-Player Game

- If ρ_n is an n-player equilibrium and $\rho_n(t)(\omega) = k/n$, then

$$\#\{Y_t^i(\omega) + c \cdot \frac{k-1}{n} \geq r\} = k \quad \text{and}$$
$$\#\{Y_t^i(\omega) + c \cdot \frac{k}{n} < r\} = n - k.$$

- This is also sufficient for the existence of ρ_n

Minimal and Maximal Equilibria

Theorem: There exists an n-player equilibrium ρ_n^m such that

$$\rho_n^m(t) = \frac{k}{n} \Longleftrightarrow \begin{cases} \#\{Y_t^i + c \cdot \frac{k}{n} \geq r\} = k \\ \#\{Y_t^i + c \cdot \frac{k-l}{n} \geq r\} \geq k - l + 1, \quad 1 \leq l \leq k. \end{cases}$$

This equilibrium is minimal: $\rho_n^m(t) \leq \rho_n(t)$ $\forall$ n-player equilibrium ρ_n.

- Similarly, there exists a maximal equilibrium ρ_n^M
- The set of all equilibria $\rho_n(t) = \#\{i : \tau^i \leq t\}/n$ can be constructed recursively:

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Recursive Construction

1. Suppose that at time τ_0, a group $K \subsetneq I$ of agents has already stopped. Then every remaining agent $i \notin K$ examines her criterion

$$\theta^i_K = \inf\{t:\ Y^i_t + c \cdot \frac{\#K}{n} \geq r\}.$$

   If $\theta^i_K \leq \tau_0$, then player i must stop immediately. We add i to the set K and repeat 1. until no further players are forced to stop. (Order does not matter.)

2. A group $J \subseteq K^c$ may be able to stop together. Indeed, suppose that

$$\theta^J_K = \inf\{t:\ Y^i_t + c \cdot \frac{\#K + \#J - 1}{n} \geq r\}$$

   satisfies $\theta^J_K \leq \tau_0$ for all $i \in J$. Then it is optimal for all these agents to stop together, but they do not have to. If they stop, we add J to K and repeat from 1.

Recursive Construction Cont'd

3. After all remaining groups of agents have decided whether to stop at time τ_0, we increment time until there exists a group or individual agent wanting to stop, and start again at 1.

- Multiplicity of equilibria arises because of the choices taken by the groups J
- "Always no" leads to ρ_n^m, "always yes" leads to ρ_n^M

Outline

Mean Field Game Equilibria

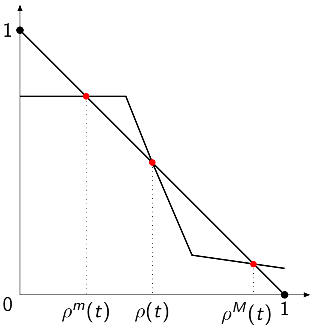
- Note $\rho^{-i}(t) = \rho(t)$ and recall $\tau^i = \inf\{t : Y_t^i + c\rho(t) \geq r\}$
- Fix $t \geq 0$. If $\rho(t)$ is an equilibrium,

$$\begin{aligned}\rho(t) = \lambda\{i : \tau^i \leq t\} &= \lambda\{i : Y_t^i + c\rho(t) \geq r\} \\ &= P\{Y_t^i + c\rho(t) \geq r\} \\ &= P\{Y_t^i \geq r - c\rho(t)\} \\ &= 1 - F_t(r - c\rho(t))\end{aligned}$$

$\Rightarrow$ Fixed point equation for $u = \rho(t)$:

$$F_t(r - cu) = 1 - u$$

Characterization of Mean Field Equilibria



Theorem: A real function $\rho : \mathbb{R}_+ \to [0,1]$ is a mean field game equilibrium if and only if it is increasing, right-continuous and

$$F_t(r - c\rho(t)) = 1 - \rho(t), \quad t \geq 0.$$

There exist minimal and maximal equilibria ρ_+^m, ρ^M.

Outline

Limits of n-Player Equilibria

Theorem:
Let $t \geq 0$ and $\mathcal{U}(t) = \{u : 1 - u = F_t(r - cu)\}$. If $(\rho_n)_{n\geq 1}$ are n-player equilibria, $(\rho_n(t))$ is asymptotically concentrated on $\mathcal{U}(t)$.

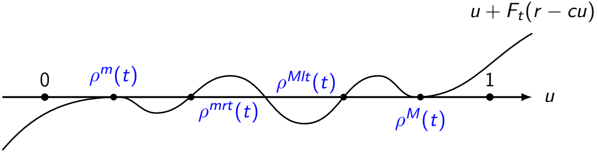
(I.e., any weak cluster point of $(\rho_n(t))$ is concentrated on $\mathcal{U}(t)$.)

Corollary:
If the mean field game has a unique equilibrium, any sequence of n-player equilibria converges to it.

- "Limits of n-player equilibria are (randomized) mean field equilibria"
- Converse?

Limit of the Minimal n-Player Equilibrium

Lemma: Let $t \geq 0$. The equation $u + F_t(r - cu) = 1$ has the solutions:



Theorem: For all $t \geq 0$, $\rho_n^m(t)$ is asymptotically concentrated on

$$\mathcal{U}(t) \cap [\rho^m(t), \rho^{mrt}(t)].$$

The Good (and Generic) Case

Theorem: Assume that $\rho^m(t)$ is not a local max, for a dense set of t.

Then the minimal n-player equilibrium ρ_n^m "Fatou converges" in probability to the minimal mean field equilibrium $\rho^m(t+)$.

- Assumption is "generic"
- Cannot have convergence at every t
- Right-continuity might be a philosophical matter in the first place
- Similar result for the maximal equilibrium

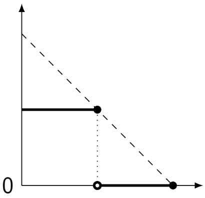
A Bad Case

Example: Let $r = c = 1$ and let Y^i_t be i.i.d. increasing processes such that $\text{Law}(Y^i_t) = \frac{1}{2}\delta_{\frac{1}{2}} + \frac{1}{2}\delta_2$ for all $0 \leq t < T$ (and $Y^i_t > r$ later). Then

$$\text{Law}((\rho^m_n(t)) \to \frac{1}{2}\delta_{\frac{1}{2}} + \frac{1}{2}\delta_1, \quad t < T.$$

- Here $\rho^m(t) \equiv 1/2$ and $\rho^{mrt}(t) \equiv 1$
- The limit is a mixture of these equilibria

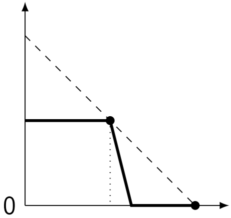
Corollary: $\rho^m(t)$ is not the limit of n-player equilibria

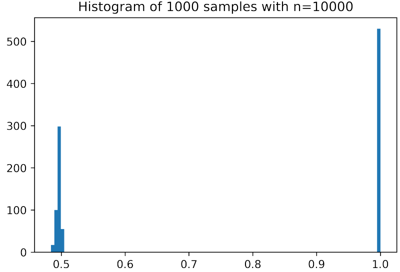


Bad Case with Density

Example: As above, but with density $f(y) = 4\,\mathbf{1}_{[\frac{3}{8},\frac{1}{2}]}(y) + \mathbf{1}_{[\frac{3}{2},2]}(y)$.

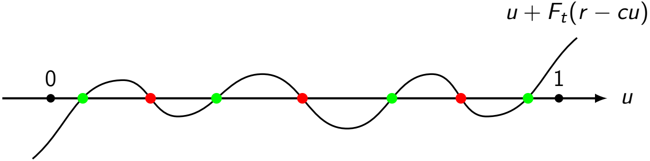
- Again, $\rho^m(t) \equiv 1/2$ and $\rho^{mrt}(t) \equiv 1$
- The limit is a mixture of these equilibria





Outline

Interior Equilibria



- We exclude the "tangential" case (positive and negative examples)

Increasing-Transversal Equilibria:

Theorem: Let ρ be a mean field equilibrium. Suppose that for all t in a dense subset $D \subseteq \mathbb{R}_+$, the solution $x := \rho(t)$ is increasing-transversal. Then there exist n-player equilibria $(\rho_n)_{n\geq 1}$ which Fatou converge in probability to ρ.

Decreasing-Transversal Equilibria

- Assume that F_t admits a continuous density f_t
- Call a solution x of $u + F_t(r - cu) = 1$ strongly decreasing-transversal if $\partial_u|_{u=x}[u + F_t(r - cu)] < 0$; i.e.,

$$cf_t(r - cx) > 1.$$

Theorem: Let ρ be a mean field equilibrium and suppose that the set

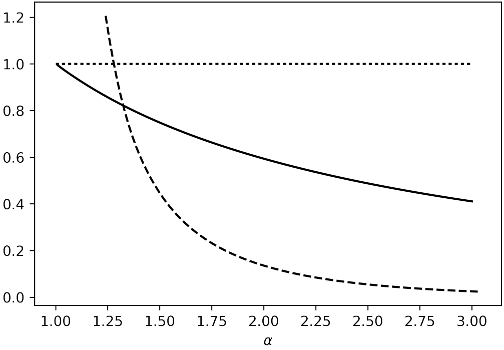
$$\{t \geq 0 : \rho(t) \text{ is strongly decreasing-transversal}\}$$

has nonempty interior. Then there does not exist a sequence of n-player equilibria ρ_n Fatou converging to ρ in probability.

Decreasing-Transversal Equilibria: Static Result

Lemma: Fix $t \geq 0$ and let $x \in [0,1]$ satisfy $x + F_t(r - cx) = 1$. If x is strongly decreasing-transversal, then

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} P(\exists\, n\text{-player equilibrium } \varepsilon\text{-close to } x) < 1.$$



Bounds depend on $\alpha := cf_t(r - cx) = 1 - \text{slope}$

Dashed: $\frac{e^{-\alpha}}{|1-\alpha|}$

Solid: $\frac{1-\theta}{\alpha-1}$ where $\theta \in (0,1)$ is defined by $\theta e^{-\theta} = \alpha e^{-\alpha}$.

Solid Bound – Crossings of Empirical C.D.F.

- Relaxing the equilibrium condition results in different problem:
- Crossings between a certain empirical c.d.f. (related to F_t) with the theoretical uniform c.d.f.
- Nair–Shepp–Klass studied the distribution of such crossings
- Their result is used to obtain the solid bound

The Annals of Probability
1986, Vol. 14, No. 3, 877–890

ON THE NUMBER OF CROSSINGS OF EMPIRICAL DISTRIBUTION FUNCTIONS

By Vijayan N. Nair, Lawrence A. Shepp and Michael J. Klass[1]

AT&T Bell Laboratories and University of California, Berkeley

Let F and G be two continuous distribution functions that cross at a finite number of points $-\infty \leq t_1 < \cdots < t_k \leq \infty$. We study the limiting behavior of the number of times the empirical distribution function G_n crosses F and the number of times G_n crosses F_n. It is shown that these variables can be represented, as $n \to \infty$, as the sum of k independent geometric random variables whose distributions depend on F and G only through $F'(t_i)/G'(t_i)$, $i = 1, \ldots, k$. The technique involves approximating $F_n(t)$ and $G_n(t)$ locally by Poisson processes and using renewal-theoretic arguments. The implication of the results to an algorithm for determining stochastic dominance in finance is discussed.

Dashed Bound – Expected Number of Equilibria Near x

Proposition: Fix $t \geq 0$ and let $x \in (0,1)$ satisfy $x + F_t(r - cx) = 1$. Let $\alpha := cf_t(r - cx) \neq 1$. Then

$$\lim_{n\to\infty} E[\#n\text{-player equilibria close to } x] = \frac{e^{-\alpha}}{|1-\alpha|}.$$

- Implies the dashed bound
- Solutions occur in a window of size $a_n/\sqrt{n}$ for any $a_n \to \infty$

- Implies $\liminf_{n\to\infty} P(\exists\, n\text{-player equilibria close to } x) > 0$
- Thus, x is part of a mixture which is itself a limit of n-player equilibria

Conclusion

- Collective choices among agents can explain multiplicity of equilibria
- n-Player equilibria converge to randomized mean field game equilibria
- Randomization may happen even for natural choices like the minimal one
- Not all mean field game equilibria are limits of n-Player equilibria
- Identification in general games?
- Related work: Cardaliaguet–Hadikhanloo, Briani–Cardaliaguet, Delarue–Tchuendom

Thank you.

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