The Term Structure of Sharpe Ratios

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Story in a nutshell

Basic questions in asset pricing in incomplete markets

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- $\mathbb{P}$-FTAP
  Harrison Kreps 1979, Delbaen Schachermayer 1994
- #EMM=$\mathcal{Q}$
  - m-stable
    Delbaen (2006):
  - (dynamic) no arbitrage interval well understood;
    Karatzas Kou 1996, El Karoui Quenez 1995
- Choice in $\mathcal{Q}$ is at $t = 0$!
  Foellmer-Schweizer 1991, Fritelli 2000

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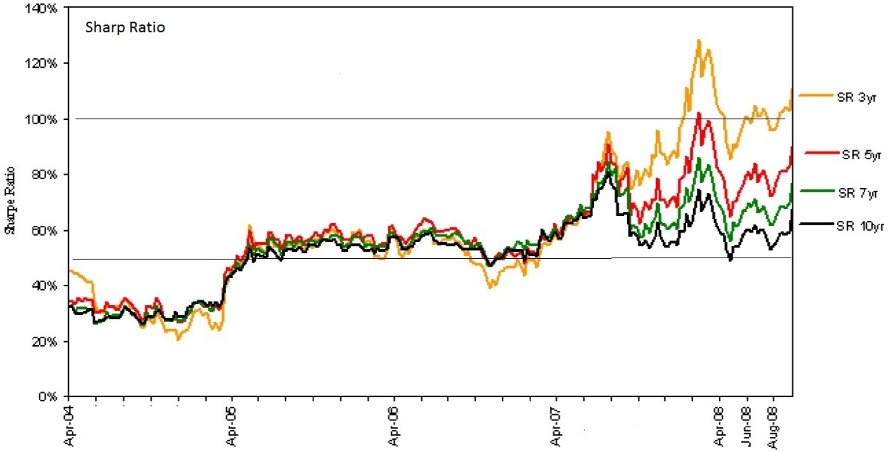
- Choice in $\mathcal{Q}$ is at $t = 0$!
  Foellmer-Schweizer 1991, Fritelli 2000
- But market employs for each maturity a different EMM!
  - New type of BSDE emerges.
  - Recent stylized fact of post GFC era.
    Van Binsbergen et. al 2012-16

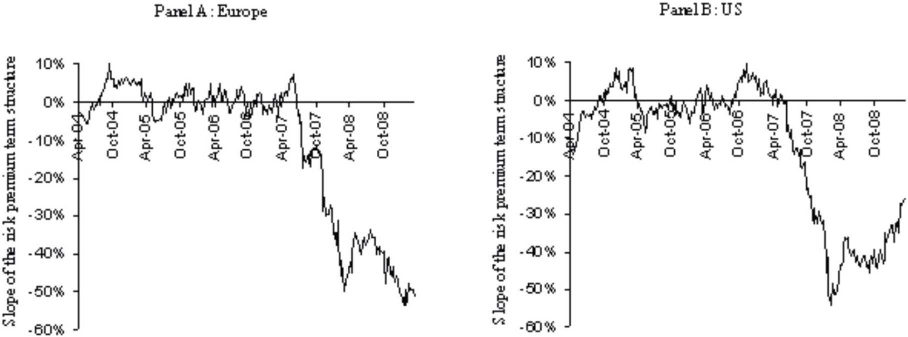
The great financial crisis I



$\theta \simeq \frac{\mu - r}{\sigma} \sim \frac{5\%}{15\%} \sim 30\%$ From Berg 2010, based on CDS data.

The great financial crisis II

slope $\approx$ SR 10yr - 3yr



Term Structures: interest rates vs. Sharpe ratios

1. Heath Jarrow Morten (1992):
   for a given forward rate $u \mapsto f_{t,u}$ the resulting **locally risk–free discount factor** $\Lambda_{t,\tau}$ in the HJM–approach:

$$\Lambda_{t,\tau} = \exp\left(-\int_t^\tau f_{t,u} \mathrm{d}u\right) \tag{1}$$

   with short rate $r_t = f_{t,t}$.

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2. Counterpart of (1) wrt the **stochastic discount factor**=SDF $\Psi_{t,\tau}$ and a given term structure of **Sharpe ratio**=SRs $u \mapsto \theta_{t,u}$:

$$\Psi_{t,\tau} = \exp\left(-\frac{1}{2}\int_t^\tau \theta_{t,u}^2 \mathrm{d}u - \int_t^\tau \theta_{t,u} \mathrm{d}B_u\right), \tag{2}$$

   SR, or market price of risk, $\frac{\mu_t - r_t}{\sigma_t} = \theta_t = \theta_{t,t}$.

Related Literature

- **BSDE's and Extensions:**

  Pardoux Peng 1990, El Karoui Peng Quenez 1997, Yong 2006, Detemple Rindisbacher 2010, Delong Imkeller 2010.

- **Term Structures:**

  HJM 1992, Schweizer Wissel 2008, Carmona Nadtochiy 2009, ..

- **Empirics:**

  Lettau Wachter 2007, Hansen et. al 2008, Van Binsbergen et. al 2012-..

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1. Introduction
2. **Reminder: complete markets – 4 cases**
3. EMM strings
4. 2 Methods: evaluation time vs. maturity 2
5. time delayed BSVIE
6. Arbitrage

Terminal Payoffs

1 SDF–Approach: The price $p_t(X)$ at time $t \in [0, T]$ of $X \in L_T = L^2(\mathbb{P})$

$$p_t(X) = E_t^{\mathbb{Q}}[X] = E_t^{\mathbb{P}}\left[\frac{\psi_T}{\psi_t} X\right],$$

with SDF $\frac{\psi_T}{\psi_t}$ where

$$\psi_t = \frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \exp\left(-\frac{1}{2}\int_0^t \theta_s^2 ds - \int_0^t \theta_s dB_s\right)$$

for some $(\theta_t)_{t\in[0,T]}$, the instantaneous SR.

2 Recursive Approach: Equivalent formulation via a linear BSDE:

$$dp_t(X) = \sigma_t dB_t^{\mathbb{Q}} = \theta_t \sigma_t dt + \sigma_t dB_t, \quad p_T(X) = X$$

Payoff Stream

3 SDF–Approach The pricing $p_t : \mathbb{L} \to L_t$ of a payoff streams $\{x_t\} \in \mathbb{L} = L^2(P \otimes dt)$:

$$p_t(x) = \int_t^T E_t^{\mathbb{P}} \left[\frac{\psi_\tau}{\psi_t} x_\tau \right] \mathrm{d}\tau$$

4 Recursive Approach: The price dynamics are given by

$$dp_t(x) = \theta_t \sigma_t - x_t dt + \sigma_t dB_t, \quad p_T(x) = 0,$$

where $\sigma_t := \sigma_t^{\mathbb{Q}}$ now depends on x, θ.

Remark

The SR only depends on one time parameter running between $[0, T]$. For pricing via EMM–strings this is no longer the case.

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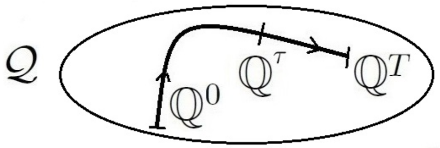
EMM strings

Definition

A function $\mathfrak{Q} : [0, T] \to \mathcal{Q}$ is called an EMM–string.



Delbaen (2006): $\mathcal{Q}$ is m-stable, or dynamic consistent, fork convex, rectangular, stable under pasting,..

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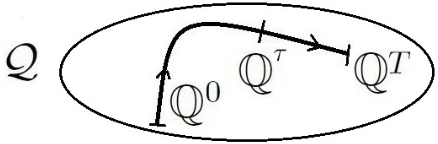

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Standing Assumption

Usual Brownian setting. Interest rate $r_t = 0$ and $(\theta_{t,\tau})_{\tau \in [t,T]}$ is

1. bounded and $t \mapsto \theta_{t,\tau}$ is Lipschitz continuous,
2. for any $t, \tau \in [0, T]$ with $t \leq \tau$, $\theta_{t,\tau} \in L_t$ and $\tau \mapsto \theta_{t,\tau}$ is prog. meas

Time inconsistencies

Example

Fix $\mathbb{Q}^0, \mathbb{Q}^1 \in \mathcal{Q}$ pricing measures of 2 gurus. Set $\psi^k = \frac{d\mathbb{Q}^k}{d\mathbb{P}}$, $k = 0, 1$. With $\alpha : [0, T] \to [0, 1]$, the EMM-string

$$\mathcal{Q}^{0,1} := \{Q_t : Q_t = \alpha_t \mathbb{Q}^0 + (1 - \alpha_t)\mathbb{Q}^1,\ t \in [0, T]\}$$

is **not m-stable**: define $\mathbb{Q}^* \notin \mathcal{Q}^{0,1}$ as $\mathbb{Q}^0$ on $[0, \tau]$ and $\mathbb{Q}^1$ on $(\tau, T]$. Moreover,

$$E_t^{\mathbb{Q}_t}[\cdot] = \rho_t E_t^{\mathbb{Q}^0}[\cdot] + (1 - \rho_t) E_t^{\mathbb{Q}^1}[\cdot], \quad \textit{where } \rho_t = \frac{\alpha_t}{\alpha_t + (1 - \alpha_t)\frac{\psi_t^0}{\psi_t^1}}.$$

from Biagini, Foellmer Nedelcu 2014

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2 Methods to employ an EMM string

Fix an EMM string $\mathfrak{Q}$.

1. **Method 1**: evaluation time
   Pricing p_t^* uses $\mathbb{Q}^t$ at evaluation time t
2. **Method 2**: maturity time
   Pricing $\hat{p}_t$ uses $\mathbb{Q}^\tau$ for payoffs x_τ with maturity $\tau \geq t$

Method 1- evaluation time; SDF perspective

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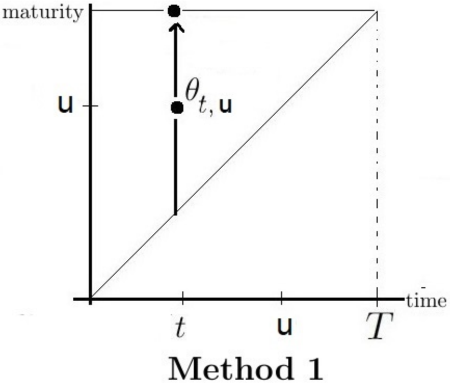


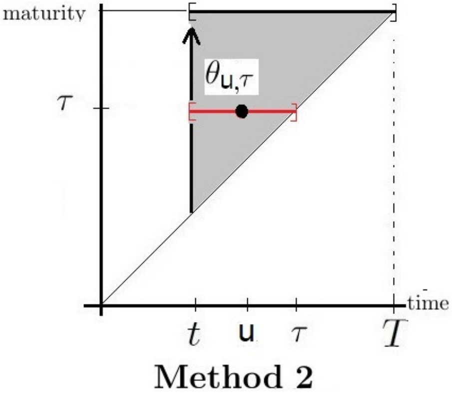
$$
\begin{aligned}
& p_t^*(x) \\
= & \int_t^T E_t^{\mathbb{Q}^t}[x_\tau] \mathrm{d}\tau \\
= & \int_t^T E_t^{\mathbb{P}}\left[\frac{\psi_{t,\tau}}{\psi_{t,t}} x_\tau\right] \mathrm{d}\tau
\end{aligned}
$$

$$
\begin{aligned}
\frac{\psi_{t,\tau}}{\psi_{t,t}} &= \frac{\frac{d\mathbb{Q}^t}{d\mathbb{P}}|\mathcal{F}_\tau}{\frac{d\mathbb{Q}^t}{d\mathbb{P}}|\mathcal{F}_t} \\
&= \exp\left(-\frac{1}{2}\int_t^\tau \theta_{t,u}^2 du - \int_t^\tau \theta_{t,u} dB_u\right)
\end{aligned}
$$

Method 2 -maturity time; SDF perspective

.

$$\hat{p}_t(x) = \int_t^T E_t^{\mathbb{Q}^\tau}[x_\tau]\mathrm{d}\tau = \int_t^T E_t^{\mathbb{P}}\left[\frac{\psi_{\tau,\tau}}{\psi_{t,\tau}} x_\tau\right]\mathrm{d}\tau$$



$$\frac{\psi_{\tau,\tau}}{\psi_{t,\tau}} = \frac{\frac{\mathrm{d}\mathbb{Q}^\tau}{\mathrm{d}\mathbb{P}}|\mathcal{F}_\tau}{\frac{\mathrm{d}\mathbb{Q}^\tau}{\mathrm{d}\mathbb{P}}|\mathcal{F}_t} = \exp\left(-\frac{1}{2}\int_t^\tau \theta_{u,\tau}^2 \mathrm{d}u - \int_t^\tau \theta_{u,\tau}\mathrm{d}B_u\right)$$

Comparison of Approaches

	Claim X		Payoff Stream x		
Method	1 **SDF**	2 **Recursive**	3 **SDF**		4 **Recursive**
p – classical	$\frac{\psi(T,T)}{\psi(t,t)}$	$BSDE$	$\frac{\psi(\tau,\tau)}{\psi(t,t)}$	$\tau \in [t,T]$	$BSDE$
p^* – time			$\frac{\psi(t,\tau)}{\psi(t,t)}$	$\tau \in [t,T]$	$BSVIE$
$\hat{p}$ – maturity			$\frac{\psi(\tau,\tau)}{\psi(t,\tau)}$	$(t,\tau): t \leq \tau$	$TD\text{–}BSVIE$

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From BSDE to BSVIE backward stochasitic Volterra integral equations

1. **BSDE**: a pair $(Y, \sigma) \in \mathbb{L} \times \mathbb{L}$ solves

$$Y_t = \int_t^T g(s, Y_s, \sigma_s) ds - \int_t^T \sigma_s dB_s,$$

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2. **BSVIE**: a pair $(Y, \sigma) \in \mathbb{L} \times \mathbb{L}_{[0,T]^2}$ solves

$$Y_t = \int_t^T g\Big(t, s, Y_s, \sigma_{t,s}\Big)ds - \int_t^T \sigma_{t,s} dB_s,$$

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$$Y_t = \int_t^T g\Big(t, s, Y_s, \sigma_{t,s}\Big) ds - \int_t^T \sigma_{t,s} dB_s,$$

where

$$\mathbb{L}_{[0,T]^2} = \left\{ \sigma : [0,T]^2 \times \Omega \to \mathbb{R} : \begin{array}{c} \sigma \text{ is } \mathcal{B} \otimes \mathcal{F}_T\text{-meas.} \\ s \mapsto \sigma_{t,s} \text{ adapted } \forall t \\ \int_{[0,T]^2} |\sigma_{t,s}|^2 dsdt \in L_T \end{array} \right\}.$$

Method 1 recursive approach

Proposition

The price $p_t^*(x)$ of the payoff stream (x_t) uniquely solves, $\sigma_{t,\tau}$, the following linear BSVIE

$$p_t^*(x) = \int_t^T x_\tau - \theta_{t,\tau}\sigma_{t,\tau}\mathrm{d}\tau + \int_t^T \sigma_{t,\tau}\mathrm{d}B_\tau.$$

We have

$$p_t^*(x) = E_t^{\mathbb{Q}^t}\left[\int_t^T x_\tau \mathrm{d}\tau\right] = E_t^{\mathbb{P}}\left[\int_t^T -\theta_{t,\tau}\sigma_{t,\tau} + x_\tau \mathrm{d}\tau\right].$$

Properties: for Method 1

Proposition

For any $t \in [0, T]$ the pricing schemes p_t^* satisfies:

1. monotonicity: $x \leq y$ implies $p_t^*(x) \leq p_t^*(y)$
2. conditional homogeneity: $p_t^*(\Lambda x) = \Lambda p_t^*(x)$ for any $\Lambda \in L_t$
3. static linearity: $p_t^*(x + y) = p_t^*(x) + p_t^*(y)$

If EMM–string is non constant, then time-consistency fails.

time delayed BSVIE

3. **time delayed BSDE**: a pair $(Y, \sigma) \in \mathbb{L} \times \mathbb{L}$ solves

$$Y_t = \int_t^T g\Big(\tau, Y_\tau, \{\sigma_{\tau+u}\}_{u\in[-T,0]}\Big) d\tau - \int_t^T \sigma_\tau dB_\tau .$$

time delayed BSVIE

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2.+3. **time delayed BSVIE**: a pair $(Y, \sigma) \in \mathbb{L} \times \mathbb{L}_{[0,T]^2}$ solves

$$Y_t = \int_t^T g\Big(t, \tau, Y_\tau, \{\sigma_{t,\tau+u}\}_{u\in[-T,0]}\Big) \mathrm{d}\tau - \int_t^T \sigma_{t,\tau} \mathrm{d}B_\tau .$$

Method 2 recursive approach

Existence of linear time delayed-BSVIE

Theorem

If T is sufficiently close to 0 then $\hat{p}_t(x) = \int_t^T E_t^{\mathbb{Q}^\tau}[x_\tau] \mathrm{d}\tau$ uniquely solves with a $\sigma \in \mathbb{L}_{[0,T]^2}$ the following time–delayed BSVIE:

$$\hat{p}_t(x) = \int_t^T \underbrace{-\int_t^\tau \theta_{u,\tau}\sigma_{u,\tau} \mathrm{d}u + x_\tau}_{=g\left(t,\tau,\{\sigma_{t,\tau+u}\}_{u\in[-T,0]}\right)} d\tau + \int_t^T \sigma_{t,\tau} dB_\tau .$$

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Dynamic Arbitrage bounds

the dynamic arbitrage interval

$$[\underline{p}_t(X), \overline{p}_t(X)] = \left[\operatorname*{ess\,inf}_{\mathbb{Q}\in\mathcal{Q}} E_t^{\mathbb{Q}} X, \operatorname*{ess\,sup}_{\mathbb{Q}\in\mathcal{Q}} E_t^{\mathbb{Q}} X \right] \subset L_t.$$

The sub- and super-hedging pricing satisfies:

$$\begin{aligned} \mathrm{d}\underline{p}_t(X) &= -\operatorname*{ess\,inf}_{\theta\in\Theta} \theta_t \underline{\sigma}_t \mathrm{d}t + \underline{\sigma}_t \mathrm{d}B_t, \quad \underline{p}_T(X) = X, \\ \mathrm{d}\overline{p}_t(X) &= -\operatorname*{ess\,sup}_{\theta\in\Theta} \theta_t \overline{\sigma}_t \mathrm{d}t + \overline{\sigma}_t \mathrm{d}B_t, \quad \overline{p}_T(X) = X, \end{aligned}$$

where $\Theta \subset \mathbb{L}$ identifies $\mathcal{Q}$ as a collection of linear BSDEs for each $\mathbb{Q} \in \mathcal{Q}$.

Question of Arbitrage

Proposition

- Method 1 is arbitrage free.
- Let for any $t \leq \tau$ and any τ, $x_t = g(X_t)$ and $\theta_{t,\tau} = \vartheta^\tau(t, X_t)$.[a] The state process follows

$$\mathrm{d}X_s = x + \int_0^s b(r, X_r)\mathrm{d}r + \int_0^s \sigma(r, X_r)\mathrm{d}B_r.$$

If the Sharpe ratio is downward sloping in maturity, then Method 2 is arbitrage-free.

[a] Let $g : \mathbb{R} \to \mathbb{R}$ be bounded, increasing, $C^{1,1}$ with bounded derivative and $\vartheta^\tau, b, \sigma : [0, T] \times \mathbb{R} \to \mathbb{R}$ be $C^{1,1}$ in space with derivatives uniformly bounded and be $\frac{1}{2}$-Hölder in time. Let also σ be uniformly elliptic, positive and $|b(\cdot, 0)|, |\sigma(\cdot, 0)|$ be uniformly bounded.

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Connections to Macro Finance

New perspective on CCAPM asset pricing a la Lucas 1978

$$U_t(e) = \mathbb{E}_t^{\mathbb{P}_t}\left[\int_t^T u(e_s)ds\right] = \mathbb{E}_t^{\mathbb{P}}\left[\int_t^T -\vartheta_{t,s}\sigma_{t,s} + u(e_s)\mathrm{d}s\right].$$

In equilibrium it is nec.+ suff asset price process satisfies a linear BSVIE:

$$S_t = \int_t^T \left(-\vartheta_{t,s}\sigma_{t,s} + u'(e_s)D_s\right)\mathrm{d}s + \int_t^T \sigma_{t,s}\mathrm{d}B_s$$

with given dividend, endowment process $D, e \in \mathbb{L}$. An EMM-string results

$$\mathbb{Q}^t = \frac{u'(e_t)}{\textit{risk free discount factor}}\mathbb{P}_t$$

Example- Robust Control of Hansen-Sargent

$$U_t(e) = \operatorname*{ess\,inf}_{P\in\mathcal{P}^\eta}\left\{E_t^P\left[\int_t^T u(e_s)ds\right] + \underbrace{\mathcal{R}_t(P\|\mathbb{P})}_{\textit{rel. Entropy}}\right\},$$

with **non** rectangular $\mathcal{P}^\eta = \left\{P_\vartheta : \frac{\mathrm{d}P_\vartheta}{\mathrm{d}\mathbb{P}} = \mathcal{E}(\vartheta)_T \textit{ for a } \vartheta \in \mathbb{L}, \mathcal{R}_0(P_\vartheta\|\mathbb{P}) \leq \eta\right\}$.

Conclusion

New perspective on asset pricing

1. recent empirical findings
2. motivation for a new type of BSDE; Volterra + time delay
3. no Arbitrage