Absolute and relative ambiguity aversion

A preferential approach

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Question

How do wealth levels impact preferences and ambiguity attitudes?

Contributions

- We provide behavioral definitions of decreasing, constant, and increasing absolute ambiguity aversion
- We characterize these notions for a large class of preferences
- We perform a similar exercise for *relative* ambiguity attitudes (in the paper)

Model uncertainty

$(\Omega, \mathcal{A})$ measurable space, $\{\mathbb{P}_s\}_{s \in S}$ collection of probability measures on $\mathcal{A}$

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that maps s to the distribution of H under $\mathbb{P}_s$

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Axiom (Consequentialism)

DM is indifferent between H and G if $H \approx G$ under $\mathbb{P}_s$ for all $s \in S$

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It is sufficient to consider preferences defined on (a subset of) $\mathcal{P}(\mathbb{R})^S$

(each H being replaced by the corresponding h)

Anscombe-Aumann setup

- (S, Σ) measurable space: *parameters* (or *models*)
- $\mathcal{X}$ simple probability measures on $\mathbb{R}$: *monetary lotteries*
- $\mathcal{F} = B_0(S, \Sigma, \mathcal{X})$ simple and measurable maps from S to $\mathcal{X}$: *acts*
- $\succsim$ complete preorder on $\mathcal{F}$: *preference*

Rational preferences (CGMMS, 2011)

Definition

A binary relation $\succsim$ on $\mathcal{F}$ is a **rational preference** iff it is a preference st

- given any $x, y, z \in \mathcal{X}$,

$$x \sim y \implies \frac{1}{2}x + \frac{1}{2}z \sim \frac{1}{2}y + \frac{1}{2}z \qquad \text{(risk independence)}$$

- given any $f, g \in \mathcal{F}$,

$$f(s) \succsim g(s) \text{ for all } s \in S \implies f \succsim g \qquad \text{(monotonicity)}$$

Regularity

Definition

A rational preference $\succsim$ on $\mathcal{F}$ is **regular** iff

- given any $x \in \mathcal{X}$, there exists a unique $r \in \mathbb{R}$ st

$$x \sim \delta_r \qquad \text{(certainty equivalents)}$$

- given any $x, y \in \mathcal{X}$,

$$x([r,\infty)) \geq y([r,\infty)) \text{ for all } r \in \mathbb{R} \implies x \succsim y \qquad \text{(dominance)}$$

- given any $f, g, h \in \mathcal{F}$, the sets

$$\{\alpha \in [0,1] : \alpha f + (1-\alpha) g \succsim h\} \text{ and } \{\alpha \in [0,1] : h \succsim \alpha f + (1-\alpha) g\}$$

are closed in $[0,1]$ (continuity)

Chisini means

Let $T \subseteq \mathbb{R}$ be a non-singleton interval

Definition

A continuous functional $I : B_0(S, \Sigma, T) \to \mathbb{R}$ is a **Chisini mean** iff

- given any $t \in T$,

$$I(t1_S) = t \qquad \text{(normalization)}$$

- given any $\varphi, \psi \in B_0(S, \Sigma, T)$,

$$\varphi \geq \psi \implies I(\varphi) \geq I(\psi) \qquad \text{(monotonicity)}$$

A representation result

For all $f \in \mathcal{F}$ and $u : \mathbb{R} \to \mathbb{R}$, set

$$u(f) = \int_{\mathbb{R}} u \, \mathrm{d}f : \begin{array}{ccc} S & \to & \mathbb{R} \\ s & \mapsto & \int_{\mathbb{R}} u \, \mathrm{d}f(s) \end{array}$$

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Theorem (CGMMS, 2011)

A binary relation $\succsim$ on $\mathcal{F}$ is a **regular rational preference** *iff there exist a strictly increasing and continuous $u : \mathbb{R} \to \mathbb{R}$ and a Chisini mean $I : B_0(S, \Sigma, u(\mathbb{R})) \to \mathbb{R}$ st*

$$f \succsim g \iff I(u(f)) \geq I(u(g))$$

for all $f, g \in \mathcal{F}$
In this case, u is cardinally unique and I is unique given u

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If $u(\mathbb{R})$ is unbounded, we replace u with $u - b$ so that $u(\mathbb{R}) - b$ is a cone

Classical examples

Random payoff are transformed into parametric expected utility profiles

$$H \mapsto h = \{h(s)\}_{s\in S} \mapsto u(h) = \left\{\int u \,\mathrm{d}h(s)\right\}_{s\in S} = \left\{\int u(H)\, \mathrm{d}\mathbb{P}_s\right\}_{s\in S}$$

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- **Subjective Expected Utility** $I(\varphi) = \int \varphi \, \mathrm{d}\mu$ hence

$$I(u(h)) = \int_S \left(\int_{\mathbb{R}} u \, \mathrm{d}h(s) \right) \mathrm{d}\mu(s)$$

  μ probability measure (Anscombe and Aumann, 1963)

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- **Robust Preferences** $I(\varphi) = \inf_{\mu \in \mathcal{M}} \int \varphi \, \mathrm{d}\mu$ hence

$$I(u(h)) = \inf_{\mu \in \mathcal{M}} \int_S \left(\int_{\mathbb{R}} u \, \mathrm{d}h(s) \right) \mathrm{d}\mu(s)$$

  $\mathcal{M}$ set of probability measures (Gilboa and Schmeidler, 1989)

More examples

- **Second Order Expected Utility** $I(\varphi) = v^{-1}\left(\int v(\varphi)\, \mathrm{d}\mu\right)$ hence

$$I(u(h)) = v^{-1}\left(\int_S v\left(\int_{\mathbb{R}} u\, \mathrm{d}h(s)\right) \mathrm{d}\mu(s)\right)$$

μ probability measure, $v : u(\mathbb{R}) \to \mathbb{R}$ strictly increasing and continuous (Klibanoff, Marinacci, and Mukerji, 2005, Neilson, 2010)

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- **Variational Preferences** $I(\varphi) = \inf_{\mu \in \mathcal{P}(S)}\left(\int \varphi \,\mathrm{d}\mu + c(\mu)\right)$ hence

$$I(u(h)) = \inf_{\mu \in \mathcal{P}(S)}\left(\int_S \left(\int_{\mathbb{R}} u \,\mathrm{d}h(s)\right) \mathrm{d}\mu(s) + c(\mu)\right)$$

$c : \mathcal{P}(S) \to [0, \infty]$ function such that $\inf_{\mu \in \mathcal{P}(S)} c(\mu) = 0$ (Maccheroni, Marinacci, and Rustichini, 2006)

Axioms for all tastes

The Subjective Expected Utility specification corresponds to

Axiom (Independence)

Given any $f, g, h \in \mathcal{F}$ *and any* $\alpha \in (0,1)$,

$$f \succsim g \iff \alpha f + (1-\alpha) h \succsim \alpha g + (1-\alpha) h$$

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The other models correspond to weakenings of independence (see papers)

Comparative ambiguity attitudes (Ghirardato and Marinacci, 2002)

Let $\succsim$ and $\succsim'$ be regular rational preferences on $\mathcal{F}$

Definition

$\succsim$ is **more ambiguity averse than** $\succsim'$ iff, given any $f \in \mathcal{F}$ and $x \in \mathcal{X}$,

$$f \succsim x \implies f \succsim' x$$

Comparative ambiguity attitudes (Ghirardato and Marinacci, 2002)

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Theorem

$\succsim$ *is* **more ambiguity averse than** $\succsim'$ *if and only if* u *and* u' *are cardinally equivalent and, after choosing* $u = u'$*, it follows* $I \leq I'$

Wealth shifts

The payoff of a DM with wealth w who makes a zero cost investment H is

$$H^w = w + H$$

which, for each $s \in S$, has distribution

$$\begin{aligned} h^w(B \mid s) &= \mathbb{P}_s \circ (w+H)^{-1}(B) = \mathbb{P}_s(\omega \in \Omega : w + H(\omega) \in B) \\ &= \mathbb{P}_s(\omega \in \Omega : H(\omega) \in B - w) = \mathbb{P}_s \circ H^{-1}(B - w) \\ &= h(B - w \mid s) \end{aligned}$$

for all $B \in \mathcal{B}$

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For all $f \in \mathcal{F}$ and $w \in \mathbb{R}$, set

$$f^w(B \mid s) = f(B - w \mid s)$$

for all $(B, s) \in \mathcal{B} \times S$

Preferences at different wealth levels

Let $\succsim$ be a regular rational preference and arbitarily choose $w \in \mathbb{R}$

Given any $f, g \in \mathcal{F}$ set

$$f \succsim^w g \iff f^w \succsim g^w$$

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Lemma

If $\succsim$ is a regular rational preference,
then $\succsim^w$ is a regular rational preference for all $w \in \mathbb{R}$

Absolute Ambiguity Aversion (AAA): definition

Definition

Let $\succsim$ be a regular rational preference

- $\succsim$ is **decreasing absolute ambiguity averse** iff, for all $w < w'$ in $\mathbb{R}$

$$\succsim^{w} \text{ is more ambiguity averse than } \succsim^{w'}$$

- $\succsim$ is **constant absolute ambiguity averse** iff, for all $w < w'$ in $\mathbb{R}$

$$\succsim^{w} \text{ is as ambiguity averse as } \succsim^{w'}$$

- $\succsim$ is **increasing absolute ambiguity averse** iff, for all $w < w'$ in $\mathbb{R}$

$$\succsim^{w} \text{ is less ambiguity averse than } \succsim^{w'}$$

$\succsim$ is **classifiable (in terms of AAA)** iff one of the three above holds

CARA

Lemma

If a regular rational preference is classifiable, then $\succsim^{w}$ *coincides with* $\succsim^{w'}$ *on* $\mathcal{X}$ *(absolute risk aversion is constant) for all* $w, w' \in \mathbb{R}$*. In particular, there exists* $\alpha \in \mathbb{R}$ *and* $\beta > 0$ *such that they can be represented by*

$$u^{w}(r) = u^{w'}(r) = u(r) = \begin{cases} -\beta e^{-\alpha r} & \text{if } \alpha > 0 \\ \beta r & \text{if } \alpha = 0 \\ \beta e^{-\alpha r} & \text{if } \alpha < 0 \end{cases}$$

for all $r \in \mathbb{R}$

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for all $r \in \mathbb{R}$

Classifiable regular rational preferences are *CARA* and

- *risk averse* iff $\alpha > 0$
- *risk neutral* iff $\alpha = 0$
- *risk loving* iff $\alpha < 0$

Positive homegeneity

Definition

Let T be a cone in $\mathbb{R}$ and $I : B_0(S, \Sigma, T) \to \mathbb{R}$

- I is **positively superhomogeneous** iff, given any $\varphi \in B_0(S, \Sigma, T)$

$$I(\lambda\varphi) \geq \lambda I(\varphi) \qquad \forall\lambda \in (0,1)$$

- I is **positively homogeneous** iff, given any $\varphi \in B_0(S, \Sigma, T)$

$$I(\lambda\varphi) = \lambda I(\varphi) \qquad \forall\lambda \in (0,1)$$

- I is **positively subhomogeneous** iff, given any $\varphi \in B_0(S, \Sigma, T)$

$$I(\lambda\varphi) \leq \lambda I(\varphi) \qquad \forall\lambda \in (0,1)$$

Positive homegeneity

Definition

Let T be a positive cone in $\mathbb{R}$ and $I : B_0(S, \Sigma, T) \to \mathbb{R}$

- I is **constant superadditive** iff, given any $\varphi \in B_0(S, \Sigma, T)$

$$I(\varphi + \lambda) \geq I(\varphi) + \lambda \qquad \forall \lambda \in (0, \infty)$$

- I is **constant additive** iff, given any $\varphi \in B_0(S, \Sigma, T)$

$$I(\varphi + \lambda) = I(\varphi) + \lambda \qquad \forall \lambda \in (0, \infty)$$

- I is **constant subadditive** iff, given any $\varphi \in B_0(S, \Sigma, T)$

$$I(\varphi + \lambda) \leq I(\varphi) + \lambda \qquad \forall \lambda \in (0, \infty)$$

Absolute ambiguity aversion: characterization

Theorem

Let $\succsim$ be a regular rational preference on $\mathcal{F}$. Then $\succsim$ is **decreasing absolute ambiguity averse** *iff one of the three following statements is satisfied:*

(i) *u is CARA, risk averse, and I is positively superhomogeneous*

(ii) *u is CARA, risk neutral, and I is constant superadditive*

(iii) *u is CARA, risk loving, and I is positively subhomogeneous*

DAAA, CAAA, IAAA

CARA risk averse	**CARA risk neutral**	**CARA risk loving**	$\succsim$
I sup homo	I cost sup add	I sub homo	**DAAA**
I homo	I cost add	I homo	**CAAA**
I sub homo	I cost sub add	I sup homo	**IAAA**

Example: the variational case

Corollary

Let $\succsim$ be a CARA variational preference

- *If it is risk neutral or a robust preference, then it is CAAA*
- *Else*

  — if it is risk averse, then it is DAAA

  — if it is risk loving, then it is IAAA

More in the project

- Characterization of AAA for many other special cases (eg, SOEU)
- General characterization of RAA (plus special cases)
- Quadratic approximations, ambiguity premia, and ambiguity attitudes
- Beyond CARA/CRRA