Pricing without martingale measure

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Aim of the paper

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- Investors trading in a multi-period and discrete-time financial market.
- Analyse from scratch the set of super-hedging prices and its infimum value.
- Use the convex duality instead of the usual financial duality based on martingale measures under the (NA) condition.
- Study the link between Absence of Immediate Profit (AIP), (NA) and the absence of weak immediate profit (AWIP) conditions.
- Give some numerical illustrations : calibrate historical data of the french index CAC 40 to our model and implement the super-hedging strategy for a call option.

Framework and notations

- Let $(\Omega, (\mathcal{F}_t)_{t\in\{0,\dots,T\}}, \mathcal{F}_T, P)$ be a complete filtered probability space, where T is the time horizon.
- Let $S := \{S_t,\ t \in \{0,\dots,T\}\}$ be a $(\mathcal{F}_t)_{t\in\{0,\dots,T\}}$-adapted, real-valued, non-negative process.
- Trading strategies are given by $(\mathcal{F}_t)_{t\in\{0,\dots,T\}}$-adapted processes $\theta := \{\theta_t, t \in \{0,\dots,T-1\}\}$
- Trading is self-financing and the riskless asset's price is a constant equal to 1. The value at time t of a portfolio θ starting from initial capital $x \in \mathbb{R}$ is then given by

$$V_t^{x,\theta} = x + \sum_{u=1}^{t} \theta_{u-1} \Delta S_u.$$

Framework and notations

- For any σ-algebra $\mathcal{H}$ and any $k \geq 1$, we denote by $L^0(\mathbb{R}^k, \mathcal{H})$ the set of $\mathcal{H}$-measurable and $\mathbb{R}^k$-valued random variables.
- Let $h : \Omega \times \mathbb{R}^k \to \mathbb{R}$. The effective domain of $h(\omega, \cdot)$ is $\operatorname{dom} h(\omega, \cdot) = \{x \in \mathbb{R}^k,\ h(\omega, x) < \infty\}$.
- $h(\omega, \cdot)$ is proper if dom $h(\omega, \cdot) \neq \emptyset$ and $h(\omega, x) > -\infty$ for all $x \in \mathbb{R}^k$.

Framework and notations

- Consider two complete sub-σ-algebras of $\mathcal{F}_T$: $\mathcal{H} \subseteq \mathcal{F}$ and two non negative random variables $y \in L^0(\mathbb{R}, \mathcal{H})$ and $Y \in L^0(\mathbb{R}, \mathcal{F})$.
- Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$. The set $\mathcal{P}(g)$ of super-hedging prices of the contingent claim $g(Y)$ consists in the initial values of super-hedging strategies θ :

$$\mathcal{P}(g) = \{x \in L^0(\mathbb{R}, \mathcal{H}), \exists\, \theta \in L^0(\mathbb{R}, \mathcal{H}),\ x + \theta(Y - y) \geq g(Y) \text{ a.s.}\}.$$

- Bensaid, B., Lesne J.P., Pagès H. and J. Scheinkman (1992).
- The infimum super-hedging cost of $g(Y)$ is defined as

$$p(g) := \operatorname{ess\,inf}_{\mathcal{H}} \mathcal{P}(g).$$

- An infimum super-hedging cost is not necessarily a price !

Conditionnal essential supremum

- Let $\Gamma = (\gamma_i)_{i \in I}$ be a family of real-valued $\mathcal{F}$-measurable random variables. There exists a unique $\mathcal{H}$-measurable random variable $\gamma_{\mathcal{H}} \in L^0(\mathbb{R} \cup \{\infty\}, \mathcal{H})$ denoted $\operatorname{ess\,sup}_{\mathcal{H}} \Gamma$ which satisfies the following properties :
  1. For every $i \in I$, $\gamma_{\mathcal{H}} \geq \gamma_i$ a.s.
  2. If $\zeta \in L^0(\mathbb{R} \cup \{\infty\}, \mathcal{H})$ satisfies $\zeta \geq \gamma_i$ a.s. $\forall i \in I$, then $\zeta \geq \gamma_{\mathcal{H}}$ a.s.
- Barron, E.N, Cardaliaguet, P. and R. Jensen (2003), Kabanov Y. and E. Lépinette (2013).
- $x \in \mathcal{P}(g) \iff \exists \theta \in L^0(\mathbb{R}, \mathcal{H})$ s.t. $x - \theta y \geq g(Y) - \theta Y$ a.s.

$$\mathcal{P}(g) = \left\{\operatorname{ess\,sup}_{\mathcal{H}} \left(g(Y) - \theta Y\right) + \theta y,\ \theta \in L^0(\mathbb{R}, \mathcal{H})\right\} + L^0(\mathbb{R}_+, \mathcal{H}).$$

Conditionnal support

- Let $X \in L^0(\mathbb{R}^d, \mathcal{F})$, conditional support of X with respect to $\mathcal{H}$

$$\mathrm{supp}_{\mathcal{H}} X(\omega) := \bigcap \left\{ A \subset \mathbb{R}^d, \text{ closed}, \ P(X \in A|\mathcal{H})(\omega) = 1 \right\}.$$

- $\mathrm{supp}_{\mathcal{H}} X$ is
  1. non-empty, closed-valued,
  2. $\mathcal{H}$-measurable : $\{\omega \in \Omega, \, O \cap \mathrm{supp}_{\mathcal{H}} X(\omega) \neq \emptyset\} \in \mathcal{H}$, $\forall O$ open set,
  3. graph-measurable random set : $Graph(\mathrm{supp}_{\mathcal{H}} X) \in \mathcal{H} \otimes \mathcal{B}(\mathbb{R}^d)$.
- Assume that $\mathrm{dom}\,\mathrm{supp}_{\mathcal{H}} X = \Omega$ and let $h : \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a $\mathcal{H} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable function which is lower semi-continuous (l.s.c.) in x. Then,

$$\mathrm{ess\,sup}_{\mathcal{H}} h(X) = \sup_{x \in \mathrm{supp}_{\mathcal{H}} X} h(x) \text{ a.s.}$$

- Recall that if h is $\mathcal{H}$-normal integrand then h is $\mathcal{H} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable and is l.s.c. in x. The converse holds true if $\mathcal{H}$ is complete for some measure.

First results I.

- Suppose that g is a $\mathcal{H}$-normal integrand. Then

$$\mathrm{ess\,sup}_{\mathcal{H}} \left(g(Y) - \theta Y\right) = \sup_{z \in \mathrm{supp}_{\mathcal{H}} Y} \left(g(z) - \theta z\right) = f^*(-\theta) \quad \text{a.s.}$$

where f^* is the Fenchel-Legendre conjugate of f i.e.

$$\begin{aligned} f^*(\omega, x) &= \sup_{z \in \mathbb{R}} \left(xz - f(\omega, z)\right) \\ f(\omega, z) &= -g(\omega, z) + \delta_{\mathrm{supp}_{\mathcal{H}} Y}(\omega, z), \end{aligned}$$

where $\delta_C(\omega, z) = 0$ if $z \in C(\omega)$ and $+\infty$ else. $f^*(\omega, \cdot)$ is proper, convex and f^* is a $\mathcal{H}$-normal integrand. Moreover, we have that

$$\begin{aligned} p(g) &= \mathrm{ess\,inf}_{\mathcal{H}} \left\{ \mathrm{ess\,sup}_{\mathcal{H}} \left(g(Y) - \theta Y\right) + \theta y, \ \theta \in L^0(\mathbb{R}, \mathcal{H}) \right\} \\ &= -\mathrm{ess\,sup}_{\mathcal{H}} \left\{ \theta y - f^*(\theta), \ \theta \in L^0(\mathbb{R}, \mathcal{H}) \right\} = \\ &= -\sup_{z \in \mathbb{R}} \left\{ zy - f^*(z) \right\} = -f^{**}(y) \quad \text{a.s.} \end{aligned}$$

where f^{**} is the Fenchel-Legendre biconjugate of f i.e.

$$f^{**}(\omega, x) = \sup_{z \in \mathbb{R}} \left(xz - f^*(\omega, z)\right).$$

First results II.

- The classical biduality result states that if the concave envelop $\mathrm{conv}\, f$ is proper, then f^{**} is also proper, convex and l.s.c. and

$$f^{**} = \underline{\mathrm{conv}\, f}$$

$\mathrm{conv}\, h(x) = \sup\{u(x),\ u$ convex and $u \leq h\}$ $\underline{h}(x) = \liminf_{y \to x} h(y)$.

- Pennanen T. and Perkkio A-P (2017)
- Suppose that g is a $\mathcal{H}$-normal integrand and that there exists some concave function φ such that $g \leq \varphi$ on $\mathrm{supp}_{\mathcal{H}} Y$ and $\varphi < \infty$ on $\mathrm{convsupp}_{\mathcal{H}} Y$. Then,

$$p(g) = -\underline{\mathrm{conv}} f(y) = \overline{\mathrm{conc}}(g, \mathrm{supp}_{\mathcal{H}} Y)(y) - \delta_{\mathrm{convsupp}_{\mathcal{H}} Y}(y) \quad \text{a.s.}$$

where $\mathrm{convsupp}_{\mathcal{H}} Y$ is the smallest convex set that contains $\mathrm{supp}_{\mathcal{H}} Y$ and the relative concave envelop is

$$\mathrm{conc}(g, \mathrm{supp}_{\mathcal{H}} Y)(x) = \inf\{v(x),\ v \text{ is concave and } v(z) \geq g(z),\ \forall z \in \mathrm{supp}_{\mathcal{H}} Y\}.$$

(AIP)

(AIP)

- There is an immediate profit (IP) if $p(0) \leq 0$ with $P(p(0) < 0) > 0$. On the contrary case, we say that the Absence of Immediate Profit (AIP) condition holds if $p(0) = 0$ a.s.
- As $p(0) = -\delta_{\mathrm{convsupp}_{\mathcal{H}} Y}(y)$ a.s. (AIP) holds true if and only if $y \in \mathrm{convsupp}_{\mathcal{H}} Y = [\mathrm{ess\,inf}_{\mathcal{H}} Y, \mathrm{ess\,sup}_{\mathcal{H}} Y] \cap \mathbb{R}$ a.s.
- (AIP) condition holds true if and only if the infimum super-hedging cost of some European call option is non-negative.
- (AIP) holds true if and only $\mathcal{P}(0) \cap L^0(\mathbb{R}_-, \mathcal{H}) = \{0\}$.
- If there is an IP $x \in \mathcal{P}(0) \cap L^0(\mathbb{R}_-, \mathcal{H})$, with $P(x < 0) > 0$. Write $0 = -x + x$ and make the immediate profit $-x$ while you get 0 at time 1 from $x \in \mathcal{P}(0)$.

(NA) and (AIP)

(NA) and (AIP)

- The No Arbitrage (NA) condition holds true if for $\theta \in L^0(\mathbb{R}, \mathcal{H})$, $\theta(Y-y) \geq 0$ a.s. implies that $\theta(Y-y)=0$ a.s. or equivalently $\mathcal{P}(0) \cap L^0(\mathbb{R}_-, \mathcal{F}) = \{0\}$ since

$$\mathcal{P}(0) = \left\{-\theta(Y-y)+\epsilon^+,\ \theta \in L^0(\mathbb{R}, \mathcal{H}),\ \epsilon^+ \in L^0(\mathbb{R}_+, \mathcal{F})\right\}.$$

- The (AIP) condition is striclty weaker than the (NA) one. It is clear that (NA) implies (AIP). We now provide some examples where (AIP) holds true and is strictly weaker than (NA).
  1. If $\operatorname{ess\,inf}_{\mathcal{H}} Y = 0$ and $\operatorname{ess\,sup}_{\mathcal{H}} Y = \infty$.
  2. If there exists $Q_1, Q_2 << P$ such that Y is a Q_2-super martingale and a Q_1-sub martingale. Using the FTAP, (NA) does not have to hold true but (AIP) holds true. Indeed let $Z_1 = dQ_1/dP$. As $\operatorname{ess\,sup}_{\mathcal{H}} Y \geq Y$ a.s. and $\operatorname{ess\,sup}_{\mathcal{H}} Y$ is $\mathcal{H}$-measurable,

$$\operatorname{ess\,sup}_{\mathcal{H}} Y \geq \frac{E(Z_1 Y|\mathcal{H})}{E(Z_1|\mathcal{H})} = E_{Q_1}(Y|\mathcal{H}) \geq y.$$

(NA) and (AIP)

- Last example. Assume that $Y = yZ$ where $Z > 0$ is such that $\operatorname{supp}_{\mathcal{H}} Z = [0,1]$ a.s. (or $\operatorname{supp}_{\mathcal{H}} Z = [1, \infty)$ a.s.) and $y > 0$.
- Then (AIP) holds true :

$$\operatorname{ess\,inf}_{\mathcal{H}} Y = y \operatorname{ess\,inf}_{\mathcal{H}} Z = 0 \leq y \text{ and } \operatorname{ess\,sup}_{\mathcal{H}} Y = y \operatorname{ess\,sup}_{\mathcal{H}} Z = y \geq y.$$

- Nevertheless, this kind of model does not admit a risk-neutral probability measure and the (NA) condition does not hold true using the FTAP.
- Indeed, in the contrary case, there exists a $\rho_1 > 0$ with $1 = E_P(\rho_1|\mathcal{H})$ such that $E_P(\rho_1 Y|\mathcal{H}) = y$ or equivalently $E_P(\rho_1 Z|\mathcal{H}) = 1$.
- We deduce that $E_P(\rho_1(1-Z)|\mathcal{H}) = 0$. Since $Z \leq 1$ a.s. $\rho_1(1-Z) = 0$ a.s. hence $Z = 1$ which yields a contradiction.

Last results

- Suppose that (AIP) holds true, g is a $\mathcal{H}$-normal integrand and there exists some concave function φ such that $g \leq \varphi$ on $\mathrm{supp}_{\mathcal{H}} Y$ and $\varphi < \infty$ on $\mathrm{convsupp}_{\mathcal{H}} Y$. Then,

$$\begin{aligned} p(g) &= \overline{\mathrm{conc}}(g, \mathrm{supp}_{\mathcal{H}} Y)(y) \\ &= \inf \{\alpha y + \beta,\ \alpha,\ \beta \in \mathbb{R},\ \alpha x + \beta \geq g(x),\ \forall x \in \mathrm{supp}_{\mathcal{H}} Y\}. \end{aligned}$$

- Beiglböck, M. and M. Nutz (2014)
- If g is concave and u.s.c., we get under (AIP) that $p(g) = g(y)$ a.s.
- If g is convex and $\lim_{x\to\infty} x^{-1} g(x) = M \in \mathbb{R}$, the relative concave envelop of g is the affine function that coincides with g on the extreme points of the interval $\mathrm{convsupp}_{\mathcal{H}} Y$ i.e. a.s.

$$\begin{aligned} p(g) &= \theta^* y + \beta^* = g(\mathrm{ess\,inf}_{\mathcal{H}} Y) + \theta^* \left(y - \mathrm{ess\,inf}_{\mathcal{H}} Y\right), \\ \theta^* &= \frac{g(\mathrm{ess\,sup}_{\mathcal{H}} Y) - g(\mathrm{ess\,inf}_{\mathcal{H}} Y)}{\mathrm{ess\,sup}_{\mathcal{H}} Y - \mathrm{ess\,inf}_{\mathcal{H}} Y}, \end{aligned}$$

  with the conventions $\theta^* = \frac{0}{0} = 0$ if $\mathrm{ess\,sup}_{\mathcal{H}} Y = \mathrm{ess\,inf}_{\mathcal{H}} Y$ a.s. and $\theta^* = \frac{g(\infty)}{\infty} = M$ if $\mathrm{ess\,inf}_{\mathcal{H}} Y < \mathrm{ess\,sup}_{\mathcal{H}} Y = +\infty$ a.s.
- Here $p(g) + \theta^*(Y - y) \geq g$ a.s. and $p(g) \in \mathcal{P}(g)$.

Multi-periods hedging prices I

- For every $t \in \{0, \ldots, T\}$ the set of all claims that can be super-replicated from 0 initial endowment at time t is

$$\mathcal{R}_t^T := \left\{ \sum_{u=t+1}^{T} \theta_{u-1} \Delta S_u - \epsilon_T^+,\ \theta_{u-1} \in L^0(\mathbb{R}, \mathcal{F}_{u-1}),\ \epsilon_T^+ \in L^0(\mathbb{R}_+, \mathcal{F}_T) \right\}.$$

- Let $g_T \in L^0(\mathbb{R}, \mathcal{F}_T)$, then

$$\begin{aligned} \Pi_{T,T}(g_T) &= \{g_T\} \text{ and } \pi_{T,T}(g_T) = g_T \\ \Pi_{t,T}(g_T) &= \{x_t \in L^0(\mathbb{R}, \mathcal{F}_t),\ \exists R \in \mathcal{R}_t^T,\ x_t + R = g_T \text{ a.s.}\} \\ \pi_{t,T}(g_T) &= \mathrm{ess\,inf}_{\mathcal{F}_t} \Pi_{t,T}(g_T). \end{aligned}$$

- Again, the infimum super-hedging cost is not necessarily a price as $\pi_{t,T}(g_T) \notin \Pi_{t,T}(g_T)$ when $\Pi_{t,T}(g_T)$ is not closed.
- Note that for all $t \in \{0, \ldots, T-1\}$

$$\Pi_{t,T}(g_T) = \{x_t,\ \exists \theta_t,\ \exists p_{t+1} \in \mathcal{P}_{t+1,T}(g_T),\ x_t + \theta_t \Delta S_{t+1} \geq p_{t+1} \text{ a.s.}\}.$$

Multi-periods hedging prices II

- Local version of super-hedging prices. Let $g_{t+1} \in L^0(\mathbb{R}, \mathcal{F}_{t+1})$,

$$\mathcal{P}_{t,t+1}(g_{t+1}) = \{x_t \in L^0(\mathbb{R}, \mathcal{F}_t), \exists \theta_t \in L^0(\mathbb{R}, \mathcal{F}_t),\ x_t + \theta_t \Delta S_{t+1} \geq g_{t+1} \text{ a.s.}\}$$
$$\pi_{t,t+1}(g_{t+1}) = \operatorname{ess\,inf}_{\mathcal{F}_t} \mathcal{P}_{t,t+1}(g_{t+1}).$$

- Let $g_T \in L^0(\mathbb{R}, \mathcal{F}_T)$ and $t \in \{0, \dots, T-1\}$.
- Then $\mathcal{P}_{t,T}(g_T) \subset \mathcal{P}_{t,t+1}(\pi_{t+1,T}(g_T))$.
- If $\pi_{t+1,T}(g_T) \in \Pi_{t+1,T}(g_T)$, then $\mathcal{P}_{t,T}(g_T) = \mathcal{P}_{t,t+1}(\pi_{t+1,T}(g_T))$ and $\pi_{t,T}(g_T) = \pi_{t,t+1}(\pi_{t+1,T}(g_T))$.
- DPP. Under (AIP), if at each step, $\pi_{t+1,T}(g_T) \in \Pi_{t+1,T}(g_T)$ and if $\pi_{t+1,T}(g_T) = g_{t+1}(S_{t+1})$ for some "nice" $\mathcal{F}_t$-normal integrand g_{t+1}, we will get that $\pi_{t,T}(g_T) = \overline{\mathrm{conc}}(g_{t+1}, \mathrm{supp}_{\mathcal{F}_t} S_{t+1})(S_t)$ a.s.

Multi-period (AIP) I

- Fix $t \in \{0, \dots, T\}$. (AIP) condition holds at time t if there is no global IP at t, i.e. if $\Pi_{t,T}(0) \cap L^0(\mathbb{R}_-, \mathcal{F}_t) = \{0\}$.
- We say that (ALIP) condition holds at time t if there is no local IP at t, i.e. if $\mathcal{P}_{t,t+1}(0) \cap L^0(\mathbb{R}_-, \mathcal{F}_t) = \{0\}$.
- Finally we say that the (AIP) condition holds true if the (AIP) condition holds at time t for all $t \in \{0, \dots, T\}$.
- As $\Pi_{t,T}(0) = (-\mathcal{R}_t^T) \cap L^0(\mathbb{R}, \mathcal{F}_t)$, (AIP) reads as $\mathcal{R}_t^T \cap L^0(\mathbb{R}_+, \mathcal{F}_t) = \{0\}$, for all $t \in \{0, \dots, T\}$.
- Equivalence between (ALIP) at time t and (AIP) at time t.
- (AIP) holds if and only if one of the the following assertions holds :
  1. $S_t \in \mathrm{convsupp}_{\mathcal{F}_t} S_{t+1}$ a.s., for all $t \in \{0, \dots, T-1\}$.
  2. $\operatorname{ess\,inf}_{\mathcal{F}_t} S_{t+1} \leq S_t \leq \operatorname{ess\,sup}_{\mathcal{F}_t} S_{t+1}$ a.s., for all $t \in \{0, \dots, T-1\}$.
  3. $\pi_{t,T}(0) = 0$ a.s. for all $t \in \{0, \dots, T-1\}$.

Multi-period (AIP), (NA) and (AWIP) I

- The (NA) condition holds true if $\mathcal{R}_t^T \cap L^0(\mathbb{R}_+, \mathcal{F}_T) = \{0\}$ for all $t \in \{0, \ldots, T\}$.
- The (AIP) condition holds true if $\mathcal{R}_t^T \cap L^0(\mathbb{R}_+, \mathcal{F}_t) = \{0\}$, for all $t \in \{0, \ldots, T\}$.
- The absence of weak immediate profit (AWIP) condition holds true if $\overline{\mathcal{R}_t^T} \cap L^0(\mathbb{R}_+, \mathcal{F}_t) = \{0\}$ for all $t \in \{0, \ldots, T\}$, where the closure of $\mathcal{R}_t^T$ is taken with respect to the convergence in probability.
- The following statements are equivalent :
  1. (AWIP) holds.
  2. For every $t \in \{0, \ldots, T\}$, there exists $Q << P$ with $E(dQ/dP|\mathcal{F}_t) = 1$ such that $(S_u)_{u \in \{t, \ldots, T\}}$ is a Q-martingale.
  3. (AIP) holds and $\overline{\mathcal{R}_t^T} \cap L^0(\mathbb{R}, \mathcal{F}_t) = \mathcal{R}_t^T \cap L^0(\mathbb{R}, \mathcal{F}_t)$ for every $t \in \{0, \ldots, T\}$.

Multi-period (AIP), (NA) and (AWIP) II

- Suppose that $P(\operatorname{ess\,inf}_{\mathcal{F}_t} S_{t+1} = S_t) = P(\operatorname{ess\,sup}_{\mathcal{F}_t} S_{t+1} = S_t) = 0$ for all $t \in \{0 \ldots, T-1\}$. Then, (AWIP) is equivalent to (AIP) and, under these equivalent conditions, $\mathcal{R}_t^T$ is closed in probability for every $t \in \{0 \ldots, T-1\}$.The infimum super-hedging cost is a super-hedging price.
- The (AIP) condition is not necessarily equivalent to (AWIP).

Explicit Dynamic programming under (AIP)

Suppose that the model is defined by $\operatorname{ess\,inf}_{\mathcal{F}_{t-1}} S_t = k^d_{t-1} S_{t-1}$ and $\operatorname{ess\,sup}_{\mathcal{F}_{t-1}} S_t = k^u_{t-1} S_{t-1}$ where $k^d_0, \cdots, k^d_{T-1}$ and $k^u_0, \cdots, k^u_{T-1}$ are deterministic non negative numbers. Then :

- The (AIP) condition holds true if and only if $k^d_t \in [0, 1]$ and $k^u_t \in [1, +\infty]$ for all $0 \le t \le T - 1$.
- Suppose (AIP). If $h : \mathbb{R} \to \mathbb{R}$ is a non-negative convex function with $\mathrm{Dom}\,\mathrm{h} = \mathbb{R}$ such that $\lim_{z \to +\infty} \frac{h(z)}{z} \in [0, \infty)$, then $\pi_{t,T}(h) = h(t, S_t) \in \mathcal{P}_{t,T}(h(S_T))$ a.s. where

$$\begin{aligned} h(T, x) &= h(x) \\ h(t-1, x) &= \lambda_{t-1} h\left(t, k^d_{t-1} x\right) + (1 - \lambda_{t-1}) h\left(t, k^u_{t-1} x\right), \end{aligned}$$

  where $\lambda_{t-1} = \frac{k^u_{t-1} - 1}{k^u_{t-1} - k^d_{t-1}} \in [0, 1]$.
- The infimum super-hedging cost of $h(S_T)$ is the binomial price when $S_t \in \{k^d_{t-1,t} S_{t-1}, k^u_{t-1,t} S_{t-1}\}$ a.s., $t = 1, \cdots, T$.
- Carassus, L., Gobet, E. and E. Temam (06) and Carassus L. and T. Vargiolu.

Asymptotic behaviour I

- Study the asymptotic behaviour of the super-hedging costs when the number of discrete dates converges to ∞.
- Use the discretization $t^n_i = (T/n)i$, $i \in \{0, 1, \cdots, n\}$ and assume that $k^u_{t^n_{i-1}} = 1 + \sigma_{t^n_{i-1}} \sqrt{\Delta t^n_i}$ and $k^d_{t^n_{i-1}} = 1 - \sigma_{t^n_{i-1}} \sqrt{\Delta t^n_i} \ge 0$ where $t \mapsto \sigma_t$ is a positive Lipschitz-continuous function on $[0, T]$.
- The assumptions on the multipliers $k^u_{t^n_{i-1}}$ and $k^d_{t^n_{i-1}}$ imply that

$$\left| \frac{S_{t^n_{i+1}}}{S_{t^n_i}} - 1 \right| \le \sigma_{t^n_i} \sqrt{\Delta t^n_{i+1}}, \text{a.s.}$$

Asymptotic behaviour II

- For every $n \geq 1$, we get a function h^n, s.t. $h^n(T,x) = (x-K)_+$ and

$$\begin{aligned} h^n(t_{i-1}^n, x) &= \lambda_{t_{i-1}^n} h^n(t_i^n, k^d_{t_{i-1}^n} x) + (1-\lambda_{t_{i-1}^n}) h^n(t_i^n, k^u_{t_{i-1}^n} x). \\ \lambda_{t_{i-1}^n}(x) &= \frac{k^u_{t_{i-1}^n} - 1}{k^u_{t_{i-1}^n} - k^d_{t_{i-1}^n}} = \frac{1}{2}. \end{aligned}$$

- Extend h^n on $[0,T]$ in such a way that h^n is constant on each interval $[t_i^n, t_{i+1}^n[$, $i \in \{0, \cdots, n\}$.
- Such a scheme is proposed by Milstein, G.N. (2002). The sequence of functions $(h^n(t,x))_n$ converges uniformly to $h(t,x)$, solution to the diffusion equation :

$$\partial_t h(t,x) + \sigma_t^2 \frac{x^2}{2} \partial_{xx} h(t,x) = 0, \quad h(T,x) = (x-K)_+.$$

- Baptiste J. and E. Lépinette (2018) for payoff function not smooth provided that the successive derivatives of the P.D.E.'s solution do not explode *too much*.

Numerical experiment : Calibration I

- If Δt_i^n is closed to 0, the observed prices of the Call option are assumed to be given by the solution $h(t, S_t)$ of the diffusion equation.
- By calibration, deduce an evaluation of the the deterministic function $t \mapsto \sigma_t$ and test

$$\left| \frac{S_{t_{i+1}^n}}{S_{t_i^n}} - 1 \right| \leq \sigma_{t_i^n} \sqrt{\Delta t_{i+1}^n}, \text{a.s.} \qquad (1)$$

- The data set is composed of historical values of the french index CAC 40 from the 23rd of October 2017 to the 19th of January 2018. For several strikes, we compute the proportion of observations satisfying (1).

Numerical experiment : Calibration II

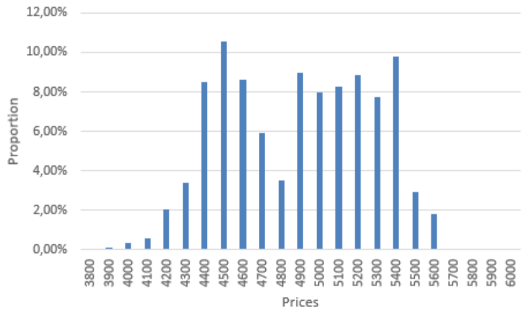

Figure : Distribution of the observed prices.

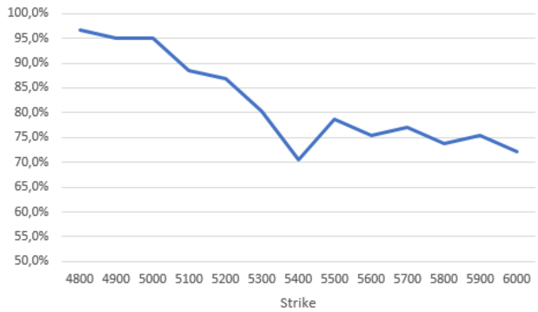

Figure : Ratio of observations satisfying (1) as a function of the strike.

Numerical experiment : super hedging I

- Test the infimum super-hedging cost on some data set composed of historical daily closing values of the french index CAC 40 from the 5th of January 2015 to the 12th of March 2018.
- The interval $[0,T]$ corresponds to one week composed of 5 days so that the discrete dates are t_i, $i \in \{0,\cdots,4\}$.

$$\sigma_{t_i} = \overline{\max}\left(\left|\frac{S_{t_{i+1}}}{S_{t_i}} - 1\right| / \sqrt{\Delta t_{i+1}},\right) \quad i \in \{0,\cdots,3\},$$

  where $\overline{\max}$ is the empirical maximum taken over a one year sliding sample window of 52 weeks.
- $k^u_{t_i} = 1 + \sigma_{t_i}\sqrt{\Delta t_{i+1}}$ and $k^d_{t_i} = 1 - \sigma_{t_i}\sqrt{\Delta t_{i+1}}$.
- Estimation does not depend on the strike as before.
- Estimate the volatility on 52 weeks and implement our hedging strategy on the fifty third one.
- Repeat the procedure by sliding the window of one week, i.e. on each of the weeks from the 11th of January 2015 to the 5th of March 2018.

Numerical experiment : super hedging II

- We study below the super-hedging error

$$\varepsilon_T = h(0, S_0) + \sum_{i=0}^{3} \theta^*_{t_i^4} \Delta S_{t_{i+1}^4} - (S_T - K)^+$$

- Case $K = 4700$. The empirical average of ε_T is 12.63 and its standard deviation is 21.65 (empirical mean of $S_0 = 4044$). The empirical probability of $\{\varepsilon_T < 0\}$ is equal to 15.18% but the Value at Risk at 95 % is -10.33 which confirms that our strategy is conservative.

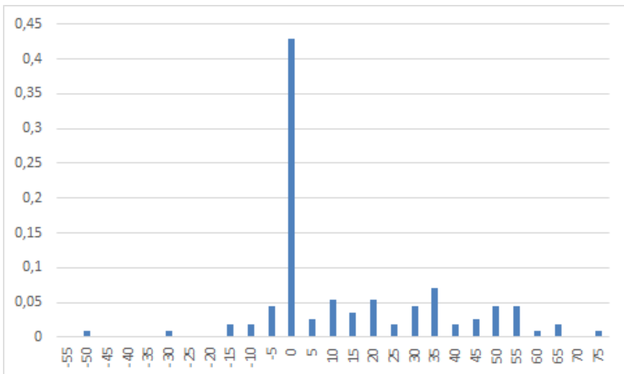

Figure : Distribution of the super-hedging error ε_T for $K = 4700$.

Numerical experiment : super hedging III

- The empirical average of V_0/S_0 is 5.63% and its standard deviation is 5.14%.

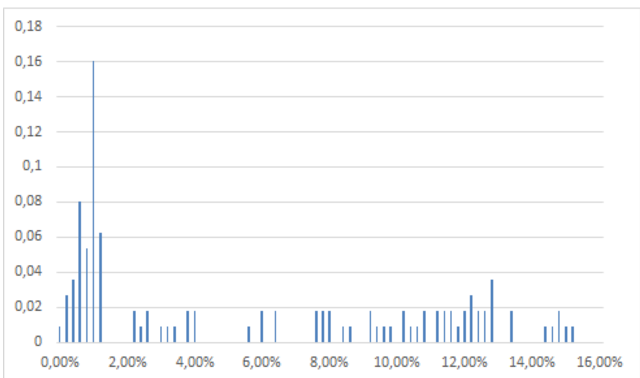

Figure : Distribution of the ratio V_0/S_0.

Conclusion

- New approach to the superreplication price, based on convex duality.
- (AIP) condition instead of (NA) condition.
- Extend the Binomial model to a more general one where the prices at the next instant may take an infinite number of values : For convex payoffs, the prices are the same than the one of the Binomial model keeping only the conditional essup and essinf under the weak (AIP) condition.
- Confirmed by real data.
- The implementation of the super-hedging strategy is very simple and efficient on real data.

Robust extension I

- Let $X : \Omega \to \mathbb{R}^d$ r.v, $\mathcal{P}$ a set of probability on $(\Omega, \mathcal{F})$.
- The robust support is defined as

$$\mathrm{supp}_{\mathcal{P}} X := \bigcap \left\{ F \subset \mathbb{R}^d, \text{ closed}, \; P(X \in F) = 1, \; \forall P \in \mathcal{P} \right\}.$$

- $\mathrm{supp}_{\mathcal{P}} X$ is a closed set, $\mathrm{supp}_{\mathcal{P}} X = \overline{\{x_n, \, n \in \mathbb{N}\}}$ and $X(\cdot) \in \mathrm{supp}_{\mathcal{P}} X$ $\mathcal{P}-$q.s.
- Let $\mathcal{X} = (X_i)_{i \in I}$ be a family of random variables $X_i : \Omega \to \mathbb{R}$ and $\mathcal{P}$ a set of probability defined on $(\Omega, \mathcal{F})$.
- Then, there exists an unique number $x \in \mathbb{R} \cup \{\infty\}$ denoted by $\mathrm{ess\,sup}_{\mathcal{P}} \mathcal{X}$ which satisfies the following properties :
  1. For every $i \in I$, $x \geq X_i(\cdot)$ $\mathcal{P}$-q.s.
  2. If for some number y, $y \geq X_i(\cdot)$ $\mathcal{P}$-q.s. $\forall i \in I$, then $y \geq x$.

Robust extension II

- Let $h \ : \mathbb{R} \to \mathbb{R}^+$ be l.s.c. Then

$$\text{ess sup}_{\mathcal{P}} h(X) = \sup_{x \in \text{supp}\,_{\mathcal{P}} X} h(x) = \sup_n h(x_n),$$

- Super-hedging prices of $g(Y)$:

$$\begin{aligned} \mathcal{P}(g) &:= \{x \in \mathbb{R}, \exists\, \theta \in \mathbb{R},\ x + \theta(Y - y) \geq g(Y)\, \mathcal{P}\text{q.s.}\} \\ &= \{\text{ess sup}_{\mathcal{P}} \left(g(Y) - \theta Y\right) + \theta y,\ \ \theta \in \mathbb{R}\} + \mathbb{R}_+. \end{aligned}$$

- Super-hedging cost of $g(Y)$ $p(g) := \text{ess inf}_{\mathcal{P}} \mathcal{P}(g)$.
- Suppose that g is lsc. Then

$$\text{ess sup}_{\mathcal{P}} \left(g(Y) - \theta Y\right) = \sup_{z \in \text{supp}_{\mathcal{P}} Y} \left(g(z) - \theta z\right) = f^*(-\theta),$$

  where f^* is the Fenchel-Legendre conjugate of
  $f = -g(z) + \delta_{\text{supp}_{\mathcal{P}} Y}(z)$ i.e. $f^*(x) = \sup_{z \in \mathbb{R}} \left(xz - f(z)\right).$
- Moreover, $p(g) = -f^{**}(y)$.