Robust utility maximization in markets with transaction costs

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Outline

1. Introduction
2. Utility functions on the positive real line
3. Utility functions on the whole real line

Literature review

Uncertainty is usually modeled by a family of prior measures $\mathcal{P}$ on the same canonical space. The dominated case: Quenez (2004), Schied (2006), etc. The non-dominated case: Tevzadze et al. (2013), Matoussi et al. (2015), etc.

Model-free approach: Hou and Obloj (2015), Cox et al. (2016), Burzoni et al. (2016), Burzoni et al. (2017), Acciaio et al. (2016), Bouchard and Nutz (2015) etc.

Existence results in a fairly general class of models are available only in discrete time: Nutz (2016), Blanchard and Carassus (2018), Neufeld and Šikić (2017), Bartl (2017), Bartl et al. (2017) and Rásonyi and Meireles-Rodrigues (2018).

Formulation of the problem

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ a filtered probability space.

Θ be a (non-empty) set.

There are two assets: a riskless asset $S^0 = 1$ and a risky asset, whose dynamics is unknown.

A family S^θ, $\theta \in \Theta$ of adapted, continuous positive processes.

No condition is imposed on Θ and on S^θ.

Example: The robust Black-Scholes market model

$$dS_t^{(\mu,\sigma)} = S_t^{(\mu,\sigma)}(\mu dt + \sigma dW_t).$$

$$\Theta = \{\theta = (\mu, \sigma) \in \mathbb{R}^2 : \underline{\mu} \le \mu \le \overline{\mu},\ \underline{\sigma} \le \sigma \le \overline{\sigma}\}.$$

See also Biagini and Pinar (2017), Neufeld and Nutz (2018) (Lévy processes), Lin and Riedel (2014).

Advantages

This approach

- is particularly tractable and easily implemented when it comes to calibration.
- simplifies technical issues: the canonical setting with problems concerning null events, filtration completion, etc. The measurable selection arguments, the analytic properties, see Bouchard and Nutz (2015), Biagini et al. (2017) or Nutz (2016).
- In Rásonyi and Meireles-Rodrigues (2018): an example lies outside the framework of prior measures $\mathcal{P}$, since lack of the analytic graph $\mathcal{P}_t$.

Drawbacks and solutions

- Consider: $\sup_H \inf_\theta EU(W_T(\theta, H))$ and $\sup_W \inf_{Q\in\mathcal{P}} E^Q U(W_T)$.
- No "abstract" versions.
- Komlós-type arguments on the space L^0 cannot be employed.
- No convexity in θ. Dual problem?
- The candidate dual problem in this setting does not, in general, admit a solution (see Remark 2.3 of Bartl (2017)).
- Work with the primal problem.
- Under proportional transaction costs.
- Komlós-type arguments for the space of finite variation processes and in an Orlicz space context.

A topological space of FV processes

$\mathcal{V}$: the family of non-decreasing, right-continuous functions on $[0, T]$ which are 0 at 0.

Let r_k, $k \in \mathbb{N}$ be an enumeration of $Q := (\mathbb{Q} \cap [0, T]) \cup \{T\}$ with $r_0 = T$. For $f, g \in \mathcal{V}$, define a metric

$$\rho(f, g) := \sum_{k=0}^{\infty} 2^{-k} |f(r_k) - g(r_k)|.$$

Let $\mathbf{V}$ denote the set of $H = (H^{\uparrow}, H^{\downarrow})$ where $H_t^{\uparrow}, H_t^{\downarrow}$, $t \in [0, T]$ are optional processes, $H^{\uparrow}(\omega), H^{\downarrow}(\omega) \in \mathcal{V}$.

We equip $\mathbf{V}$ with the metric

$$\varrho(H, G) := E[\rho(H^{\uparrow}, G^{\uparrow}) \wedge 1] + E[\rho(H^{\downarrow}, G^{\downarrow}) \wedge 1], \; H, G \in \mathbf{V}.$$

A compactness result

Lemma

Let $H(n) \in \mathbf{V}$, $n \in \mathbb{N}$ *be such that*

$$\sup_{n \in \mathbb{N}} E^Q[H^{\uparrow}_T(n) + H^{\downarrow}_T(n)] < \infty$$

for some $Q \sim P$. *Then there is* $H \in \mathbf{V}$ *and there are convex weights* $\alpha^n_j \geq 0$, $j = n, \ldots, M(n)$, $\sum_{j=n}^{M(n)} \alpha^n_j = 1$, $n \in \mathbb{N}$ *such that*

$$\tilde{H}(n) := \sum_{j=n}^{M(n)} \alpha^n_j H(j)$$

satisfy, for each $t \in [0, T]$, $\tilde{H}^{\uparrow}(n)_t \to H^{\uparrow}_t$ *and* $\tilde{H}^{\downarrow}(n)_t \to H^{\downarrow}_t$, $n \to \infty$ *almost surely. In particular,* $\tilde{H}^{\uparrow}(n) \to H^{\uparrow}$ *and* $\tilde{H}^{\downarrow}(n) \to H^{\downarrow}$, $n \to \infty$ *almost surely in* $\mathcal{V}$. □

Kabanov (1999), Guasoni (2002), Campi and Schachermayer (2006)

Consistent price systems

Definition

A λ-*consistent price system* (λ-CPS) for S is a pair $(\tilde{S}, Q)$ of a probability measure $Q \sim P$ and a Q local martingale $\tilde{S}$ such that

$$(1-\lambda)S_t \leq \tilde{S}_t \leq S_t, \qquad a.s. \qquad \forall t \in [0,T]. \tag{1}$$

A λ-*strictly consistent price system* (λ-SCPS) is a CPS such that the inequalities are strict in (1).

See also Kabanov and Safarian (2009), Guasoni et al. (2010), and Guasoni et al. (2008).

Trading strategies

- Trading strategies: $H \in \mathbf{V}$.
- Denote: $H^{\uparrow}$ for buying and $H^{\downarrow}$ for selling.
- The position in the risky asset $\phi = H^{\uparrow} - H^{\downarrow}$.
- The liquidation value is defined by

$$\begin{aligned} W_t^{x,\text{liq}}(\theta, H) &:= x - \int_0^t S_u^\theta dH_u^{\uparrow} + \int_0^t (1-\lambda) S_u^\theta dH_u^{\downarrow} \\ &+ \phi_t^+ (1-\lambda) S_t^\theta - \phi_t^- S_t^\theta . \end{aligned}$$

- Neither concave nor convex in H. Assume $\phi_T = 0$: to recover concavity.
- $V^x(\theta, H) = x - \int_0^t S_u^\theta dH_u^{\uparrow} + \int_0^t (1-\lambda) S_u^\theta dH_u^{\downarrow} + \phi_t \tilde{S}_t^\theta$

Definition (Admissibility)

Let $x > 0$. Denote

$$\mathcal{A}_0^{\theta}(x) := \{H \in \mathcal{A}^{\theta}(x) : \; W_t^{x,\mathrm{liq}}(\theta, H) \geq 0 \text{ a.s.}, \phi_T = H_T^{\uparrow} - H_T^{\downarrow} = 0\},$$

and $\mathcal{A}(x) = \bigcap_{\theta \in \Theta} \mathcal{A}_0^{\theta}(x)$.

Utility functions on $\mathbb{R}_+$

Definition (Admissibility)

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and $\mathcal{A}(x) = \bigcap_{\theta \in \Theta} \mathcal{A}_0^\theta(x)$.

Investors want to find the optimizer for

$$u(x) := \sup_{H \in \mathcal{A}(x)} \inf_{\theta \in \Theta} EU(W_T^{x,liq}(\theta, H)).$$

Finiteness of the value function

- No uncertainty. In discrete time, Rásonyi and Stettner (2006) prove that $NA + "u(x) < \infty" \Rightarrow \exists H^*$.
- Uncertainty. Nutz (2016): an example with $u(x) < \infty$ but there is no optimizer. Since the lack of upper-semicontinuity property in one model.
- Condition: $E^P U^+(x + h\Delta S) < \infty, \forall h, P$ from Nutz (2016), Blanchard and Carassus (2018).

Some notations

- Dual function $V(y) := \sup_{x>0}(U(x) - xy), \qquad y > 0.$
- The primal and dual value functions for the θ-model are

$$u^\theta(x) := \sup_{f \in \mathcal{C}^\theta(x)} EU(f), \qquad v^\theta(y) := \inf_{h \in \mathcal{D}^\theta(y)} EV(h).$$

- $u^\theta(x) \leq v^\theta(y) + xy.$

The first result

Theorem

Let $U : (0, \infty) \to \mathbb{R}$ be a nondecreasing, concave function and $U(\infty) > 0$. Assume that

- *For each $\theta \in \Theta$ and for all $0 < \mu < \lambda$, the price process S^θ admits a μ-CPS.*
- *The dual problem for the model θ, is finite for all $\theta \in \Theta$.*

The robust utility maximization problem admits a solution.

$u(x) < \infty$? Candidate for H^*? Admissibility? Upper semicontinuity?

Utility functions on $\mathbb{R}$

Assumption

$U : \mathbb{R} \to \mathbb{R}$ is bounded from above, nondecreasing, concave, $U(0) = 0$. Define the convex conjugate by

$$V(y) := \sup_{x \in \mathbb{R}} (U(x) - xy), \qquad y > 0.$$

We also assume that

$$\lim_{x \to -\infty} \frac{U(x)}{x} = \infty, \tag{2}$$

$$\limsup_{y \to \infty} \frac{V(2y)}{V(y)} < \infty. \tag{3}$$

Admissibility

- $X_t \geq -C, \forall t$. Too small when S is non locally bounded.
- $X_t > -cW$ where $EU(-\alpha W) > -\infty$. Biagini and Frittelli (2005).
- supermartingale property. Ansel and Stricker (1994): $H \cdot S$ is a supermartingale iff $(H \cdot S)^-$ is dominated by a martingale.
- Six Authors' paper, Kabanov and Stricker (2002) (exponential U), Schachermayer (2003) (general U), Owen and Žitković (2009) (random endowment).
- Biagini and Sîrbu: "Moreover, realistic market models are incomplete and thus the description of the whole $\mathcal{M}_\sigma \cap \mathcal{P}_V$ is often impossible. Consequently, checking admissibility with respect to this definition is practically unfeasible".

Admissibility

Define

$$\mathcal{M}_V^\theta := \{Q^\theta : (\tilde{S}^\theta, Q^\theta) \text{ is a consistent price system, } EV(dQ^\theta/dP) < \infty\},$$

$$V^x(\theta, H) = x - \int_0^t S_u^\theta dH_u^\uparrow + \int_0^t (1-\lambda) S_u^\theta dH_u^\downarrow + \phi_t \tilde{S}_t^\theta$$

Definition

We define

$$\mathcal{A}^\theta(x) = \Big\{H \in \mathbf{V} : \phi_T = 0,\ V^x(\theta, H) \textit{ is a } Q^\theta\textit{-supermartingale for each consistent price system } (\tilde{S}^\theta, Q^\theta) \textit{ such that } Q^\theta \in \mathcal{M}_V^\theta\Big\}$$

and $\mathcal{A}(x) := \bigcap_{\theta \in \Theta} \mathcal{A}^\theta(x)$.

The second result

The optimization problem

$$u(x) = \sup_{H \in \mathcal{A}(x)} \inf_{\theta \in \Theta} EU(W_T^{x,liq}(\theta, H)).$$

Theorem

Let Assumption 5 hold, and suppose that for each $\theta \in \Theta$, the price process S^θ admits a λ-SCPS $(Q^\theta, \tilde{S}^\theta)$ such that $Q^\theta \in \mathcal{M}_V^\theta$. Then the robust optimization problem admits a solution.

$u(x) < \infty$? Candidate for H^*? Admissibility? Upper semicontinuity?

Ansel and Stricker (1994): $H \cdot S$ is a supermartingale iff $(H \cdot S)^-$ is dominated by a martingale.

$U^-(W)$ to W^-?

Orlicz spaces

$\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a *Young function* if it is convex with $\Phi(0) = 0$ and $\lim_{x\to\infty} \Phi(x)/x = \infty$.
The set

$$L^\Phi := \{X \in L^0 : E\Phi(\gamma|X|) < \infty \text{ for some } \gamma > 0\}$$

is a Banach space with the following norm

$$||X||_\Phi := \inf\{\gamma > 0 : X \in \gamma B_\Phi\}$$

where $B_\Phi := \{X \in L^0 : E\Phi(|X|) \leq 1\}$, the unit ball of L^Φ.
Define the conjugate function $\Phi^*(y) := \sup_{x\geq 0}(xy - \Phi(x)), y \in \mathbb{R}_+$. This is also a Young function and $(\Phi^*)^* = \Phi$.
Φ is of class Δ_2 if $\lim_{x\to\infty} \frac{\Phi(2x)}{\Phi(x)} < \infty$.

A compactness result

Lemma (Delbaen, Owari 2018)

Let Φ be a Young function of class Δ_2 and let $\xi_n, n \geq 1$ be a norm-bounded sequence in L^{Φ^*}. Then there are convex weights $\alpha_j^n \geq 0$, $n \leq j \leq M(n)$, $\sum_{j=n}^{M(n)} \alpha_j^n = 1$ such that

$$\xi_n' := \sum_{j=n}^{M(n)} \alpha_j^n \xi_j$$

converges almost surely to some $\xi \in L^{\Phi^*}$ as $n \to \infty$, and $\sup_n |\xi_n'|$ is in L^{Φ^*}. □

Recall: $\lim_{x \to -\infty} \frac{U(x)}{x} = \infty, \qquad \limsup_{y \to \infty} \frac{V(2y)}{V(y)} < \infty.$
Supermartingale property: control the losses, use Fatou's Lemma.

An example

Let us consider

$$S_t = 1 + t + \frac{1}{2\pi} \arctan(W_t), \qquad t \in [0,1].$$

If $\lambda < 3/7$ then $(1-\lambda)S_1 > 1$ a.s, therefore, there is no consistent price system. If $\lambda \geq 2/3$, then

$$S_t(1-\lambda) \leq 3/4 \leq S_t,\ \ t \in [0, T].$$

In other words, $(\tilde{S} \equiv 3/4, P)$ is a consistent price system.

Conclusion

- The existence results, no passing to dual problems
- Future: duality?

Thank you for your attention!

Acciaio, B., Beiglböck, M., Penkner, F., and Schachermayer, W. (2016). A model-free version of the fundamental theorem of asset pricing and the super-replication theorem. *Mathematical Finance*, 26(2):233–251.

Bartl, D. (2017). Exponential utility maximization under model uncertainty for unbounded endowments. *Preprint*. Retrieved from https://arxiv.org/abs/1610.00999v2.

Bartl, D., Cheridito, P., and Kupper, M. (2017). Robust utility maximization with medial limits. *Preprint, arXiv:1712.07699*.

Biagini, S., Bouchard, B., Kardaras, C., and Nutz, M. (2017). Robust fundamental theorem for continuous processes. *Mathematical Finance*, 27(4):963–987.

Biagini, S. and Pinar, M. C. (2017). The robust Merton problem of an ambiguity averse investor. *Mathematics and Financial Economics*, 11(1):1–24.

Blanchard, R. and Carassus, L. (2018). Multiple-priors optimal investment in discrete time for unbounded utility function. *Annals of Applied Probability*. Retrieved from http://www.imstat.org/aap/future_papers.html.

Bouchard, B. and Nutz, M. (2015). Arbitrage and duality in nondominated discrete-time models. *Annals of Applied Probability*, 25(2):823–859.

Burzoni, M., Frittelli, M., and Maggis, M. (2016). Universal arbitrage aggregator in discrete-time markets under uncertainty. *Finance and Stochastics*, 20(1):1–50.

Burzoni, M., Frittelli, M., Maggis, M., et al. (2017). Model-free superhedging duality. *The Annals of Applied Probability*, 27(3):1452–1477.

Campi, L. and Schachermayer, W. (2006). A super-replication theorem in Kabanov's model of transaction costs. *Finance and Stochastics*, 10(4):579–596.

Cox, A. M., Hou, Z., and Obłój, J. (2016). Robust pricing and hedging under trading restrictions and the emergence of local martingale models. *Finance and Stochastics*, 20(3):669–704.

Guasoni, P. (2002). Optimal investment with transaction costs and without semimartingales. *The Annals of Applied Probability*, 12(4):1227–1246.

Guasoni, P., Rásonyi, M., and Schachermayer, W. (2008). Consistent price systems and face-lifting pricing under transaction costs. *Annals of Applied Probability*, pages 491–520.

Guasoni, P., Rásonyi, M., and Schachermayer, W. (2010). The fundamental theorem of asset pricing for continuous processes under small transaction costs. *Annals of Finance*, 6(2):157–191.

Hou, Z. and Obloj, J. (2015). On robust pricing-hedging duality in continuous time. *arXiv preprint arXiv:1503.02822*.

Kabanov, Y. and Safarian, M. (2009). *Markets with transaction costs*. Springer.

Kabanov, Y. M. (1999). Hedging and liquidation under transaction costs in currency markets. *Finance and Stochastics*, 3(2):237–248.

Lin, Q. and Riedel, F. (2014). Optimal consumption and portfolio choice with ambiguity. *Preprint, arXiv:1401.1639*.

Matoussi, A., Possamaï, D., and Zhou, C. (2015). Robust utility maximization in nondominated models with 2BSDE: the uncertain volatility model. *Mathematical Finance*, 25(2):258–287.

Neufeld, A. and Nutz, M. (2018). Robust utility maximization with Lévy processes. *To appear in Mathematical Finance*.

Neufeld, A. and Šikić, M. (2017). Robust utility maximization in discrete-time markets with friction. *Preprint*. Retrieved from https://arxiv.org/abs/1610.09230v1.

Nutz, M. (2016). Utility maximization under model uncertainty in discrete time. *Mathematical Finance*, 26(2):252–268.

Owen, M. P. and Žitković, G. (2009). Optimal investment with an unbounded random endowment and utility-based pricing. *Mathematical Finance*, 19(1):129–159.

Quenez, M.-C. (2004). Optimal portfolio in a multiple-priors model. In *Seminar on Stochastic Analysis, Random Fields and Applications IV*, pages 291–321. Springer.

Rásonyi, M. and Meireles-Rodrigues, A. (2018). On utility maximisation under model uncertainty in discrete-time markets. *Preprint, arXiv:1801.06860*.

Schachermayer, W. (2003). A super-martingale property of the optimal portfolio process. *Finance and Stochastics*, 7(4):433–456.

Schied, A. (2006). Risk measures and robust optimization problems. *Stochastic Models*, 22(4):753–831.

Tevzadze, R., Toronjadze, T., and Uzunashvili, T. (2013). Robust utility maximization for a diffusion market model with misspecified coefficients. *Finance and Stochastics*, 17(3):535–563.