Uncertainty Robust Spaces

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The limitations of functional analysis

The main difficulty in our endeavor is that $\mathcal{P}$ [a class of probability measures] can be nondominated, which leads to the failure of various tools of probability theory and functional analysis ... As a consequence, we have not been able to reach general results by using separation arguments in appropriate function spaces ...

(Bouchard & Nutz: ARBITRAGE AND DUALITY IN NONDOMINATED DISCRETE-TIME MODELS)

Why exactly does functional analysis not work?

What would such a function space look like?

$\rightarrow$ 'generalised' L^p-space

- $\mathcal{P} \neq \emptyset$ set of priors (probability measures on $(\Omega, \mathcal{F})$)
- $\sigma : \mathcal{P} \to (0, \infty)$ bounded weight function
- $L^0_{\mathcal{P}} :=$ space of $\mathcal{P}$-q.s. equivalence classes of $\mathbb{R}$-valued random variables, $\preceq_{\mathcal{P}}$ $\mathcal{P}$-q.s. order
- $j_{\mathbb{P}} : L^0_{\mathcal{P}} \to L^0_{\mathbb{P}}$, $\mathbb{P} \in \mathcal{P}$, projection
- $1 \leq p \leq \infty$, $X \in L^0_{\mathcal{P}}$.

$$\|X\|_{p,\sigma} := \sup_{\mathbb{P} \in \mathcal{P}} \sigma(\mathbb{P}) \|j_{\mathbb{P}}(X)\|_{L^p_{\mathbb{P}}} \in [0, \infty]$$

$$L^p_{\mathcal{P},\sigma} := \{X \in L^0_{\mathcal{P}} \mid \|X\|_{p,\sigma} < \infty\}$$

- $(L^p_{\mathcal{P},\sigma}, \preceq_{\mathcal{P}}, \|\cdot\|_{p,\sigma})$ is a Banach lattice!

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What can we do with these Banach lattices?

- Define a canonical model space for capital requirements/risk measures
- Study their order properties
- ...

Risk measures as capital requirements

- (Generalised) one-period market model (Riesz space $(\mathcal{X}, \preceq)$)
- Tomorrow: Does loss profile X pass the capital adequacy test?
  $\leftrightarrow X \in \mathcal{A}$ or $X \notin \mathcal{A}$, $\mathcal{A} :=$ (acceptance) set of adequately capitalised losses
  ($\emptyset \neq \mathcal{A} \subsetneq \mathcal{X}$ convex, monotone ($X \preceq Y, Y \in \mathcal{A} \implies X \in \mathcal{A}$))
- Today: What remedial action renders adequate capitalisation of X?
  $\leftrightarrow (\mathcal{S}, \mathfrak{p})$ security market; choose $Z \in \mathcal{S}$ at price $\mathfrak{p}(Z) \in \mathbb{R}$ s.t. $X - Z \in \mathcal{A}$
  ($\mathcal{S} \subsetneq \mathcal{X}$, $\dim(\mathcal{S}) < \infty$, $\mathcal{S} \cap \mathcal{X}_+ \supsetneq \{0\}$, $\mathfrak{p} : \mathcal{S} \to \mathbb{R}$, positive, linear, $\exists\, U \in \mathcal{S} \cap \mathcal{X}_+$: $\mathfrak{p}(U) = 1$)

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Risk measures as capital requirements

- Risk := capital requirement := minimal cost for securing X
- $\mathcal{R} := (\mathcal{A}, \mathcal{S}, \mathfrak{p})$ RISK MEASUREMENT REGIME if

$$\boxed{\forall X \in \mathcal{X} : \ \sup\{\mathfrak{p}(Z) \mid Z \in \mathcal{S}, X + Z \in \mathcal{A}\} < \infty}$$

RISK MEASURE associated to $\mathcal{R}$:

$$\boxed{\rho_{\mathcal{R}}(X) := \inf\{\mathfrak{p}(Z) \mid Z \in \mathcal{S}, X - Z \in \mathcal{A}\} \in (-\infty, \infty]}$$

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Risk measures as capital requirements

Advantage: Operational interpretation, generalisation of cash-additive risk measures!

Further properties of $\rho_{\mathcal{R}}$

- convexity and properness
- $\preceq$-monotonicity: $X \preceq Y \implies \rho_{\mathcal{R}}(X) \leq \rho_{\mathcal{R}}(Y)$
- $\mathcal{S}$-additivity: $X \in \mathcal{X}, Z \in \mathcal{S} \implies \rho_{\mathcal{R}}(X + Z) = \rho_{\mathcal{R}}(X) + \mathfrak{p}(Z)$

Model free risk measurement

- **Aim:** Find maximal model space/domain of definition.
- Start in model free setting $\rightarrow$ infer suitable analytic structure
- **completely model free** Banach lattice $\mathcal{L}^\infty := \mathcal{L}^\infty(\Omega, \mathcal{F})$ – bounded random variables
- pointwise order
- $|\cdot|_\infty : X \mapsto \sup_{\omega\in\Omega} |X(\omega)|$ norm

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Weak reference probability models

Definition

A finite risk measure $\rho_{\mathcal{R}} : \mathcal{L}^\infty \to \mathbb{R}$ is (SEQUENTIALLY) CONTINUOUS FROM ABOVE if

$$(X_n)_{n\in\mathbb{N}\cup\{\infty\}} \subset \mathcal{X},\ X_n \downarrow X_\infty \text{ in order } \implies \rho_{\mathcal{R}}(X_n) \downarrow \rho_{\mathcal{R}}(X_\infty).$$

Theorem (Liebrich & Svindland, '17, also c.f. Maccheroni et al., '14)

If $\rho_{\mathcal{R}}$ is finite and continuous from above, $\exists$ WEAK REFERENCE PROBABILITY MODEL $\mathbb{P}$, *i.e. probability measure* $\mathbb{P}$ *on* $(\Omega, \mathcal{F})$ *s.t.*

- $\mathbb{P} \approx \operatorname{dom}(\rho^*_{\mathcal{R}})$
- $\exists \gamma > 0 :\ \rho^*_{\mathcal{R}}(\gamma\mathbb{P}) < \infty$

($\rho^*_{\mathcal{R}}(\mu) := \sup_{Y\in\mathcal{L}^\infty} \int Y\, d\mu - \rho_{\mathcal{R}}(Y)$) DUAL CONJUGATE

Towards canonical model spaces

- $\rho_{\mathcal{R}} : L^{\infty}_{\mathbb{P}} \to \mathbb{R}$ continuous from above
- $\rho_{\mathcal{R}}(0) = 0$
- $\mathbb{P}$ weak reference probability model
- $\text{dom}\,(\rho^*_{\mathcal{R}}) \subset (\mathbf{ca}_{\mathbb{P}})_+$
- Norming functional ($\leftrightarrow$ "Young function")

$$\rho(|X|) := \sup_{\mu \in \text{dom}\,(\rho^*_{\mathcal{R}})} \int |X|\, d\mu - \rho^*_{\mathcal{R}}(\mu) \in (-\infty, \infty], \; X \in L^0_{\mathbb{P}}.$$

- $\|X\|_{\mathcal{R}} := \left(\sup\left\{\lambda > 0 \mid \rho\left(\lambda |X|\right) \leq 1\right\}\right)^{-1} \in [0, \infty], \quad X \in L^0_{\mathbb{P}}$
  ($\leftrightarrow$ "Luxemburg norm")
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The Minkowski domain

Theorem (Liebrich & Svindland, '17)

(i) *Set* $\mathcal{P} := \{\frac{\mu}{\mu(\Omega)} \mid \mu \in \mathrm{dom}\,(\rho_{\mathcal{R}}^*)\}$ *and* $\sigma(\mathbb{P}) = \frac{\mu(\Omega)}{1+\rho_{\mathcal{R}}^*(\mu)}$ *if* $\frac{\mu}{\mu(\Omega)} = \mathbb{P}$*. Then* $L^{\mathcal{R}} = L^1_{\sigma,\mathcal{P}}$ *and* $\|\cdot\|_{\mathcal{R}} = \|\cdot\|_{1,\sigma}$*.*

(ii) $L^\infty_{\mathbb{P}} \subset L^{\mathcal{R}}$*.*

(iii) **Reference model robustness:** $L^{\mathcal{R}}$ *independent of choice of* $\mathbb{P}$ *among weak reference probability models.*

- $L^{\mathcal{R}}$ Minkowski domain associated to $\rho_{\mathcal{R}}$
- Geometry completely determined by $\rho_{\mathcal{R}}$

Extending $\rho_{\mathcal{R}}$

- Why is $(L^{\mathcal{R}}, \|\cdot\|_{\mathcal{R}})$ a canonical model space for $\rho_{\mathcal{R}}$?
- Extend **acceptability** on $L^\infty_{\mathbb{P}}$ to $L^{\mathcal{R}}$:

  Dually: $\mathcal{A}_1 := \left\{ X \mid \forall \mu \in \mathrm{dom}\,(\rho^*_{\mathcal{R}}) : \int X \, d\mu \le \rho^*_{\mathcal{R}}(\mu) \right\},$

  à la Delbaen: $\mathcal{A}_2 := \{ X \mid \inf_{n \in \mathbb{N}} \sup_{m \in \mathbb{N}} \rho_{\mathcal{R}}((-n) \vee X \wedge m) \le 0 \}.$
- $\mathcal{A}_i \cap L^\infty_{\mathbb{P}} = \{ X \in L^\infty_{\mathbb{P}} \mid \rho_{\mathcal{R}}(X) \le 0 \}$

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Extending $\rho_{\mathcal{R}}$

Theorem (Liebrich & Svindland, '17)

(i) $\mathcal{R}_i := (\mathcal{A}_i, \mathcal{S}, \mathfrak{p})$ *risk measurement regime on* $L^{\mathcal{R}}$,
$\rho_{\mathcal{R}_i}|_{L^\infty_{\mathbb{P}}} = \rho_{\mathcal{R}}$.

(ii) $\rho_{\mathcal{R}_1}$ *order lower semicontinuous because*

$$\rho_{\mathcal{R}_1}(X) = \sup_{\mu \in \text{dom}\,(\rho^*_{\mathcal{R}})} \int X\, d\mu - \rho^*_{\mathcal{R}}(\mu), \quad X \in L^{\mathcal{R}}.$$

(iii) $\forall U \in L^\infty_{\mathbb{P}}\, \forall X \in L^{\mathcal{R}}$:

$$\rho_{\mathcal{R}_1}(U + X) = \sup\nolimits_{m \in \mathbb{N}} \inf\nolimits_{n \in \mathbb{N}} \rho_{\mathcal{R}}(U + (-n) \vee X \wedge m),$$
$$\rho_{\mathcal{R}_2}(U + X) = \inf\nolimits_{n \in \mathbb{N}} \sup\nolimits_{m \in \mathbb{N}} \rho_{\mathcal{R}}(U + (-n) \vee X \wedge m).$$

- Often, if X^+ and X^- not bounded: $\rho_{\mathcal{R}_1}(X) < \rho_{\mathcal{R}_2}(X)$.

Beyond domination

- order structure of Minkowski domains is an **almost sure** one
- $\Rightarrow$ **Order complete Riesz space, sequences are sufficient!**
- $(L^p_{\mathcal{P},\sigma}, \preceq_{\mathcal{P}})$ may **NOT** be order complete, 'robust essential suprema' may not exist
- Order completeness is key: set $\sigma \equiv 1$, $L^\infty_{\mathcal{P}} := L^\infty_{\mathcal{P},\sigma}$.
  - *Maggis, Meyer-Brandis, Svindland ('18)*: $L^\infty_{\mathcal{P}}$ order complete iff $(L^\infty_{\mathcal{P}}, ca_{\mathcal{P}})$ is weak* pairing; Krein-Šmulian based tricks are possible
  - *Cohen ('12)*: Hahn property (special instance of order completeness of $L^\infty_{\mathcal{P}}$) allows to define robust conditional expectations
- $\Rightarrow$ study $(L^\infty_{\mathcal{P}}, \preceq_{\mathcal{P}})$ as a Riesz space

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Understanding the consequences of order completeness

Definition

A measure $\mu \ll \mathcal{P}$ has a $\mathcal{P}$-QUASI-SURE SUPPORT if there is $S(\mu) \in \mathcal{F}$ s.t. $\mu(S(\mu)) = \mu(\Omega)$ and $\mu(A) = \mu(\Omega)$ implies $\mathbf{1}_{S(\mu)} \preceq_{\mathcal{P}} \mathbf{1}_A$.

Theorem (Liebrich, Munari & Svindland)

Suppose $L^\infty_{\mathcal{P}}$ is order complete.

(i) *Each $\mu \in (\mathbf{ca}_{\mathcal{P}})_+$ (e.g. $\mathbb{P} \in \mathcal{P}$) has a quasi-sure support $S(\mu)$.*

(ii) *$L^\infty_{\mathcal{P}}$ has the* **aggregation property:** *for all $(X^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}}$ with*

- $X^{\mathbb{P}} = X^{\mathbb{P}}\mathbf{1}_{S(\mathbb{P})}$,
- $X^{\mathbb{P}}\mathbf{1}_{S(\mathbb{P})\cap S(\mathbb{Q})} = X^{\mathbb{Q}}\mathbf{1}_{S(\mathbb{P})\cap S(\mathbb{Q})}$,
- $\sup_{\mathbb{P}\in\mathcal{P}} \|X^{\mathbb{P}}\|_{\infty,\mathcal{P}} < \infty$,

there is $X \in L^\infty_{\mathcal{P}}$ s.t. $X^{\mathbb{P}} = X\mathbf{1}_{S(\mathbb{P})}$, $\mathbb{P} \in \mathcal{P}$.

Moreover, (i) and (ii) together imply order completeness.

Two consequences

Corollary

The following are equivalent:

(i) $L^\infty_{\mathcal{P}}$ *has countable sup-property* *(every supremum is attained by countable subset)* *and every* $\mathbb{P} \in \mathcal{P}$ *has a q.s. support*

(ii) $\mathcal{P}$ *is dominated*

Proposition

If every $\mathbb{P} \in \mathcal{P}$ *has a q.s. support and*

$$\mathcal{F}^{\mathcal{P}} := \bigcap_{\mathbb{P} \in \mathcal{P}} \sigma(\mathcal{F} \cup \{\mathbb{P}\text{-polar sets}\})$$

is the $\mathcal{P}$*-augmentation of* $\mathcal{F}$*, then* $L^\infty(\Omega, \mathcal{F}, \mathcal{P}) \subset L^\infty(\Omega, \mathcal{F}^{\mathcal{P}}, \mathcal{P})$ *is order dense, and order completeness of the first space implies order completeness of the latter.*

Thank you for your attention!