Structure of martingale transports in Banach spaces

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Take $\mathbb{B}$ Banach space, μ, ν proba on $\mathbb{B}$ with finite first moment

Transports from μ to ν

$$\pi := \pi(\mu,\nu) := \left\{\theta \text{ law of } (X, Y) : X \sim \mu, Y \sim \nu\right\} =$$

$$= \left\{\theta \ : \theta(dx, dy) = \mu(dx)K_x(dy), K_x \text{ proba}, \int \mu(dx)K_x = \nu\right\}$$

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Martingale transports

$$\mathcal{M} := \mathcal{M}(\mu,\nu) := \{\theta \text{ law of } (X, Y) : X \sim \mu, Y \sim \nu, \mathbb{E}^\theta[Y|X] = X\} =$$

$$= \{\theta = \mu \otimes K \in \pi(\mu,\nu) : \int K_x(dy)y = x \text{ for all } x\}$$

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Theorem (Strassen)

If $\mathbb{B}$ is separable then

$$\mathcal{M}(\mu,\nu) \neq \emptyset \iff \mu \preceq_c \nu$$

So we will always assume $\mathbb{B}$ is separable and $\mu \preceq_c \nu$, i.e. $\int \phi d(\nu - \mu) \geq 0$ for all ϕ convex (and Lipschitz)

Structure of martingale transports if $\mathbb{B} = \mathbb{R}$

If $\mathbb{B} = \mathbb{R}$,

$$\mu \preceq_c \nu \iff u_\mu(x) \leq u_\nu(x) \text{ for all } x$$

where $u_\lambda(x) := \int |x - y| \lambda(dy)$ is the potential of λ.

$(I_k)_k$:= the disjoint open intervals s.t. $\{u_\mu < u_\nu\} = \cup_k I_k$,

$$C_x \quad := \quad \begin{cases} \overline{I_k} & \text{if } x \in I_k \\ \{x\} & \text{if } x \in \{u_\mu = u_\nu\} \end{cases}$$

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Theorem (Beiglböck, Nutz, Touzi '15)

If $\theta = \mu \otimes K \in \mathcal{M}$ then K_x is concentrated on C_x for μ a.e. x (and not on $D_x \subsetneq C_x$)

Intuitively: mass transported with $\theta \in \mathcal{M}$ from x must stay in C_x

Can we define some C_x for general $\mathbb{B}$ so that prev. thm holds?
How? One cannot characterize $\mu \preceq_c \nu$ with some $u_\mu \leq u_\nu$...

For $\mathbb{B} = \mathbb{R}^N$ see also preprint by De March and Touzi (...), and partial results by Ghoussoub, Kim And Lim.

Jensen inequality

- Denote with ⓜ be the barycentre $\int_{\mathbb{B}} x\mu(dx)$ of μ.
- Given $a, \phi : \mathbb{B} \to \mathbb{R}$ with ϕ convex, we say that *a supports* ϕ *at x* if a is affine and continuous, $a \leq \phi$ and $a(x) = \phi(x)$.
- If a supports ϕ at x then

$$\int (\phi - a) d\mu \geq 0$$

and = holds $\iff$ μ is concentrated on $\{\phi - a = 0\}$.

Jenses Inequality

If ϕ is convex then

$$\int \phi d\mu \geq \phi(ⓜ),$$

and = $\iff$ μ is conc. on $\{\phi = a\}$ for some a supporting ϕ at ⓜ.

Proof: $\exists a$ supporting ϕ at ⓜ, and $\phi(ⓜ) = a(ⓜ) = \int a\, d\mu$.

Equality in Jensen inequality

Theorem

If $\mu \otimes K \in \mathcal{M}(\mu, \nu)$, ϕ convex s.t. $\int \phi d(\nu - \mu) = 0$ and a_x supports ϕ at x then K_x is conc. on $\{\phi = a_x\}$ for μ a.e. x.

Proof: since

$$\int \mu(dx) \int K_x(dy)\phi(y) - \phi(x) = 0$$

and $-\nabla\phi(x) \cdot \int K_x(dy)(y - x) = 0$ we get

$$\int \mu(dx) \int K_x(dy)(\phi - a_x)(y) = 0$$

where $a_x(y) := \phi(x) + \nabla\phi(x) \cdot (y - x)$ supports ϕ at x. $\square$

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Corollary

If $\exists\phi$ *strictly* convex s.t. $\int \phi d(\nu - \mu) = 0$ then $\mu = \nu$.

Proof: $\exists \mu \otimes K \in \mathcal{M}$, K_x is conc. on $\{\phi = a_x\} = \{x\}$, so $K_x = \delta_x$.

K_x is conc. on $\{\phi = a_x\}$, but $\{\phi - a_x = 0\}$ depends on a_x

For D convex, $x \in D$, denote with
$F(x, D)$ the smallest face of D containing x

Replace $\int \phi d(\nu - \mu) = 0, \{\phi - a_x = 0\}$
with $\int \phi^n d(\nu - \mu) \to 0, \{\phi^n - a_x^n \to 0\}$

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If $\theta = \mu \otimes K \in \mathcal{M}$, ϕ^n convex s.t. $\int \phi^n d(\nu - \mu) \to 0$ then $A((\phi^n)_n)_x := \overline{F(x, \{\phi^n - a_x^n \to 0\})}$ does not depend on a_x^n, and K_x is concentrated on $A((\phi^n)_n)_x$ for μ a.e. x

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Corollary

If $\mathbb{B} = \mathbb{R}$ then C_x equals
$S_x := \cap\{A((\phi^n)_n)_x : \phi^n \text{ convex s.t. } \int \phi^n d(\nu - \mu) \to 0\}$

Can we use the eq. above as a *definition* of C_x when $\mathbb{B} \neq \mathbb{R}$?
Is it true that the mass transported with $\theta \in \mathcal{M}$ from x must stay in S_x (and not in $D_x \subsetneq S_x$) ?

Thrilling climax ;-)

Surprisingly NO!

Explicit examples show that S_x can be too small,
i.e. K_x is not always conc. on S_x for μ a.e. x for all $\mu \otimes K \in \mathcal{M}$

What should we do?

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What should we do?

Consider these clues:

- In some examples we can only prove that $C_x \subseteq A((\phi^n)_n)_x$ for μ a.e. x
- We want to prove that for μ a.e. x, K_x is conc. on C_x. This also suggests that C_x should only be defined for μ a.e. x
- In order to have

$$K_x \text{ conc. on } A((\phi^n)_n)_x \Rightarrow K_x \text{ conc. on } \cap \{A((\phi^n)_n)_x : \phi_n ...\}$$

we should have a *countable* intersection

C_x as an essential infimum

We aim to define C as

$$\mu\text{-}ess \bigcap \{A((\phi^n)_n) : \phi_n \text{ is convex s.t. } \int \phi^n d(\nu - \mu) \to 0\} \quad (1)$$

Is this possible? Can one take equiv. countable intersection?
If so then any K_x is conc. on C_x for μ a.e. x, we are done.

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For $Y, Z, X_i \in A := L^0(\mathbb{P}; [0,1])$ let $Y \leq Z$ if $\mathcal{Y} \leq \mathcal{Z}$ $\mathbb{P}$ a.e., then by definition $X := \mathbb{P}\text{-}essinf_{i \in I} X_i$ is simply $A\text{-}\inf_{i \in I} X_i$,

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Definition

We say that $(A, \leq)$ has the *Countable Inf Property* if $\forall D \subseteq A$ bounded below $\exists \inf D$ and $d^n \in D$ with $\inf_{n \in \mathbb{N}} d^n = \inf D$.

$\exists \mathbb{P}\text{-}essinf_{i \in I} X_i$ means that $L^0(\mathbb{P}; [0,1])$ satisfies the CIP

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For $A, B \in CC(\mathbb{B}) := \{V \subseteq \mathbb{B} : V \text{ is closed and convex}\}$
define $A \leq B$ if $A \subseteq B$, so $\inf\{A, B\} = A \cap B$, so
to conclude, let's prove that $L^0(\mu; CC(\mathbb{B}))$ satisfies the CIP

The Countable Inf Property

Theorem

Let A, B be ordered, $h : A \to B$ strictly increasing. If B satisfies CIP and any $(a_n)_{n\in\mathbb{N}} \subseteq A$ bounded below has an infimum, then A satisfies CIP.

Take $A := L^0(\mathbb{P}; [0,1])$, $B := [0,1]$ and $h = \mathbb{E}^{\mathbb{P}} \Rightarrow \exists\mathbb{P}\text{-}essinf_{i\in I} X_i$

If $\mathbb{B} = \mathbb{R}^N$, M. Larsson showed $\exists\phi : CC(\mathbb{B}) \to [0,1]$ strictly increasing and measurable, so can take $A := L^0(\mu; CC(\mathbb{B}))$, $B := [0,1]$ and $h := \mathbb{E}^{\mu} \circ \phi \Rightarrow \exists\mu\text{-}essinf_{(\phi^n)_n} A((\phi^n)_n)$

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Let $B_k := \{x \in \mathbb{B} : ||x|| \le k\}$, take $A := L^0(\mu; CC(\mathbb{B}) \cap B_k)$, $B := CC(\mathbb{B}) \cap B_k$, let h be the selection expectation, then if $\mathbb{B}$ is reflexive h is strictly increasing and B satisfies CIP $\Rightarrow$ $\exists \mu\text{-}essinf_{(\phi^n)_n} A((\phi^n)_n) \cap B_k \Rightarrow \exists \mu\text{-}essinf_{(\phi^n)_n} A((\phi^n)_n =: C_x$ (and $L^0(\mu; CC(\mathbb{B}))$ satisfies CIP)