Robust Modelling of Financial Markets in Discrete time

Johannes Wiesel
Mathematical Institute
University of Oxford

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Setting

- Financial market: Let $T \in \mathbb{N}$, $d \in \mathbb{N}$, S_t the $\mathbb{R}^d$-valued canonical process $S_t(\omega) = \omega_t$ for all $t \in 1, ..., T$ and $S_0 = s_0 \in \mathbb{R}$. The natural filtration is $\mathbb{F} = (\mathcal{F}_t^S)_{t=0,\dots,T}$ and its universal completion $\mathbb{F}^{\mathcal{U}} = (\mathcal{F}_t^{\mathcal{U}})_{t=0,\dots,T}$.
- Trading: $\mathcal{H}(\mathbb{F}^{\mathcal{U}})$ are defined as the set of $\mathbb{F}^{\mathcal{U}}$-predictable $\mathbb{R}^d$-valued processes. Given $H \in \mathcal{H}(\mathbb{F}^{\mathcal{U}})$ we set $H \circ S_t = \sum_{u=1}^t H_u \Delta S_u$.
- It is possible to include a finite number of statically traded options.
- No market frictions.

Motivation

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Universally acceptable setting: $\Omega = (\mathbb{R}^d)^T, \mathfrak{P} = \mathcal{P}((\mathbb{R}^d)^T)$

Quasisure approach: Add probability measures to arrive at $\mathfrak{P}$

We are here!

Pathwise robust approach: Start with $(\mathbb{R}^d)^T$, rule out impossible paths to get Ω

Model-specific approach: Fixed $\mathbb{P}$

Notions of Arbitrage

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Definition

Fix a filtration $\mathbb{F}$, a set $\mathfrak{P}$ and a set $\Omega \subseteq (\mathbb{R}^d)^T$.

- A One-Point Arbitrage is a strategy $H \in \mathcal{H}(\mathbb{F})$ such that $H \circ S_T \geq 0$ on Ω with strict inequality for some $\omega \in \Omega$.
- A Strong Arbitrage is a strategy $H \in \mathcal{H}(\mathbb{F})$ such that $H \circ S_T > 0$ on Ω.
- A $\mathfrak{P}$-quasisure Arbitrage is a strategy $H \in \mathcal{H}(\mathbb{F}^{\mathcal{U}})$ such that $H \circ S_T \geq 0 \quad \mathfrak{P}$-q.s. and $\mathbb{P}(H \circ S_T > 0) > 0$ for some $\mathbb{P} \in \mathfrak{P}$.

Martingale measures (1)

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For a set Ω and a filtration $\mathbb{F}$ we define

$$\mathcal{M}_\Omega^f(\mathbb{F}) = \{\mathbb{Q} \in \mathcal{P}^f((\mathbb{R}^d)^T) \mid S \text{ is an } \mathbb{F}\text{-martingale under } \mathbb{Q},\ \mathbb{Q}(\Omega) = 1\}.$$

Furthermore let

$$\Omega^* = \bigcup_{\mathbb{Q} \in \mathcal{M}_\Omega^f(\mathbb{F}^S)} \mathrm{supp}(\mathbb{Q}).$$

We also define

$$\mathbb{F}^M = (\mathcal{F}_t^M)_{t \in \{0,\dots,T\}}, \quad \text{where } \mathcal{F}_t^M = \bigcap_{\mathbb{Q} \in \mathcal{M}_\Omega(\mathbb{F}^S)} \mathcal{F}_t^S \vee \mathcal{N}^{\mathbb{Q}}(\mathcal{F}_T^S),$$

where $\mathcal{N}^{\mathbb{Q}}(\mathcal{F}_T^S) := \{N \subseteq A \in \mathcal{F}_T^S \mid \mathbb{Q}(A) = 0\}$.

Martingale measures (2)

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Given a set of measures $\mathfrak{P}$ we define

$$\mathcal{Q} = \{\mathbb{Q} \in \mathcal{P}(X) \mid S \text{ is a } \mathbb{F}^{\mathcal{U}}\text{-martingale under } \mathbb{Q},\ \exists \mathbb{P} \in \mathfrak{P} \text{ s.t. } \mathbb{Q} \ll \mathbb{P}\}.$$

This is the natural counterpart to the case $\mathfrak{P} = \{\mathbb{P}\}$.

Dynamic programming

To establish their Multistep results we essentially paste together their One-Step counterpants. For this to work we need certain measurability assumptions: We assume Ω is analytic and $\mathfrak{P}$ satisfies (APS):

Definition

The set $\mathfrak{P}$ satisfies the Analytic Product Structure condition (APS), if

$$\mathfrak{P} = \{\mathbb{P}_0 \otimes \cdots \otimes \mathbb{P}_{T-1} \mid \mathbb{P}_t \text{ is } \mathcal{F}_t^{\mathcal{U}}\text{-measurable selector of } \mathfrak{P}_t\},$$

where the sets $\mathfrak{P}_t(\omega) \subseteq \mathcal{P}((\mathbb{R}^d)^T)$ are nonempty, convex and

$$\text{graph}(\mathfrak{P}_t) = \{(\omega, \mathbb{P}) \mid \omega \in (\mathbb{R}^d)^t,\ \mathbb{P} \in \mathfrak{P}_t(\omega)\}$$

is analytic.

Fundamental Theorem of Asset pricing (1)

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Theorem

For $\mathfrak{P}$ satisfying (APS) there exists a set $\Omega \in \mathcal{F}^{\mathcal{U}}$ such that $\mathbb{P}(\Omega) = 1$ for all $\mathbb{P} \in \mathfrak{P}$ and a filtration $\tilde{\mathbb{F}}$ with $\mathbb{F}^S \subseteq \tilde{\mathbb{F}} \subseteq \mathbb{F}^{\mathcal{M}}$, such that the following are equivalent:

1. $\mathcal{Q} \neq \emptyset$.
2. $\mathbb{P}(\Omega^*) > 0$ *for some* $\mathbb{P} \in \mathfrak{P}$.
3. $\mathcal{M}_\Omega \neq \emptyset$.
4. $\Omega^* \neq \emptyset$.
5. *There is no Strong Arbitrage in* $\mathcal{H}(\tilde{\mathbb{F}})$ *on* Ω.

Conversely, for an analytic set Ω there exists a set $\mathfrak{P}^\Omega$ satisfying (APS), such that (1.)-(5.) are equivalent.

Fundamental Theorem of Asset pricing (2)

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Theorem

Let $\mathfrak{P}$ be a set of probability measures satisfying (APS). Then there exists an analytic set of scenarios $\Omega^{\mathfrak{P}}$ with $\mathbb{P}(\Omega^{\mathfrak{P}}) = 1$ for all $\mathbb{P} \in \mathfrak{P}$, such that the following are equivalent:

1. $\mathbb{P}((\Omega^{\mathfrak{P}})^*) = 1$ *for all* $\mathbb{P} \in \mathfrak{P}$.
2. *For all* $\mathbb{P} \in \mathfrak{P}$ *there exists* $\mathbb{Q} \in \mathcal{Q}$ *such that* $\mathbb{P} \ll \mathbb{Q}$.
3. *NA*($\mathfrak{P}$) *holds.*

Conversely, if Ω is an analytic set, then there exists a set $\mathfrak{P}^{\Omega}$ of probability measures satisfying (APS) with $\mathbb{P}(\Omega) = 1$ for all $\mathbb{P} \in \mathfrak{P}^{\Omega}$ such that the following are equivalent:

1. $\Omega = \Omega^*$.
2. *For all* $\mathbb{P} \in \mathfrak{P}^{\Omega}$ *there exists* $\mathbb{Q} \in \mathcal{Q}$ *such that* $\mathbb{P} \ll \mathbb{Q}$.
3. *NA*($\mathfrak{P}^{\Omega}$) *holds.*

Idea of proof (1)

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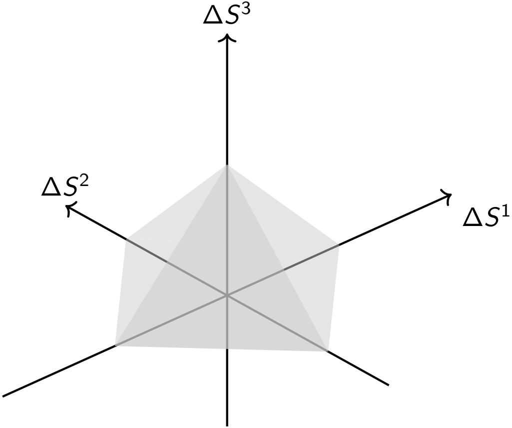
Take T=1 and $S_0 = 0$. Given $\mathfrak{P}$ define

$$\Omega^{\mathfrak{P}} = \mathrm{supp}(\mathfrak{P}) = \bigcap \{A \subseteq \mathbb{R}^d \text{ closed} \mid \mathbb{P}(A) = 1 \ \forall \mathbb{P} \in \mathfrak{P}\}.$$

Obviously $\mathbb{P}(\Omega^{\mathfrak{P}}) = 1$ for all $\mathbb{P} \in \mathfrak{P}$. Check that $\Omega^{\mathfrak{P}}$ is analytic and show that NA($\mathfrak{P}$) is equivalent to $\Omega^{\mathfrak{P}} = (\Omega^{\mathfrak{P}})^*$.

Idea of proof (2)



Idea of proof (3)

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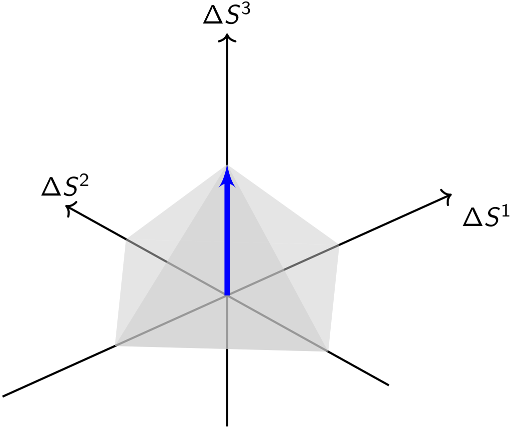
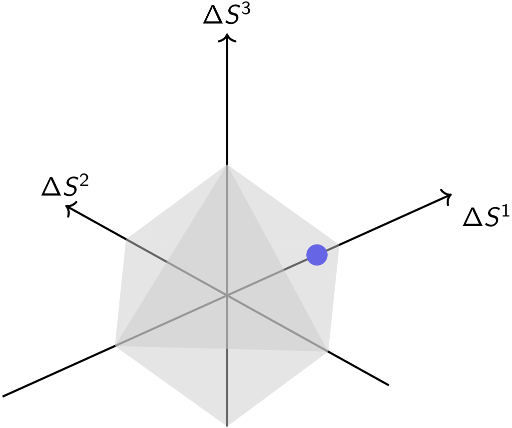
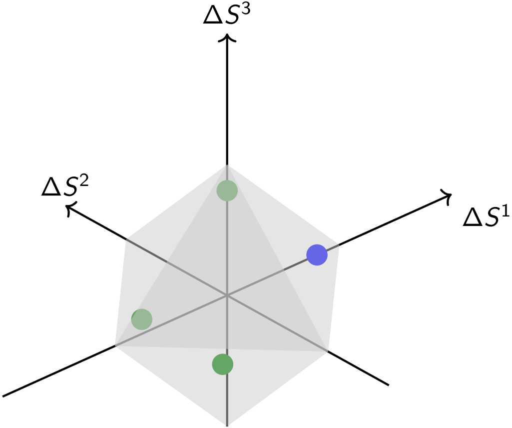


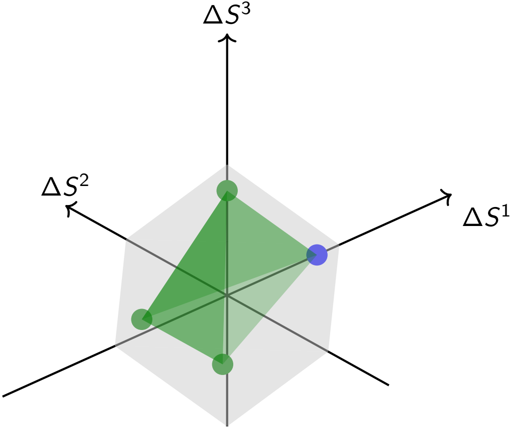
Idea of proof (3)



Idea of proof (4)



Idea of proof (5)



Idea of proof (6)

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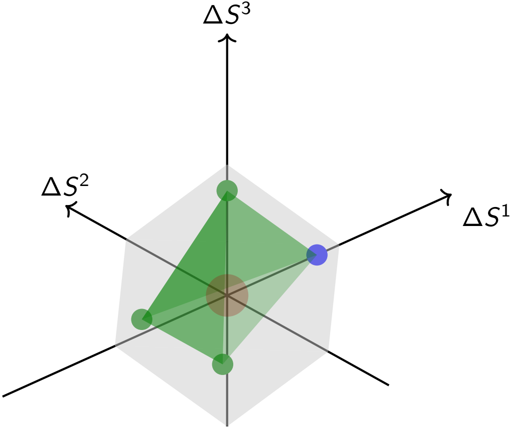

Superhedging Theorem (1)

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For a set $\Omega \subseteq (\mathbb{R}^d)^T$ we denote the pathwise superhedging price on Ω^* by

$$\pi_{\Omega^*}(g) := \inf\{x \in \mathbb{R} \mid \exists H \in \mathcal{H}(\mathbb{F}^{\mathcal{U}}) \text{ s. t. } x + (H \circ S_T) \geq g \text{ on } \Omega^*\}$$

and denote the $\mathfrak{P}$-q.s. superhedging price by

$$\pi^{\mathfrak{P}}(g) := \inf\{x \in \mathbb{R} \mid \exists H \in \mathcal{H}(\mathbb{F}^{\mathcal{U}}) \text{ s. t. } x + (H \circ S_T) \geq g \ \mathfrak{P}\text{-q.s.}\}.$$

Superhedging Theorem (2)

Theorem

Let $\mathfrak{P}$ satisfy (APS). Let NA($\mathfrak{P}$) hold and let $g : X \to \mathbb{R}$ be upper semianalytic. Then there exists a measure $\hat{\mathbb{P}} = \hat{\mathbb{P}}_0 \otimes \cdots \otimes \hat{\mathbb{P}}_{T-1}$, an $\mathcal{F}^{\mathcal{U}}$-measurable function $\mathfrak{g}$ such that $\mathfrak{g} = g$ $\mathfrak{P}$-q.s. and an $\mathcal{F}^{\mathcal{U}}$-measurable set $\Omega_g^{\mathfrak{P}}$ such that $\mathbb{P}(\Omega_g^{\mathfrak{P}}) = 1$ for all $\mathbb{P} \in \mathfrak{P}$ such that duality holds:

$$\pi^{\mathfrak{P}}(g) = \pi^{\hat{\mathbb{P}}}(g) = \pi_{(\Omega_g^{\mathfrak{P}})^*}(g) = \sup_{\mathbb{Q} \in \mathcal{M}_{\Omega^{\mathfrak{P}}}} \mathbb{E}_{\mathbb{Q}}(\mathfrak{g}) = \sup_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_{\mathbb{Q}}(g).$$

Conversely, let Ω be an analytic subset of X with $\Omega^* \neq \emptyset$ and let $g : X \to \mathbb{R}$ be upper semianalytic. For any set $\mathfrak{P} \subseteq \mathcal{P}((\mathbb{R}^d)^T)$, which satisfies (APS) and has the same polar sets as $\mathcal{M}_\Omega^f$, we have

$$\sup_{\mathbb{Q} \in \mathcal{M}_\Omega^f} \mathbb{E}_{\mathbb{Q}}(g) = \pi_{\Omega^*}(g) = \pi^{\mathfrak{P}}(g) = \sup_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_{\mathbb{Q}}(g).$$

Idea of proof (1)

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Again $T = 1$, $S_0 = 0$. Take $(\mathbb{P}_n)_{n\in\mathbb{N}}$ with $\mathbb{P}_n \in \mathfrak{P}$ such that NA$(\mathbb{P}_n)$ holds and

$$\sup_{\mathbb{Q}\sim\mathbb{P}_n} \mathbb{E}_{\mathbb{Q}}(g) \uparrow \sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}_{\mathbb{Q}}(g).$$

Define $\hat{\mathbb{P}} := \sum_{k=1}^{\infty} 2^{-k}\mathbb{P}_k$ and note that NA$(\hat{\mathbb{P}})$ holds . By superhedging duality for $\hat{\mathbb{P}}$

$$sup_{\mathbb{Q}\in\mathcal{Q}}\mathbb{E}_{\mathbb{Q}}(g) = \sup_{n\in\mathbb{N}} \sup_{\mathbb{Q}\sim\mathbb{P}_n,\ \mathbb{Q}\in\mathcal{M}_{X_1}} \mathbb{E}_{\mathbb{Q}}(g) \leq \sup_{\mathbb{Q}\sim\hat{\mathbb{P}},\ \mathbb{Q}\in\mathcal{M}} \mathbb{E}_{\mathbb{Q}}(g)$$
$$= \pi^{\hat{\mathbb{P}}}(g) \leq \pi^{\mathfrak{P}}(g).$$

Argue $\pi^{\mathbb{P}}(g) = sup_{\mathbb{Q}\in\mathcal{Q}}\mathbb{E}_{\mathbb{Q}}(g)$ by contradiction argument.
The idea of reducing to one probability measure can also be applied to $\mathfrak{P}$-q.s. quantile hedging.

References

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Thank you!