Pairs trading under model uncertainty

Daisuke Yoshikawa
joint work with Mark H.A. Davis

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Background

- The long history of the pairs trading has to be date back to 80's, when Nunzio Tartaglia's trading group at Morgan Stanley developed this trading method.
- The attractiveness of this method is characterized, so-called, "market neutrality".
- However, although the market neutral trading strategy should stably make a profit, it sometimes makes a big loss.

Distance method and cointegration method

- Distance method: when the spread between stocks widens, the agent takes the position by selling the expensive stock and purchasing the cheaper stock. The agent will liquidate the position, when the spread is shrunk.
- Cointegration method: the agent chooses the pair of stocks specified by a cointegration test, like Philips-Ouliaris model, Engle-Granger model, Johansen model and VECM (vector error correction model). The cointegration test implies the mean-reverted point.

Optimal stopping problem for pairs trading

- The formulation of the stochastic processes of the pairs value leads the point further most from the mean-reverted point.
- The stochastic process should be mean-reverting, like Ornstein-Uhlenbeck process (OU process) (Elliott et al. (2005)), the OU process with jumps (Larsson et al. (2013)).
- For given stochastic processes of the pairs value, we can formulate the optimal stopping problem.
- The solution of this implies the point further most from the mean-reverted point.

Reducing model uncertainty

- To derive the mean-reverted point and the point further most from the mean-reverted point, we may use the cointegration method which depends on a model.
- Hence, if we incorrectly specify the model parameters, it may leads the big loss. This possibility is called “model uncertainty” or “model risk”.
- Methods for resolution of model uncertainty:
  - statistical sophistication (the introduction of the reinforcement learning method (Fallahpour et al. (2016); Hwang et al. (2016)) or machine lerning (Huang et al. (2018)) into the pairs trading is developed.)
  - stop-loss
  - entropic approach

Stop-loss

- Empirical researches on stop-loss for pairs trading: Huang and Martin (2017), Caldeira and Moura (2013), Herlemont and Alexander (2003), Nath (2003), Baviera and Baldi (2018).
- Since the pairs trading often consist of the long and short positions of stocks, the generation of the margin call may be crucial.
- In this case, for a given loss cut line, it may be helpful to define the optimal trading strategy including stop-loss (Leung and Li (2015); Ekström et al. (2011); Larsson et al. (2013)).

Entropic approach

- The strategy of pairs trading is based on the expectation of the pairs value.
- Taking into account the model uncertainty, we can estimate the probability measure (reference measure).
- Entropic approach suggests to define a penalty function for the misspecification of the reference measure (Hansen et al. (2006); Cartea et al. (2017)) and derive the optimal measure for the evaluation of the pairs value.
- In this talk, we show a boundary which is used for the exit or entry points for pairs trading, taking into account the model uncertainty.

Processes of the pair value

- The probability space is given by $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T)}$. We assume that $\mathcal{F}_0$ is the trivial σ-field and that $\mathcal{F}_T = \mathcal{F}$.
- The process of the pair value is given by Ornstein–Uhlenbeck process $\hat{X}_t$ such that

$$d\hat{X}_t = -\alpha(\hat{X}_t - \mu)dt + \sigma dW_t,\ \hat{X}_0 = \mu + x,$$

  where α, σ, x are positive constants and μ is the mean-reverted point. Further, W_t is a P-Brownian motion.
- Letting $X_t := \hat{X}_t - \mu$, it follows that

$$dX_t = -\alpha X_t dt + \sigma dW_t,\ X_0 = x. \tag{1}$$

Optimal stopping problem

- For the process X_t, we will find the point further most from the mean-reverted point; i.e., we consider the following optimal stopping problem:

$$v^0(t,x) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}[e^{-\rho(\tau-t)} X_\tau^{t,x}], \ (t,x) \in [0,T] \times \mathbb{R} \tag{2}$$

  where $\mathcal{T}_{t,T}$ is the set of stopping times valued in $[0,T]$, ρ is the discount rate and the process $\{X_\tau^{t,x}, \tau \geq t\}$ is the solution of (1) starting from x at t.
- We also assume that $\mathbb{E}\tau < \infty$ for all $\tau \in \mathcal{T}_{t,T}$.

Free boundary problem

- According to Peskir and Shiryaev (2006) (or other literatures, like Pham (2009)), the optimal stopping problem is reduced to the followings:

$$-\tilde{v}^0_\tau - \alpha x \tilde{v}^0_x + \frac{1}{2}\sigma^2 \tilde{v}^0_{xx} = \rho \tilde{v}^0 \quad \text{on } C^0, \tag{3}$$

$$\tilde{v}^0(\tau, \tilde{b}(\tau)) = \tilde{b}(\tau) \text{ and } \tilde{v}^0_x(\tau, \tilde{b}(\tau)) = 1 \text{ on } \partial C^0, \tag{4}$$

$$\tilde{v}^0(0, x) = x, \tag{5}$$

  where $\tau := T - t$ and $\tilde{v}^0(\tau, x) = v^0(t, x)$.
- The continuation region C^0:

$$C^0 := \{x \in \mathbb{R} : v^0(t, x) > x\},$$

- The complement set of C^0 is the stopping region

$$\mathcal{S}^0 := \{x \in \mathbb{R} : v^0(t, x) = x\}.$$

Laplace transform

- Applying Laplace transform and substituting the terminal condition into other conditions, the free boundary problem is rewritten as follows:

$$x - \alpha x \hat{v}^0_x + \frac{1}{2}\sigma^2 \hat{v}^0_{xx} = (\rho + s)\hat{v}^0 \text{ on } \hat{\mathcal{C}}^0, \tag{6}$$

$$\hat{v}^0(s, s\hat{b}(s)) = s\hat{b}(s) \text{ and } \hat{v}^0_x(s, s\hat{b}(s)) = 1 \text{ on } \partial\hat{\mathcal{C}}^0, \tag{7}$$

  where $\hat{b} := \mathcal{L}[\tilde{b}]$.
- Here, we define a continuation region $\hat{\mathcal{C}}^0$ such that

$$\hat{\mathcal{C}}^0 := \{x \in \mathbb{R} : \hat{v}^0(s, x) > x/s\},$$

  and the stopping region $\hat{\mathcal{S}}^0$ as follows,

$$\hat{\mathcal{S}}^0 := \{x \in \mathbb{R} : \hat{v}^0(s, x) = x/s\},$$

  for all $s \in \mathbb{C}$, where $\mathbb{C}$ is the set of complex number.

Optimal boundary

- This type of ODE (6) can be solved by

$$\hat{v}^0(s,x) = a\int_0^\infty u^{\frac{\rho+s}{\alpha}-1} e^{\frac{\sqrt{2\alpha}}{\sigma}ux-\frac{u^2}{2}}du + \frac{x}{\alpha+\rho+s}.$$

- By boundary conditions, it follows that

$$\hat{v}^0(s,s\hat{b}) = a\int_0^\infty u^{\frac{\rho+s}{\alpha}-1} e^{\frac{\sqrt{2\alpha}}{\sigma}us\hat{b}-\frac{u^2}{2}}du + \frac{s\hat{b}}{\alpha+\rho+s} = s\hat{b},$$

$$\hat{v}^0_x(s,s\hat{b}) = a\frac{\sqrt{2\alpha}}{\sigma}\int_0^\infty u^{\frac{\rho+s}{\alpha}} e^{\frac{\sqrt{2\alpha}}{\sigma}us\hat{b}-\frac{u^2}{2}}du + \frac{1}{\alpha+\rho+s} = 1.$$

- Thus, we attain the boundary $\hat{b}(s)$ as follows;

$$\hat{b}(s) = \frac{\sigma}{s\sqrt{2\alpha}}\frac{\int_0^\infty u^{\frac{\rho+s}{\alpha}-1} e^{\frac{\sqrt{2\alpha}}{\sigma}us\hat{b}(s)-\frac{u^2}{2}}du}{\int_0^\infty u^{\frac{\rho+s}{\alpha}} e^{\frac{\sqrt{2\alpha}}{\sigma}us\hat{b}(s)-\frac{u^2}{2}}du}.$$

- From this and the inverse Laplace transform $\tilde{b}^*(\tau) = \mathcal{L}^{-1}[\hat{b}]$, the optimal boundary $b^*(t)$ is given by

$$b^*(t) = \tilde{b}^*(\tau).$$

Model uncertainty and Maxmin Expected Utility

- A class $\mathcal{Q}$ of probability measures equivalent to the reference measure P is defined on $(\Omega, \mathcal{F})$.
- The most intuitive way for avoiding the misspecification of the reference measure P is to derive the multiple-prior expected optimal reward is given by

$$\sup_{\tau \in \mathcal{T}} \inf_{Q \in \mathcal{Q}} \mathbb{E}_x^Q [e^{-\rho(\tau - t)} X_\tau].$$

- This is often called the robust utility or the maxmin expected utility.

Extension of maximin expected utility

- We consider the following optimal stopping problem conditioned on $x = X_t$ at time t:

$$v(t,x) := \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}^{Q}[e^{-\rho(\tau-t)} X_\tau^{t,x}], \tag{8}$$

where Q is the solution to the following:

$$\inf_{Q \in \mathcal{Q}} \left\{ \mathbb{E}^{Q}[e^{-\rho(\tau-t)} X_\tau^{t,x}] + \lambda e^{-\rho(\tau-t)} H_x[Q|P] \right\}. \tag{9}$$

- λ is a positive constant and $H(\cdot)$ is the relative entropy defined by

$$H_x(Q|P) := \begin{cases} \mathbb{E}_x^{Q}\left[\ln\left(\frac{dQ}{dP}\right)\right], & Q \in \mathcal{Q} \\ \infty, & \text{otherwise} \end{cases}$$

- The constant λ reflects how accurate the agent believes the reference measure P to be, i.e., when $\lambda \uparrow \infty$, the agent has complete trust in the reference measure P, whereas when $\lambda \downarrow 0$, the agent has no confidence in P.

Main result

Theorem

For $t \in [0, T)$, the optimal boundary $b(t)$ for (8) is given by the solution to the following free boundary problem,

$$-\rho v + v_t - \left(\alpha x + \frac{\sigma^2}{\lambda} e^{-\alpha t}\right) v_x + \frac{1}{2}\sigma^2 v_{xx} = 0 \text{ on } C, \tag{10}$$

$$v(t, b(t)) = b(t) \text{ and } v_x(t, b(t)) = 1 \text{ on } \partial C, \tag{11}$$

$$v(T, x) = x, \tag{12}$$

where C is a continuation region.

Brief sketch of the proof

- The measure Q taking into account the model uncertainty is given by (9): i.e.,

$$\inf_{Q \in \mathcal{Q}} \left\{ \mathbb{E}^Q[e^{-\rho(\tau-t)} X_\tau^{t,x}] + \lambda e^{-\rho(\tau-t)} H_x[Q|P] \right\}.$$

- The solution to this problem is given by

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_\tau} = \frac{e^{-X_\tau^{t,x}/\lambda}}{\mathbb{E}\left[e^{-X_\tau^{t,x}/\lambda}\right]},$$

  where we used Theorem 4.5 of Föllmer and Penner (2006).

- By definition of the OU process $X^{t,x}$, it follows that

$$\left.\frac{dQ}{dP}\right|_{\mathcal{F}_\tau} = \frac{e^{-\frac{1}{\lambda}\int_t^\tau \sigma e^{\alpha(u-t)} dW_u}}{\mathbb{E}\left[e^{-\frac{1}{\lambda}\int_t^\tau \sigma e^{\alpha(u-t)} dW_u}\right]}.$$

Brief sketch of the proof (cont.)

- From this and Girsanov, Cameron and Martin theorem, Q-Brownian motion W^* is given by

$$W^*_\tau = W_\tau + \frac{\sigma}{\lambda\alpha}\left(1 - e^{-\alpha\tau}\right).$$

- Substituting W^* into the optimal stopping problem (8), that is, $v(t,x) := \sup_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}^Q[e^{-\rho(\tau-t)}X^{t,x}_\tau]$ and applying the procedure given by Peskir and Shiryaev (2006), the following free boundary problem is reduced:

$$-\rho v + v_t - \left(\alpha x + \frac{\sigma^2}{\lambda}e^{-\alpha t}\right)v_x + \frac{1}{2}\sigma^2 v_{xx} = 0 \text{ on } C,$$
$$v(t,b(t)) = b(t) \text{ and } v_x(t,b(t)) = 1 \text{ on } \partial C,$$
$$v(T,x) = x,$$

where C is a continuation region.

Implication of the theorem

- For deriving the optimal boundary, we have to numerically solve the problem (10)-(12).
- The more tractable formula of $b(t)$ is available by introducing the following assumption:

$$\alpha < 1, \tag{13}$$

  where α is the speed of mean-reversion.
- If the speed of the mean-reversion is too fast, then it may be difficult to capture the opportunity for making profit by pairs trading.
- As we will see later, the numerical example using market data supports this assumption.

Proposition

Under the same assumption of the above Theorem and the assumption (13), the optimal boundary is given by the inverse Laplace transform of $\hat{b}(s)$ where the approximated value of $\hat{b}(s)$ is given by

$$\hat{b}(s) = \frac{\sigma}{s\sqrt{2\alpha}} \frac{\int_0^\infty u^{\frac{\rho+s}{\alpha}-1} e^{\frac{\sqrt{2\alpha}}{\sigma}\left(s\hat{b}(s)+\frac{\sigma^2}{\alpha\lambda}\left(\frac{s}{s-\alpha}\right)^2 e^{-\alpha T}\right)u-\frac{u^2}{2}} du}{\int_0^\infty u^{\frac{\rho+s}{\alpha}} e^{\frac{\sqrt{2\alpha}}{\sigma}\left(s\hat{b}(s)+\frac{\sigma^2}{\alpha\lambda}\left(\frac{s}{s-\alpha}\right)^2 e^{-\alpha T}\right)u-\frac{u^2}{2}} du} + \frac{s}{(\rho+s+\alpha-1)(\rho+s)(s-\alpha)^2}\frac{\sigma^2}{\lambda}e^{-\alpha T}.$$

Brief sketch of the proof

- Our target is to solve the problem (10)-(12); i.e.,

$$-\rho v + v_t - \left(\alpha x + \frac{\sigma^2}{\lambda} e^{-\alpha t}\right) v_x + \frac{1}{2}\sigma^2 v_{xx} = 0 \text{ on } \mathcal{C},$$

$$v(t, b(t)) = b(t) \text{ and } v_x(t, b(t)) = 1 \text{ on } \partial\mathcal{C},$$

$$v(T, x) = x,$$

- Applying Laplace transform to the above PDE, it follows that

$$-\rho\hat{v}(s,x) - s\hat{v}(s,x) + x - \alpha x\hat{v}_x(s,x) + \frac{\sigma^2}{\lambda} e^{-\alpha T}\hat{v}_x(s-\alpha, x) + \frac{1}{2}\sigma^2\hat{v}_{xx}(s,x) = 0 \text{ on } \hat{\mathcal{C}}, \tag{14}$$

$$\hat{v}(s, s\hat{b}(s)) = s\hat{b}(s) \text{ and } \hat{v}_x(s, s\hat{b}(s)) = 1 \text{ on } \partial\hat{\mathcal{C}}, \tag{15}$$

where $\mathcal{C}$ is a Laplace transform of the continuation region $\mathcal{C}$.

Brief sketch of the proof (cont.)

- We write (14) again,

$$-\rho\hat{v}(s,x) - s\hat{v}(s,x) + x - \alpha x\hat{v}_x(s,x) + \frac{\sigma^2}{\lambda}e^{-\alpha T}\hat{v}_x(s-\alpha,x) + \frac{1}{2}\sigma^2\hat{v}_{xx}(s,x) = 0.$$

- Under the assumption $\alpha < 1$, we have the approximation of the 5-th term $\hat{v}_x(s-\alpha,x) \approx \left(\frac{s}{s-\alpha}\right)^2 \hat{v}_x(s,x)$. Inserting it into (14), we have the following ODE.

$$-(\rho + s)\hat{v}(s,x) + x - \left(\alpha x - \frac{\sigma^2}{\lambda}e^{-\alpha T}\left(\frac{s}{s-\alpha}\right)^2\right)\hat{v}_x(s,x) + \frac{1}{2}\sigma^2\hat{v}_{xx}(s,x) = 0 \text{ on } \hat{C},$$

$$\hat{v}(s, s\hat{b}(s)) = s\hat{b}(s) \text{ and } \hat{v}_x(s, s\hat{b}(s)) = 1 \text{ on } \partial\hat{C},$$

Brief sketch of the proof (cont.)

- This ODE can be solved as $\hat{v} = \hat{v}_G + \hat{v}_P$, where $\hat{v}_G$ and $\hat{v}_P$ are given by

$$\hat{v}_G(x) = c_1 \int_0^\infty u^{\frac{\rho+s}{\alpha}-1} e^{\frac{\sqrt{2\alpha}}{\sigma}\left(x-\frac{\sigma^2}{\alpha\lambda}\left(\frac{s}{s-\alpha}\right)^2 e^{-\alpha T}\right)u-\frac{u^2}{2}} du,$$

$$\hat{v}_P(x) = \frac{1}{\rho+s+\alpha}x - \frac{1}{\rho+s+\alpha}\frac{1}{\rho+s}\left(\frac{s}{s-\alpha}\right)^2 \frac{\sigma^2}{\lambda}e^{-\alpha T}.$$

- By using boundary conditions, we have

$$\hat{v}(s, s\hat{b}(s)) = c_1 \int_0^\infty u^{\frac{\rho+s}{\alpha}-1} e^{\frac{\sqrt{2\alpha}}{\sigma}\left(s\hat{b}(s)-\frac{\sigma^2}{\alpha\lambda}\left(\frac{s}{s-\alpha}\right)^2 e^{-\alpha T}\right)u-\frac{u^2}{2}} du + \frac{1}{\rho+s+\alpha}s\hat{b}(s)$$
$$- \frac{1}{\rho+s+\alpha}\frac{1}{\rho+s}\left(\frac{s}{s-\alpha}\right)^2 \frac{\sigma^2}{\lambda}e^{-\alpha T} = s\hat{b}(s),$$

$$\hat{v}_x(s, s\hat{b}(s)) = c_1 \frac{\sqrt{2\alpha}}{\sigma} \int_0^\infty u^{\frac{\rho+s}{\alpha}} e^{\frac{\sqrt{2\alpha}}{\sigma}\left(s\hat{b}(s)-\frac{\sigma^2}{\alpha\lambda}\left(\frac{s}{s-\alpha}\right)^2 e^{-\alpha T}\right)u-\frac{u^2}{2}} du + \frac{1}{\rho+s+\alpha} = 1.$$

- Thus, the claim of the proposition is proved.

Boundary for the infinite horizon

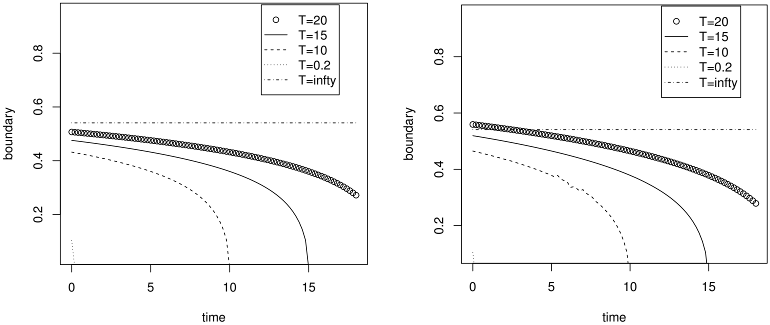
- The boundaries for finite horizon $b(t)$ should be close to the boundary for the infinite horizon, when the time horizon is sufficiently long.
- The boundary b^* for the infinite horizon is given as follows (Yoshikawa (2017)):

$$b^* = \frac{\sigma}{\sqrt{2\alpha}} \frac{\int_0^\infty u^{\frac{\rho}{\alpha}-1} e^{\frac{\sqrt{2\alpha}}{\sigma} u b^* - \frac{u^2}{2}} du}{\int_0^\infty u^{\frac{\rho}{\alpha}} e^{\frac{\sqrt{2\alpha}}{\sigma} u b^* - \frac{u^2}{2}} du},$$

where, to derive the boundary for the infinite horizon, it does not need Laplace transform.

Comparison to the boundary for the infinite horizon

- Following illustrations show boundaries b^* and $b(t)$ for infinite and finite horizons of $T = 1, 25, 50, 100$ and ∞ and parameters are given by $\alpha = 0.9, \sigma = 0.3$.
- In the left panel, the boundaries for infinite horizons are calculated not taking account the model uncertainty, that is, $\lambda = \infty$ and the boundaries in the right panel takes into account the model uncertainty, setting $\lambda = 10$.



Basic statistics of the Market data

- The application of our pairs trading strategy for the stock data listed on Dow-Jones Industrial Average.
- Market data from 1 June, 2017 to 30 September, 2017.
- By the Philips-Ouliaris test, we searched pairs with mean-reversion.

Pairs	μ	α	σ
”CSCO” ”PFE”	14.62	0.21	0.28
”KO” ”UTX”	59.42	0.24	0.22
”GE” ”MCD”	86.01	0.33	0.59
”GE” ”UNH”	64.71	0.23	0.35
”MCD” ”PG”	70.27	0.24	1.35
”MCD” ”UNH”	76.58	0.36	1.41
”MCD” ”UTX”	215.19	0.23	1.52
”MCD” ”V”	96.00	0.33	1.41

Table: The ticker codes is given such that “CSCO” is Cisco Systems, “PFE” is Pfizer, “KO” is Coca-Cola, “UTX” is United Technologies, “GE” is General Electric, “MCD” is McDonalsd’s, “UNH” is UnitedHealth Group, “PG” is Procter &Gamble, and “V” is Visa.

Simulation 1

The application for the out-of-sample (1 October, 2017 to 31 December 2017).

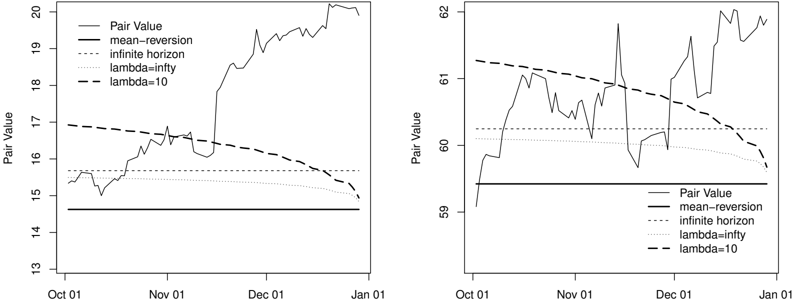


Figure: The left panel is (Cisco Systems, Pfizer), the right panel is (Coca-Cola, United Technologies).

Simulation 2

The application for the out-of-sample (1 October, 2017 to 31 December 2017).

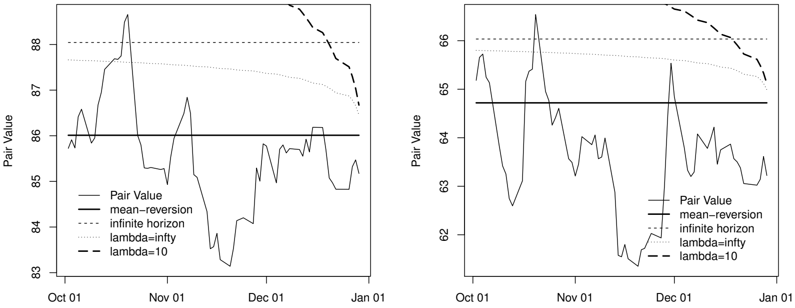


Figure: The left panel is (General Electric, McDonald's), and the right panel is (General Electric, UnitedHealth Group).

Simulation 3

The application for the out-of-sample (1 October, 2017 to 31 December 2017).

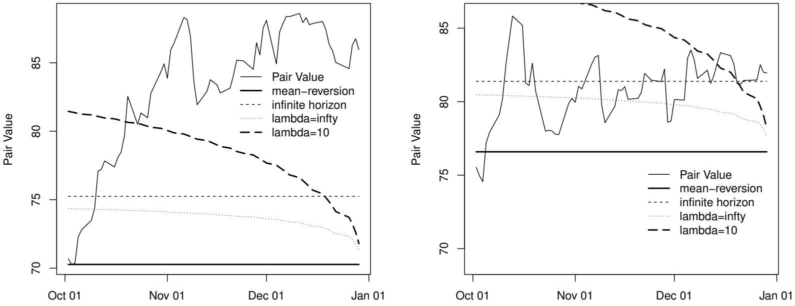


Figure: The left panel is (McDonald's, Procter & Gamble), the right panel is (McDonald's, UnitedHealth Group)

Simulation 4

The application for the out-of-sample (1 October, 2017 to 31 December 2017).

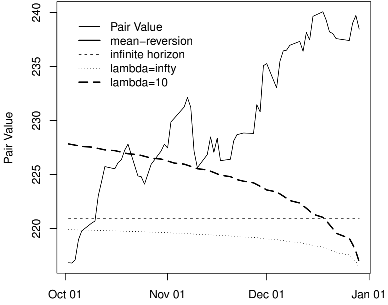


Figure: The left panel is (McDonald's, United Technologies), and the right panel is (McDonald's, Visa).

Implication of Simulations

- We can observe that the distance between mean-reverted point and boundary with $\lambda = 10$ is larger than that with $\lambda = \infty$.
- This implies that even if the pair value is getting higher, the less confidence investor (λ is small) does not consider that the pair value approaches the maximum value.
- Hence, the boundary with small λ is more diverted from the mean-reverted point than that with large λ.
- We also calculate the return for pairs trading with difference boundaries by defining the following trading code:
  - The position is set when the pair value touches either the boundary $b(t)$ or the mean-reverted point μ.
  - If the position is set for the case that the pair value touches $b(t)$, then the position is liquidated when the pair value touches the mean-reverted point, and vice versa.
  - After the liquidation, the next position is set when the pair value touches either $b(t)$ or μ and this position is liquidated by the same rule, and so forth.

Returns by pairs trading

Pairs	Infinite Horizon	$\lambda = 10$	$\lambda = \infty$
”CSCO” ”PFE”	0.25	0.15	0.30
”KO” ”UTX”	-0.96	0.01	-0.97
”GE” ”MCD”	-2.95	-0.09	-2.95
”GE” ”UNH”	1.99	1.99	-3.01
”MCD” ”PG”	0.12	0.05	0.16
”MCD” ”UNH”	0.09	-0.03	0.098
”MCD” ”UTX”	0.07	0.06	0.09
”MCD” ”V”	0.08	-0.001	0.07

Table: Returns on pairs trading strategy for boundaries with the infinite horizon, the finite horizon with $\lambda = 10$ and the finite horizon with $\lambda = \infty$.

Implication on the returns

- The second column shows the return for the pairs trading strategy assuming infinite horizon without taking into account model uncertainty, that is, $\lambda = \infty$.
- The fourth column shows the result without taking into account model uncertainty ($\lambda = \infty$), but the horizon is finite.
- By restricting the horizon, we have larger return in some cases (see (“CSCO”, “PFE”), (“MCD”, “PG”), (“MCD”,”UNH”), and (“MCD”,”UTX”)). However, we have very large loss (see (“KO”, “UTX”), (“GE”,”UNH”)).
- This implies that, just introducing the finite horizon does not mean the hedge of the big loss.
- By introduction of the model uncertainty with $\lambda = 10$, we have more stable return.
- Indeed, the third column shows it; e.g., even (“KO”,”UTX”) and (“GE”,”MCD”) does not leads big loss for the strategy with finite horizon and model uncertainty.

Summary

- We have following results:
    - The optimal boundary for the pairs trading with finite horizon, taking into account the model uncertainty.
    - The explicit form of the boundary by the incorporation of a simple approximation,
    - Comparison between returns between strategies of pairs trading with infinite horizon, finite horizon and finite horizon with model uncertainty. This comparison implies that the pairs trading strategy taking into account the model uncertainty leads the stability of the return.
- In this talk, we focus on the model uncertainty on the misspecification of parameters. However, we have other types of model uncertainty.
    - The selection of the statistical test for cointegration.
    - The selection of the model of the stochastic processes; e.g., we may choose the OU process, but the appropriate process might be the OU process with jumps.

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