Stochastic calculus without probability: Pathwise integration and functional calculus for functionals of paths with arbitrary Hölder regularity

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Model-free approaches in mathematical finance

- The classical approach to continuous-time finance theory starts by specifying a probability measure $\mathbb{P}$ on a space of scenarios (price trajectories).
- Gain of a strategy is defined as an Ito integral, constructed as a limit in probability of Riemann sums under $\mathbb{P}$.
- Stochastic calculus plays a key role in derivation of key results in continuous-time finance.
- A lot of results rely on probabilistic assumptions such as the Markov property: pricing PDEs, sensitivity analysis, HJB equations for optimal investment...
- However, $\mathbb{P}$ is not known in pratice $\rightarrow$ Knightian uncertainty.
- Also: many results are almost-sure statements about gains, losses or strategies.
- What is the role of $\mathbb{P}$? How far can one go without knowing it?

Pathwise approaches to stochastic calculus

- R Cont, N Perkowski (2018) Pathwise integration and change of variable formulas for paths with arbitrary regularity, ARXIV.
- A Ananova, R Cont (2017) Pathwise integration with respect to paths of finite quadratic variation, `Journal de Mathématiques Pures et appliquées`.
- R Cont *Functional Ito Calculus and Functional Kolmogorov Equations*, (Lectures Notes of the Barcelona Summer School on Stochastic Analysis, July 2012), Springer.
- R Cont and D Fournié (2010) Change of variable formulas for non-anticipative functional on path space, Journal of Functional Analysis, 259, 1043 - 1072.
- H Föllmer (1981) Calcul d'Ito sans probabilités, *Séminaire de Probabilités*.

Notations

$D([0,T],\mathbb{R}^d)$ space of cadlag functions (right continuous with left limits).
$C^\alpha([0,T],\mathbb{R}^d)$ $\alpha-$Holder functions
For a path $\omega \in D([0,T],\mathbb{R}^d)$, denote by

- $\omega(t) \in \mathbb{R}^d$ the value of ω at t
- $\omega_t = \omega(t \wedge .)$: path stopped at t
- $\omega_{t-} = \omega \quad 1_{[0,t[} + \omega(t-) \quad 1_{[t,T]}$

For a process X we denote

- $X(t)$ its value and
- $X_t = X(t \wedge .)$ its path stopped at t.

Young integration

$W_p([0,T],\mathbb{R}^d)$: space of path with finite p-variation
Let $p,q>0$ with $\frac{1}{p}+\frac{1}{q}>1$. Then for any $H\in W_p([0,T],\mathbb{R}^d)$, $S\in W_q([0,T],\mathbb{R}^d)$ and any sequence $\pi^n=(t_0^n=0<..<t_j^n<...<t_{m(n)}^n=T)$. $[0,T]$ of partitions of $[0,T]$ with $|\pi^n|\to 0$, the Riemann sums

$$\sum_{t_i^n\in\pi_n} H(t_i^n).(S(t_{i+1}^n)-S(t_n^i))$$

converge as $n\to\infty$ to a limit $\int_0^T H.dS$ which does not depend on the choice of the partition (π,α) and satisfies

$$|\int_0^T H.dS|\leq C_{p,q}\left(\|H\|_\infty+\|H\|_{p-var}\right)\|S\|_{q-var}$$

Young integration

Young integration allows to define $\int HdS$ when $H \in W_p([0,T],\mathbb{R}^d)$, $S \in W_q([0,T],\mathbb{R}^d)$ with $\frac{1}{p}+\frac{1}{q}>1$.
This fails for typical Brownian or semimartingale paths for which $p=q=2$.
Rough path theory has provided an integration concepts which extends to such situations by *adding an extra structure to the path S*: a (non-unique) 'rough path' associated with S. This construction, however, is neither canonical nor pathwise: one needs to first define multiple integrals of S wrt itself, which is usually done probabilistically.
Moreover, its physical meaning is not always obvious.
In this work we exploit a different, a 'canonical' and strictly pathwise approach to integration against such irregular paths, purely based on the path S itself.

"Calcul d'Itô Sans Probabilités" (Föllmer 1981)

Let $X \in C^0([0,T],\mathbb{R}^d)$ and $f \in C^2(\mathbb{R}^d,\mathbb{R})$. The main idea in the proof of the Ito formula is to consider a sequence of partitions $\pi_n = (0 = t_0^n < t_1^n .. < t_{N(\pi_n)}^n = T)$ of $[0,T]$ with step size decreasing to zero and expand increments of $f(X(t))$ along the partition using a 2nd order Taylor expansion:

$$f(X(t)) - f(X(0)) = \sum_{\pi_n} f(X(t_{i+1}^n)) - f(X(t_i^n))$$

$$= \sum_{\pi_n} \nabla f(X(t_i^n)).(X(t_{i+1}^n) - X(t_i^n))$$

$$+\frac{1}{2}{}^t(X(t_{i+1}^n) - X(t_i^n))\nabla^2 f(X(t_i^n)).(X(t_{i+1}^n) - X(t_i^n)) + \ r(X(t_{i+1}^n), X(t_i^n))$$

Summing over π_n we get

$$f(X(t)) - f(X(0)) = S_1(\pi_n, f) + S_2(\pi_n, f) + R(\pi_n, f)$$

- By uniform continuity of $r(x, y) = f(y) - f(x) - \nabla f(x).(y-x) - 0.5^t(y-x)\nabla^2 f(x)(y-x)$,

$$r(x, y) \leq \varphi(||x-y||)||x-y||^2 \quad \text{with} \quad \varphi(u) \overset{u \to 0}{\to} 0$$

  $R(\pi_n, f) = \epsilon_n \sum_{\pi_n} \|X(t_{i+1}^n) - X(t_i^n)\|^2$.
- So both this term and the 'quadratic Riemann sum'

$$S_2(\pi_n, f) = \frac{1}{2} \sum_{\pi_n} {}^t(X(t_{i+1}^n) - X(t_i^n))\nabla^2 f(X(t_i^n)).(X(t_{i+1}^n) - X(t_i^n))$$

  are controlled by the convergence of (weighted) sums of squared increments of X along π_n.

Quadratic Riemann sums

For d=1: given a path of X, pointwise convergence of 'quadratic Riemann sums'

$$S_2(\pi_n, f) = \frac{1}{2} \sum_{\pi_n} \nabla^2 f(X(t_i^n)).(X(t_{i+1}^n) - X(t_i^n))^2$$

along the path for *every* $f \in C^2(\mathbb{R}^d, \mathbb{R})$ is exactly equivalent to the weak convergence of the sequence of discrete measures

$$\mu_n = \sum_{t_j \in \pi^n} (X(t_{j+1}^n) - X(t_j^n))^2 \delta_{t_j}$$

where δ_t denotes a point mass at t. This is a joint property of the path X and (π_n).
This motivated Föllmer's (1981) definition of 'pathwise quadratic variation along a sequence of partitions.

Quadratic variation along a partition sequence

Definition (Follmer 1981)

Let $\pi_n = (0 = t_0^n < t_1^n .. < t_{N(\pi_n)}^n = T)$ be a sequence of partitions of $[0, T]$ with step $|\pi_n|$ decreasing to zero. A càdlàg function $x \in D([0, T], \mathbb{R})$ is said to have finite quadratic variation along the sequence of partitions $\pi = (\pi_n)_{n \geq 1}$ if the weak limit

$$\mu := \lim_{n \to \infty} \sum_{t_j \in \pi^n} (x(t_{j+1}^n) - x(t_j^n))^2 \delta_{t_j}$$

exists and $[x]_\pi^c$ defined by $[x]_\pi^c(t) = \mu([0, t]) - \sum_{0 < s \leq t} |\Delta x(s)|^2$ is a continuous and increasing function. We denote $[x]_\pi = \mu([0, t])$.

We denote $Q_\pi([0, T], \mathbb{R})$ the set of functions with this property

Characterization in continuous case

Proposition (R.C. & P. Das (2017))

Let $x \in C^0([0,T],\mathbb{R})$ *and define*

$$[x]_{\pi_n}(t) = \sum_{t_j \in \pi^n} (x(t_{j+1}^n \wedge t) - x(t_j^n \wedge t))^2$$

The following properties are equivalent:

1. x *has finite quadratic variation along the sequence of partitions* $(\pi_n)_{n\geq 1}$.
2. *The sequence* $[x]_{\pi_n}$ *converges uniformly on* $[0,T]$ *to a continuous function* $[x]_\pi$.
3. *The sequence* $[x]_{\pi_n}$ *converges pointwise on* $[0,T]$ *to a continuous function* $[x]_\pi$.

Multidimensional paths

Definition (Paths of finite quadratic variation)

$x \in Q^{\pi}([0,T],^{d})$ if, for all $1 \leq i,j \leq d$, $x^i, x^i + x^j$ in $Q^{\pi}([0,T],\mathbb{R})$. Then $[x]_\pi : [0,T] \to S_d^+$ defined by

$$\begin{aligned}([x]_\pi)_{i,j}(t) &= \frac{1}{2}\left([x^i + x^j]_\pi(t) - [x^i]_\pi(t) - [x^j]_\pi(t)\right) \\ &= [x^i, x^j]^c_\pi(t) + \sum_{0<s\leq t} \Delta x^i(s)\Delta x^j(s), \quad i,j = 1,\ldots,d\end{aligned}$$

is an increasing function with values positive symmetric $d \times d$ matrices.

Quadratic Riemann sums

Denote $< A, B >= \sum_{i,j} A_{ij} B_{ij} = tr({}^t A.B)$ for $d \times d$ matrices A, B
Follmer's definition guarantees the convergence of quadratic Riemann sums, which is the condition for deriving the Ito formula:
Let $\omega \in Q_\pi([0,T], \mathbb{R}^d)$ be a path with finite quadratic variation along $(\pi_n, n \geq 1)$.
Then $\forall f \in C^0_b([0,T], \mathbb{R}^{d\times d})$,

$$\sum_{\pi_n} < f(t_i^n), {}^t(\omega(t_{i+1}^n) - \omega(t_i^n))(\omega(t_{i+1}^n) - \omega(t_i^n)) > \stackrel{n\to\infty}{\to} \int_0^T < f, d[\omega] > .$$

uniformly on $[0, T]$.

Föllmer's 'pathwise Ito formula'

Proposition (Föllmer, 1981)

$\forall f \in C^2(\mathbb{R}^d, \mathbb{R}), \forall \omega \in Q_\pi([0,T], \mathbb{R}^d)$, *the non-anticipative Riemann sums along* π

$$\sum_{\pi_n} \nabla f(\omega(t_i^n)).(\omega(t_{i+1}^n) - \omega(t_i^n)) \overset{n\to\infty}{\to} \int_0^T \nabla f(\omega(t)).d^\pi \omega$$

converge pointwise and

$$f(\omega(t)) - f(\omega(0)) = \int_0^t \nabla f(\omega).d^\pi \omega + \frac{1}{2}\int_0^t < \nabla^2 f(\omega), d[\omega]_\pi^c > \\ + \sum_{s\leq t} f(\omega(s)) - f(\omega(s-)) - \nabla f(\omega(s-)).\Delta\omega(s)$$

Pathwise construction of stochastic integrals

- Föllmer's result allows to define integrals of the type

$$\int \nabla f(\omega(t))d\omega \quad f \in C^2(\mathbb{R}^d, \mathbb{R})$$

  as a (pathwise) limit of Riemann sums, for any path ω with finite quadratic variation along (π_n).
- Admissible integrands are gradients of C^2 functions.

Dependence on the partition

Consider now two sequences of partitions π, τ and a continuous path $\omega \in Q_\pi([0,T],\mathbb{R}^d) \cap Q_\tau([0,T],\mathbb{R}^d)$.
Since $\forall f \in C^2(\mathbb{R}^d)$,

$$f(\omega(t)) - f(\omega(0)) = \int_0^t \nabla f(\omega).d^\pi\omega + \frac{1}{2}\int_0^t < \nabla^2 f(\omega), d[\omega]_\pi > \\ = \int_0^t \nabla f(\omega).d^\tau\omega + \frac{1}{2}\int_0^t < \nabla^2 f(\omega), d[\omega]_\tau > \qquad (1)$$

the pathwise integrals are equal if and only if $[\omega]_\pi = [\omega]_\tau$.
But the pathwise quadratic variation **does** depend on the sequence of partition...

Dependence on the sequence of partitions

Proposition (Friedman)

Let $\omega \in C^0([0,T],\mathbb{R}^d)$*. There exists a sequence of partitions* (π_n) *such that* $[\omega]_\pi = 0$*.*

Proof: We construct recursively partitions π_n such that

$$|\pi_n| \leq \frac{1}{n} \qquad \text{and} \qquad \sum_{\pi_n} |\omega(t^n_{k+1}) - \omega(t^n_k)|^2 \leq \frac{1}{n}.$$

Assume we have constructed π_n with this property. Adding to π_n the points $k/(n+1), k = 1..n$ we obtain a partition $\sigma_n = (s^n_i, i = 0..M_n)$ with $|\sigma_n| \leq 1/(n+1)$.

For $i = 0..(M_n - 1)$, we further refine $[s_i^n, s_{i+1}^n]$ as follows. Let $J(i)$ be an integer with

$$J(i) \geq (n+1) M_n |\omega(s_{i+1}^n) - \omega(s_i^n)|^2,$$

$$\tau_{i,k+1}^n = \inf\{t \geq \tau_{i,k}^n, \quad \omega(t) = \omega(s_i^n) + \frac{k\left(\omega(s_{i+1}^n) - \omega(s_i^n)\right)}{J(i)}\}.$$

Then points $(\tau_{i,k}^n, k = 1..J(i))$ defines a partition of $[s_i^n, s_{i+1}^n]$ with

$$|\tau_{i,k+1}^n - \tau_{i,k}^n| \leq \frac{1}{n+1} \quad \text{and} \quad |\omega(\tau_{i,k+1}^n) - \omega(\tau_{i,k}^n)| = \frac{|\omega(s_{i+1}^n) - \omega(s_i^n)|}{J(i)}$$

$$\text{so} \quad \sum_{k=1}^{J(i)} |\omega(\tau_{i,k+1}^n) - \omega(\tau_{i,k}^n)|^2 \leq J(i) \frac{|\omega(s_{i+1}^n) - \omega(s_i^n)|^2}{J(i)^2} = \frac{1}{(n+1) M_n}.$$

Sorting $(\tau_{i,k}^n, i = 0..M_n, k = 1..J(i))$ gives $\pi_{n+1} = (t_j^{n+1})$ such that

$$|\pi_{n+1}| \leq \frac{1}{n+1}, \qquad \sum_{\pi_{n+1}} |\omega(t_{i+1}^n) - \omega(t_i^n)|^2 \leq \frac{1}{n+1}.$$

Definition (Well-balanced sequence of partitions)

Let $\underline{\pi_n} = \inf_{i=0..N(\pi_n)-1} |t^n_{i+1} - t^n_i|$.
The sequence of partitions $(\pi_n)_{n\geq 1}$ well-balanced if

$$\exists c > 0, \qquad \forall n \geq 1, \quad \frac{|\pi_n|}{\underline{\pi_n}} \leq c. \tag{2}$$

Theorem (R.C. & P. Das, 2017)

Let $\alpha > 0$, $f \in C^\alpha([0,T], \mathbb{R}^d)$ and $\tau = (\tau^n)_{n\geq 1}$ and $\sigma = (\sigma^n)_{n\geq 1}$ two well-balanced partition sequences such that

$$f \in Q_\tau([0,T], \mathbb{R}^d) \cap Q_\sigma([0,T], \mathbb{R}^d) \quad \text{and} \qquad [f]_\sigma > 0, \quad [f]_\tau > 0.$$

Then: $\forall t \in [0,T], \qquad [f]_\sigma(t) = [f]_\tau(t)$

p-th variation along a sequence of partitions

Define the *oscillation* of $S \in C([0,T],\mathbb{R})$ along π_n as

$$osc(S,\pi_n) := \max_{[t_j,t_{j+1}]\in\pi_n} \max_{r,s\in[t_j,t_{j+1}]} |S(s)-S(r)|.$$

Definition (p-th variation along a sequence of partitions)

Let $p>0$. A continuous path $S \in C([0,T],\mathbb{R})$ is said to have a p-th variation along a sequence of partitions $\pi = (\pi_n)_{n\geq 1}$ if $osc(S,\pi_n)\to 0$ and the sequence of measures

$$\mu^n = \sum_{[t_j,t_{j+1}]\in\pi_n} \delta(\cdot - t_j)|S(t_{j+1})-S(t_j)|^p$$

converges weakly to a measure μ without atoms. We write $S \in V_p(\pi)$ and call $[S]^p(t) := \mu([0,t])$ for $t\in[0,T]$the *p-th variation* of S.

p-th variation along a sequence of partitions

Functions in $V_p(\pi)$ do not necessarily have finite p-**variation**.
Example: If B is a fractional Brownian motion with Hurst index $H \in (0,1)$ and $\pi_n = \{kT/n : k \in \mathbb{N}_0\} \cap [0, T]$, then $B \in V_{1/H}(\pi)$ and $[B]^{1/H}(t) = t\mathbb{E}[|B_1|^{1/H}]$, while $\|B\|_{p-var} = \infty$ almost surely for $p = 1/H$.

Lemma

Let $S \in C([0,T],\mathbb{R})$. $S \in V_p(\pi)$ if and only if there exists a continuous increasing function $[S]^p$ such that

$$\forall t \in [0,T], \qquad \sum_{\substack{[t_j,t_{j+1}]\in\pi_n:\\ t_j\le t}} |S(t_{j+1}) - S(t_j)|^p \overset{n\to\infty}{\to} [S]^p(t).$$

If this property holds then the convergence is uniform.

'Rough' Change of variable formula

Theorem (R.C- Perkowski (2018))

Let $p \in 2\mathbb{N}$ be even, $S \in V_p(\pi)$. Then for every $f \in C^p(\mathbb{R}, \mathbb{R})$

$$f(S(t)) - f(S(0)) = \int_0^t < T_{p-1} f(S(s)), d^{p-1} S > + \frac{1}{p!} \int_0^t f^{(p)}(S(s)) d[S]^p(s),$$

where the integral is defined as a (pointwise) limit of compensated Riemann sums:

$$\int_0^t < T_{p-1} f(S(s)), d^{p-1} S(s) >:=$$

$$\lim_{n\to\infty} \sum_{\pi_n} \sum_{k=1}^{p-1} \frac{f^{(k)}(S(t_j))}{k!} (S(t_{j+1} \wedge t) - S(t_j \wedge t))^k$$

Pathwise integral

The pathwise integral

$$\int_0^t < T_{p-1}f(S(s)), d^{p-1}S(s) >:= \lim_n R_{p-1}(f, S, \pi_n)$$

is a pointwise limit of compensated Riemann sums

$$R_{p-1}(f, S, \pi_n) = \sum_{\pi_n} \sum_{k=1}^{p-1} \frac{f^{(k)}(S(t_j))}{k!} (S(t_{j+1} \wedge t) - S(t_j \wedge t))^k$$

It should be really seen as an integral of the $(p-1)-$**jet** of f

$$T_{p-1}f(x) = (f^{(k)}(x), k = 0, 1, ..., p-1)$$

with respect to a differential structure of order $p-1$ constructed along $S \in V_p(\pi)$ using the powers of increments up to order $p-1$.

Pathwise local time of order p

Definition (Local time of order p)

Let $p \in \mathbb{N}$ be an even integer and let $q \in [1, \infty]$. A continuous path $S \in C([0,T], \mathbb{R})$ has an L^q*-local time of order* $p-1$ along a sequence of partitions $\pi = (\pi_n)_{n\geq 1}$ if $osc(S, \pi_n) \to 0$ and

$$L_t^{\pi_n, p-1}(\cdot) = \sum_{t_j \in \pi} \mathbf{1}_{\llbracket S(t_j \wedge t), S_{t_{j+1} \wedge t} \rrbracket}(\cdot) |S(t_{j+1} \wedge t) - \cdot|^{p-1}$$

converges weakly in $L^q(\mathbb{R})$ to a weakly continuous map $L\colon [0,T] \to L^q(\mathbb{R})$ which we call the *order p local time* of S. We denote $\mathcal{L}_p^q(\pi)$ the set of continuous paths S with this property.

Intuitively, the limit $L_t(x)$ then measures the rate at which the path S accumulates p-th order variation near x.

Theorem (Higher order Tanaka-Wuermli formula)

Let $p \in 2\mathbb{N}$ be an even integer, $q \in [1, \infty]$ with conjugate exponent $q' = q/(q-1)$. Let $f \in C^{p-1}(\mathbb{R}, \mathbb{R})$ and assume that $f^{(p-1)}$ is weakly differentiable with derivative in $L^{q'}(\mathbb{R})$. Then for any $S \in \mathcal{L}_p^q(\pi)$

$$\int_0^t f'(S(s))\mathrm{d}S(s) := \lim_{n\to\infty} \sum_{[t_j,t_{j+1}]\in\pi_n} \sum_{k=1}^{p-1} \frac{f^{(k)}(S(t_j))}{k!}(S(t_{j+1}\wedge t) - S(t_j \wedge t))^k$$

exists and the following change of variable formula holds:

$$f(S(t)) - f(S(0)) = \int_0^t < T_{p-1}f(S(s)), d^{p-1}S(s) > + \frac{1}{(p-1)!}\int_{\mathbb{R}} f^{(p)}(x)L_t(x)\mathrm{d}x.$$

Multidimensional paths: Symmetric tensors

A symmetric p-tensor T on $\mathbb{R}^d$ is a p-tensor invariant under any permutation $\sigma \in \mathfrak{S}_p$ of its arguments: for $(v_1, v_2, \dots, v_p) \in (\mathbb{R}^d)^p$

$$\sigma T(v_1, \dots, v_p) := T(v_{\sigma 1}, v_{\sigma 2}, \dots, v_{\sigma p}) = T(v_1, v_2, \dots, v_p)$$

The space $\mathrm{Sym}_p(\mathbb{R}^d)$ of symmetric tensors of order p on $\mathbb{R}^d$ is naturally isomorphic to the dual of the space $\mathbb{H}_p[X_1, ..., X_d]$ of homogeneous polynomials of degree p on $\mathbb{R}^d$.

$$\mathbb{S}_p(\mathbb{R}^d) = \bigoplus_{k=0}^{p} \mathrm{Sym}_k(\mathbb{R}^d).$$

For any tensor $T \in T_p(\mathbb{R}^d)$ we define the *symmetric part*

$$\mathrm{Sym}(T) := \frac{1}{p!} \sum_{\sigma \in \mathfrak{S}_k} \sigma T \in \mathrm{Sym}_p(\mathbb{R}^d)$$

where $\mathfrak{S}_p$ of $\{1, \dots, k\}$ is the group of permutations of $\{1, 2, ..., p\}$

Extension to multidimensional functions

Consider now a continuous $\mathbb{R}^d$-valued path $S \in C([0,T],\mathbb{R}^d)$ and a sequence of partitions $\pi_n = \{t_0^n, \ldots, t_{N(\pi_n)}^n\}$ with $t_0^n = 0 < ... < t_k^n < ... < t_{N(\pi_n)}^n = T$. Then

$$\mu^n = \sum_{\pi_n} \underbrace{(S(t_{j+1}) - S(t_j)) \otimes ... \otimes (S(t_{j+1}) - S(t_j))}_{\text{p times}} \delta(\cdot - t_j)$$

defines a tensor-valued measure on $[0,T]$ with values in $\mathrm{Sym}_p(\mathbb{R}^d)$. This space of measures is in duality with the space $C([0,T],\mathbb{H}_p[X_1,...,X_d])$ of continuous functions taking values in homogeneous polynomials of degree p i.e. homogeneous polynomials of degree p with continuous time-dependent coefficients.
This motivates the following definition:

Definition (p-th variation of a multidimensional function)

Let $p \in 2\mathbb{N}$ be an (even) integer, and $S \in C([0,T],\mathbb{R}^d)$ a continuous path and $\pi = (\pi_n)_{n\geq 1}$ a sequence of partitions of $[0,T]$. $S \in C([0,T],\mathbb{R}^d)$ is said to have a p-th variation along $\pi = (\pi_n)_{n\geq 1}$ if $osc(S,\pi_n) \to 0$ and the sequence of tensor-valued measures

$$\mu_S^n = \sum_{\pi_n} (S(t_{j+1}) - S(t_j))^{\otimes p} \quad \delta(\cdot - t_j)$$

converges to a $\mathrm{Sym}_p(\mathbb{R}^d)$-valued measure μ_S without atoms in the following sense: $\forall f \in C([0,T],\mathbb{S}_p(\mathbb{R}^d))$,

$$< f, \mu_n >= \quad \sum_{\pi_n} < f(t_j), (S(t_{j+1}) - S(t_j))^{\otimes p} > \to^{n\to\infty} < f, \mu_S > . \qquad (3)$$

We write $S \in V_p(\pi)$ and call $[S]^p(t) := \mu([0,t])$ the *p-th variation* of S.

Theorem (Rough change of variable formula: multi-dim case)

Let $p \in 2\mathbb{N}$ be an even integer, let (π_n) be a sequence of partitions of $[0, T]$ and $S \in V_p(\pi) \cap C([0, T], \mathbb{R}^d)$. Then for every $f \in C^p(\mathbb{R}, \mathbb{R})$ the limit of compensated Riemann sums

$$\int_0^t < T_{p-1} f(S), d^{p-1} S >= \lim_{n\to\infty} \sum_{\pi_n} \sum_{k=1}^{p-1} \frac{1}{k!} < \nabla^k f(S(t_j)), (S(t_{j+1} \wedge t) - S(t_j \wedge t))^{\otimes k} >$$

exists for every $t \in [0, T]$ and satisfies

$$f(S(t)) - f(S(0)) = \int_0^t < T_{p-1} f(S(u)), d^{p-1} S(u) >$$

$$+ \frac{1}{p!} \int_0^t < \nabla^p f(S(t))), d[S]^p(u) > .$$

Extension to non-anticipative Functionals

Denote $\omega_t = \omega(t \wedge .)$ the *past* i.e. the path stopped at t.

Definition (Non-anticipative Functionals)

A *causal*, or *non-anticipative functional* is a functional $F : [0,T] \times D([0,T], \mathbb{R}^d) \mapsto \mathbb{R}$ whose value only depends on the past:

$$\forall \omega \in \Omega, \quad \forall t \in [0,T], \qquad F(t,\omega) = F(t,\omega_t). \tag{4}$$

Causal functional= map on the space Λ_T^d of stopped paths, defined as the quotient space:

$$\Lambda_T^d := \left([0,T] \times D([0,T], \mathbb{R}^d)\right) \Big/ \sim$$

where $(t,x) \sim (t',x') \leftrightarrow t = t', x_t = x'_t$. Λ_T^d is equipped with a metric

$$d_\infty((t,x),(t',x')) = \sup_{u \in [0,T]} |x(u \wedge t) - x'(u \wedge t')| + |t - t'|.$$

$\mathbb{C}^{0,0}(\Lambda_T^d)$ = continuous maps $(\Lambda_T^d, d_\infty) \to \mathbb{R}$.

Definition (Horizontal and vertical derivatives)

A non-anticipative functional F is said to be:

- horizontally differentiable at $(t, \omega) \in \Lambda_T^d$ if the finite limit exists

$$\mathcal{D}F(t,\omega) := \lim_{h \to 0+} \frac{F(t+h, \omega_t) - F(t, \omega_t)}{h}.$$

- vertically differentiable at $(t, \omega) \in \Lambda_T^d$ if the map

$$\mathbb{R}^d \to \mathbb{R}, \;\; e \mapsto F(t, \omega(t \wedge .) + e1_{[t,T]})$$

  is differentiable at 0; its gradient at 0 is denoted by $\nabla_\omega F(t, \omega)$.

Note that $\mathcal{D}F(t,\omega)$ is **not** the partial derivative in t:

$$\mathcal{D}F(t,\omega) \neq \partial_t F(t,\omega) = \lim_{h \to 0} \frac{F(t+h, \omega) - F(t, \omega)}{h}.$$

Smooth functionals

Definition ($\mathbb{C}_b^{1,p}(\Lambda_T^d)$ functionals)

We denote by $\mathbb{C}_b^{1,p}(\Lambda_T^d)$ the set of non-anticipative functionals $F \in \mathbb{C}_l^{0,0}(\Lambda_T^d)$, such that

- F is horizontally differentiable with $\mathcal{D}F$ continuous at fixed times,
- F is p times vertically differentiable with $\nabla_\omega^j F \in \mathbb{C}_l^{0,0}(\Lambda_T^d)$ for $j = 1..p$
- $\mathcal{D}F, \nabla_\omega^j F \in \mathbb{B}(\Lambda_T^d)$.

Examples of smooth functionals

Example (Cylindrical functionals)

For $g \in C^0(\mathbb{R}^{d\times n}), h \in C^k(\mathbb{R}^d)$ with $h(0) = 0$. Then

$$F(t,\omega) = h\left(\omega(t) - \omega(t_n-)\right) \quad 1_{t\geq t_n} \quad g(\omega(t_1-), \omega(t_2-)..., \omega(t_n-))$$

is in $\mathbb{C}_b^{1,k}$ and

$$\mathcal{D}_t F(\omega) = 0, \quad \text{and} \qquad \forall j = 1..k,$$

$$\nabla_\omega^j F(t,\omega) = h^{(j)}\left(\omega(t) - \omega(t_n-)\right) 1_{t\geq t_n} g\left(\omega(t_1-), \omega(t_2-)..., \omega(t_n-)\right)$$

$\mathbb{S}(\Lambda_T, \pi_n) :=$ space of simple predictable cylindrical functionals piecewise constant along π_n, $\mathbb{S}(\Lambda_T, \pi) := \cup_{n>1} \mathbb{S}(\Lambda_T, \pi_n)$x

Examples of smooth functionals

Example (Integral functionals)

For $g \in C_0(\mathbb{R}^d)$, $Y(t) = \int_0^t g(X(u))\rho(u)du = F(t, X_t)$ where

$$F(t,\omega) = \int_0^t g(\omega(u))\rho(u)du \tag{5}$$

$F \in \mathbb{C}_b^{1,\infty}$, with:

$$\mathcal{D}_t F(\omega) = g(\omega(t))\rho(t) \qquad \nabla_\omega^j F(t,\omega) = 0 \tag{6}$$

Conditional expectations as smooth functionals

Let $\sigma \in \mathbb{C}^{0,0}(\mathcal{W}_T)$ be such that

$$X(t) = X(0)\exp\left(\int_0^t \sigma(u)dW(u) - \frac{1}{2}\int_0^t \sigma^2(u).du\right) \qquad (*)$$

is a martingale, i.e. $E(X(T)) = 1$ and denote by $\mathbb{Q}^\sigma$ the law of (*).

Proposition (Cont & Riga 2015)

Let $h : (D([0,T],R), \|.\|_\infty) \mapsto \mathbb{R}$ be $\mathbb{Q}^\sigma$-integrable and Lipschitz. Assume that for $(t,\omega) \in \mathcal{W}_T$, the map

$$g^h(.;t,\omega) : e \in \mathbb{R}^d \to \quad g^h(e) = h\left(\omega + e1_{[t,T]})\right), \qquad (7)$$

is twice differentiable at 0, with derivatives bounded uniformly in $(t,\omega) \in \mathcal{W}_T$ in a neighborhood of 0. Then, there exists $F \in \mathbb{C}_b^{0,2}(\mathcal{W}_T)$ such that $F(t, X_t) = E^{\mathbb{Q}^\sigma}[H|\mathcal{F}_t^X]$ $\mathbb{Q}^\sigma-a.s.$

Weak Euler schemes as smooth functionals

Let $\sigma : (\Lambda_T, d_\infty) \to \mathbb{R}^{d\times d}$ be a Lipschitz map. Then

$$ {}_nX(t_{j+1},\omega) = {}_nX(t_j,\omega) + \sigma(t_j, {}_nX_{t_j}(\omega)) \cdot (\omega(t_{j+1}-) - \omega(t_j-)) \,. \quad (8) $$

defines a non-anticipative functional ${}_nX$ which approximates

$$ X(t) = X(0) + \int_0^t \sigma(u, X_u) dW(u) \quad (9) $$

For a Lipschitz functional $g : (D([0,T],\mathbb{R}^d), \|.\|_\infty) \to \mathbb{R}$, consider the 'weak Euler approximation' of $\mathbb{E}\left[g(X_T)|\mathcal{F}_t^W\right]$:

$$ F_n(t,\omega) = \mathbb{E}\left[g\left({}_nX_T(W_T)\right)|\mathcal{F}_t^W\right](\omega). \quad (10) $$

(R Cont- Yi Lu, SPA 2016): $F_n \in \mathbb{C}_b^{1,\infty}(\mathcal{W}_T)$.

Functional change of variable formula: $p = 2$

Theorem (R.C.- Fournié ,2010)

Let $\omega \in V_2(\pi)$ *and* $\omega^n := \sum_{i=0}^{m(n)-1} \omega(t_{i+1}^n -)\mathbf{1}_{[t_i^n, t_{i+1}^n)} + \omega(T)\mathbf{1}_{\{T\}}$. *Then for any* $F \in \mathbb{C}_b^{1,2}(\Lambda_T^d)$, *the pointwise limit of Riemann sums*

$$\int_0^T \nabla_\omega F(t,\omega) d^\pi \omega := \lim_{n\to\infty} \sum_{i=0}^{m(n)-1} \nabla_\omega F(t_i^n, \omega^n)(\omega(t_{i+1}^n) - \omega(t_i^n))$$

exists and

$$F(T,\omega) = F(0,\omega) + \int_0^T \nabla_\omega F(t,\omega)\cdot d^\pi\omega + \int_0^T \mathcal{D}F(t,\omega)dt + \int_0^T \frac{1}{2} tr\left(\nabla_\omega^2 F(t,\omega) d[\omega]_\pi^c(t)\right).$$

These result allows to construct $\int_0^{\cdot} \nabla_\omega F$ as a pointwise limit of non-anticipative 'Riemann sums':

$$\int_0^T \nabla_\omega F(t, \omega_{t-}) \cdot d^\pi \omega = \lim_{n \to \infty} \sum_{i=0}^{m(n)-1} \nabla_\omega F(t_i^n, \omega^n)(\omega(t_{i+1}^n) - \omega(t_i^n))$$

Remark

$$F \in \mathcal{R}(\Lambda_T^d),\ \omega \in C^{\frac{1}{2}-}([0,T], \mathbb{R}^d) \Rightarrow \nabla_\omega F(t, \omega) \in C^{\frac{1}{2}-}([0,T], \mathbb{R}^d).$$

The pathwise integral is a **strict extension** *of the Young integral.*

$$\text{Denote} \qquad C^{\frac{1}{2}-}([0,T],\mathbb{R}^d) = \bigcap_{\nu<1/2} C^{\nu}([0,T],\mathbb{R}^d)$$

Theorem (Pathwise Isometry formula, A. Ananova, R. C. 2016)

If $F \in \mathbb{C}^{1,2}(\Lambda_T)$ is Lipschitz-continuous and $\nabla_\omega F \in \mathbb{C}^{1,1}_b(\Lambda^d_T)$ then for any $\omega \in Q_\pi([0,T],\mathbb{R}) \cap C^{1/2-}([0,T],\mathbb{R}^d)$ and any sequence of partitions $\pi = (\pi_n)_{n\geq 1}$ satisfying $osc(F(.,\omega),\pi_n) \to_{n\to+\infty} 0$ we have

$$[F(t,\omega)]^\pi(t) =$$

$$\left[\int_0^{\cdot} \nabla_\omega F(s,\omega).d^\pi\omega\right]^\pi(t) = \int_0^t \langle {}^t\nabla_\omega F(s,\omega).\nabla_\omega F(s,\omega), d[\omega]^\pi(s)\rangle. \qquad \text{(Isometry)}$$

Theorem

Let p be an even integer, let $F \in \mathbb{C}_b^{1,p}(\Lambda_T)$, and let $S \in V_p(\pi)$ for a sequence of partitions (π_n) with $|\pi_n| \to 0$. Then the limit $\int_0^t < \mathbb{T}_{p-1}F(s, S_s), d^{p-1}S(s) >=$

$$\lim_{n\to\infty} \sum_{[t_j,t_{j+1}]\in\pi_n} \sum_{k=1}^{p-1} \frac{1}{k!} \nabla_\omega^k F(t_j, S^n_{t_j-})(S(t_{j+1}\wedge t) - S(t_j \wedge t))^k, \quad \text{and}$$

$$F(t, S_t) = F(0, S_0) + \int_0^t \mathcal{D}F(s, S_s)ds$$

$+ \int_0^t < \mathbb{T}_{p-1}F(s, S_s), d^p S > + \frac{1}{p!} \int_0^t \nabla_\omega^p F(s, S_s) d[S]^p(s)$

A higher order isometry formula

The following result extends the pathwise isometry formula (Ananova-C. 2017) for the pathwise integral to paths with lower regularity:

Theorem ('Isometry' formula)

Let $p \in \mathbb{N}$ be an even integer, $\alpha > ((1 + \frac{4}{p})^{1/2} - 1)/2$, (π_n) a sequence of partitions with mesh size going to zero, and $S \in V_p(\pi) \cap C^{\alpha}([0, T], \mathbb{R})$. Let $F \in \mathbb{C}_b^{1,p}(\Lambda_T) \cap \mathrm{Lip}(\Lambda_T, d_\infty)$ be such that $\nabla_\omega F \in \mathbb{C}_b^{1,p-1}(\Lambda_T)$. Then $F(\cdot, S) \in V_p(\pi)$ and

$$[F(\cdot, S)]^p(t) = \int_0^t |\nabla_\omega F(s, S_s)|^p d[S]^p(s).$$

Higher-order Dirichlet decomposition

The following result gives a Doob-Meyer/ Dirichlet decomposition for functionals of processes with arbitrary regularity:

Theorem

Let $p \in \mathbb{N}$ be an even integer, let $\alpha > ((1+\frac{4}{p})^{1/2}-1)/2$, and let $S \in V_p(\pi) \cap C^\alpha([0,T],\mathbb{R})$ be a path with strictly increasing p-th variation $[S]^p$ along (π_n). Then any $X \in \mathbb{C}_b^{1,p}(S)$ admits a unique decomposition

$$X = X(0) + A + \int_0^t < \phi, d^{p-1} S >$$

where ϕ is a $(p-1)$-form and $[A]^p = 0$.

Relation with rough path integration

Define a *control function* as a continuous map $c \colon \Delta_T \to \mathbb{R}_+$ such that $c(t,t) = 0$ and $c(s,u) + c(u,t) \le c(s,t)$.

Definition

Reduced rough path of order p Let $p \ge 1$. A *reduced rough path* of finite p-variation is a map $\mathbb{X} = (1, \mathbb{X}^1, \ldots, \mathbb{X}^{\lfloor p \rfloor}) \colon \Delta_T \longrightarrow \mathbb{S}_{\lfloor p \rfloor}(\mathbb{R}^d)$, such that

$$\sum_{k=1}^{\lfloor p \rfloor} |\mathbb{X}^k_{s,t}|^{p/k} \le c(s,t), \qquad (s,t) \in \Delta_T;$$

for some control function c and the *reduced Chen relation* holds

$$\mathbb{X}_{s,t} = \mathrm{Sym}(\mathbb{X}_{s,u} \otimes \mathbb{X}_{u,t}), \qquad (s,u),(u,t) \in \Delta_T.$$

The canonical reduced rough path for $S \in V_p(\pi)$

Lemma

Let $S \in C([0,T], \mathbb{R}^d)$ and let (π_n) be the dyadic Lebesgue partition generated by S. Let $p \geq 1$ and assume that $S \in V_p(\pi)$. Then for any $q > p$ with $\lfloor q \rfloor = \lfloor p \rfloor$ we obtain a reduced rough path of finite q-variation by setting $\mathbb{X}^0_{s,t} := 1$,

$$\mathbb{X}^k_{s,t} := \frac{1}{k!}(S(t) - S(s))^{\otimes k}, \qquad k = 1, \ldots, \lfloor p \rfloor - 1,$$

$$\mathbb{X}^{\lfloor p \rfloor}_{s,t} := \frac{1}{\lfloor p \rfloor !}(S(t) - S(s))^{\otimes \lfloor p \rfloor} - \frac{1}{\lfloor p \rfloor !}([S]^p(t) - [S]^p(s)).$$

Proposition

Let $p \geq 1$, let $\mathbb{X}$ be a reduced rough path of finite p-variation and let $Y \in \mathcal{D}_{\mathbb{X}}^{\lfloor p \rfloor / p}([0,T])$. Then the rough path integral

$$I_{\mathbb{X}}(Y)(t) = \int_0^t \langle Y(s), \mathrm{d}\mathbb{X}(s) \rangle = \lim_{\substack{\pi \in \Pi([0,t]) \\ |\pi| \to 0}} \sum_{[t_j, t_{j+1}] \in \pi} \sum_{k=1}^{\lfloor p \rfloor} \langle Y^k(t_j), \mathbb{X}^k_{t_j, t_{j+1}} \rangle,$$

defines a function in $C([0,T],\mathbb{R})$, and it is the unique function with $I_{\mathbb{X}}(Y)(0) = 0$ for which there exists a control function c with

$$\Big| \int_s^t \langle Y(r), \mathrm{d}\mathbb{X}(r) \rangle - \sum_{k=1}^{\lfloor p \rfloor} \langle Y^k(s), \mathbb{X}^k_{s,t} \rangle \Big| \lesssim c(s,t)^{\frac{\lfloor p \rfloor + 1}{p}}, \qquad (s,t) \in \Delta_T.$$

Pathwise integral as 'canonical' rough integral

Proposition (R.C- N. Perkowski 2018)

Let $p \in 2\mathbb{N}$ be an even integer, $S \in V_p(\pi)$ and $\mathbb{X}$ the canonical reduced rough path of order p associated to S, defined above. Then

$$\underbrace{\int_0^t \langle \nabla f(S(s)), \mathrm{d}\mathbb{X}(s) \rangle}_{\text{Rough integral}} = \underbrace{\int_0^t \langle T_{p-1} f(S), \mathrm{d}^{p-1} S \rangle}_{\text{Pathwise integral}},$$

where the right hand side is the pathwise integral defined as a limit of compensated Riemann sums.